Elect. Comm. in Probab. **13** (2008), 1–9

ELECTRONIC COMMUNICATIONS in PROBABILITY

A STOCHASTIC SCHEME OF APPROXIMATION FOR ORDINARY DIFFERENTIAL EQUATIONS

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Submitted July 30, 2007, *accepted in final form* November 20, 2007

AMS 2000 Subject classification: Primary: 60H10, Secondary: 34F05, 93E03

Keywords: Numerical Scheme - Convergence in law - Central limit theorem.

Abstract

In this note we provide a stochastic method for approximating solutions of ordinary differential equations. To this end, a stochastic variant of the Euler scheme is given by means of Markov chains. For an ordinary differential equation, these approximations are shown to satisfy a Large Number Law, and a Central Limit Theorem for the corresponding fluctuations about the solution of the differential equation is proven.

1 Introduction

This paper deals with stochastic approximation for ordinary differential equations. A number of numerical methods for approximating solutions of such equations have been developed, and references to this subject can be found by Fierro & Torres in [2], Kloeden & Platen in [4], and San Martín & Torres in [6], among others. The purpose of this paper is to give and analyze a variant of the Euler scheme. For a given ordinary differential equation we propose a method consisting in a sequence of Markov chains to approximate its solution, where each of these chains minus its initial condition takes values in a finite state space of rational numbers previously defined. Two important considerations have to be done when compared this method with the Euler scheme. In the Euler scheme its state space could be even uncountable and digital computers are restricted to rational numbers when doing

[1]RESEARCH SUPPORTED IN PART BY LABORATORIO DE ANÁLISIS ETOCÁSTICO, PBCT-ACT13.

[2]RESEARCH SUPPORTED IN PART BY LABORATORIO DE ANÁLISIS ETOCÁSTICO, PBCT-ACT13 AND FONDECYT GRANT N. 1070919.

calculations, which may introduce round-off error.

When approximation schemes are to be applied, their inherent round-off errors are an important subject which must be considered. Usually, the finer discretization of the approximation method is, the bigger the round-off error becomes. We refer to Henrici in [3] for a complete discussion on methods for solving differential equation problems, error propagation and rate of convergence. In this paper, we are interested in reducing the round-off error when numerical methods are used for approximating the unique solution of an ordinary differential equation. Our main consideration is that in practice, both we and digital computers are restricted to a finite number of decimal places. Hence, we introduce a method of approximation which considers a finite state space with states having few digitals. However, to carry out this simplification our scheme of approximation needs to be stochastic. Some authors interested in decreasing the round-off error have contributed in this direction. For instance, this circumstance have been considered by Bykov in [1] for systems of linear ordinary differential equations, by Srinivasu & Venkatesulu in [7] for nonstandard initial value problems, and by Wollman in [8] for the one-dimensional Vlasov-Poisson system, among others. An important difference between the algorithms introduced in these articles and our work is that the first ones are deterministic while the scheme we are introducing here aims to approximate the solutions to ordinary differential equations by means of stochastic processes taking values in finite state spaces.

On the other hand, even though the proposed approximation method does not have this disadvantage, it introduces a stochastic error as a consequence of the random choice of states considered in the approximation. In [4] (Section 6.2) an approximation scheme for stochastic differential equations is given by means of Markov chains on a countable infinite state space, however, our method is different and it can not be obtained as a particular case of the former one.

After constructing the approximation scheme, our first aim is to prove such a scheme satisfies a Law of Large Numbers, that is it converges in some sense to the unique solution to the initial value problem. The second aim of this paper is to state a Central Limit Theorem for the fluctuations of these approximations about of the mentioned solution.

The plan of this paper is as follows. In Section 2 we define the approximation scheme. Main results are stated in Section 3 and their proofs are deferred to Section 4.

2 The approximation scheme

Let us consider the following ordinary differential equation:

$$\dot{X}(t) = b(t, X(t)), X(0) = x_0, \tag{1}$$

where $x_0 \in \mathbb{R}^d$, and $b : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies sufficient conditions of regularity which ensure existence and uniqueness of the solution to (1).

The Markov chains approximating the unique solution to (1) are defined by means of their probability transitions. Let $\mathcal{H}$ be the set of all functions h from $\mathbb{R}$ into itself such that for each $x \in \mathbb{R}$, $[x] \leq h(x) \leq x$, and for each $x \in \mathbb{R} \setminus \mathbb{Z}$, h is continuous at x. In what follows, $h_1, \ldots, h_d$ denote d functions in $\mathcal{H}$ and $H : \mathbb{R}^d \to \mathbb{R}^d$ stands for the function defined by $H(y_1, \ldots, y_d) = (h_1(y_1), \ldots, h_d(y_d))$.

For each $n \in \mathbb{N}$, let $0 = t_0^n < \ldots < t_n^n = 1$ be the partition of $[0,1]$ defined as $t_k^n = k/n, i = 0, 1, \ldots, n$.

On a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and for each $n \in \mathbb{N}$, we define the random variables $\xi_1^n, \ldots, \xi_n^n$ and the Markov chain $X_0^n, X_1^n, \ldots, X_n^n$ starting at $X_0^n = x_0$, recursively as follows.

Suppose X_k^n is defined and consider ξ_{k+1}^n, $k=0,1,\dots,n-1$, such that

$$\mathbb{P}\left(\xi_{k+1}^n = e \mid X_k^n\right)(\omega) = \mu_k^1(\{e_1\},\omega)\cdots\mu_k^d(\{e_d\},\omega), \tag{2}$$

where $e=(e_1,\dots,e_d)\in\{0,1\}^d$, $\omega\in\Omega$ and, $\mu_k^i(\cdot,\omega)$, $i=1,\dots,d$, is the Bernoulli law of parameter $p_i^n(t_k^n)=b_i(t_k^n,X_k^n(\omega))-h_i(b_i(t_k^n,X_k^n(\omega)))$.
Next we define

$$X_{k+1}^n = X_k^n + \frac{1}{n}(H(b(t_k,X_k^n))+\xi_{k+1}^n). \tag{3}$$

In what follows I_A stands for the indicator function of the set A and p_i^n denotes the function from [0, 1] into itself defined as

$$p_i^n(t)=\sum_{k=0}^{n-1} p_i^n(t_k^n)\mathrm{I}_{[t_k^n,t_{k+1}^n[}(t)+p_i^n(t_n^n)\mathrm{I}_{\{1\}}(t).$$

Note that if H is defined as $H(x)=x$, then $\xi_{k+1}^n=0,\mathbb{P}-$a.s. Hence, in this case, $X_0^n,\dots,X_n^n$ is the well-known deterministic Euler scheme for the solution to (1). Another extreme case is obtained when H is defined as $H(x)=([x_1],\dots,[x_d])$, where for $x\in\mathbb{R}$, $[x]$ denotes the integral part of x. In this situation, the Markov chain takes values in a finite state space, however this scheme is not deterministic. Intermediate situations can be obtained for arbitrary $H=(h_1,\dots,h_d)\in\mathcal{H}^d$.
For $k=1,\dots,n$, let us define

$$m_k^n=\xi_k^n-\left(b(t_{k-1},X_{k-1}^n)-H(b(t_{k-1},X_{k-1}^n))\right).$$

By defining $L_r^n=\sum_{k=1}^{r} m_k^n$, it is easy to see that for each $n\in\mathbb{N}$, $(L_r^n; r=1,\dots,n)$ is a $\mathcal{F}^n$-martingale with mean zero, where $\mathcal{F}^n=\{\mathcal{F}_0^n,\dots,\mathcal{F}_n^n\},\mathcal{F}_0^n=\{\emptyset,\Omega\}$ and $\mathcal{F}_k^n=\sigma\{\xi_1^n,\dots,\xi_k^n\}$ $k=1,\dots,n$.
For approximating X, we define the process $\{X^n(t);0\le t\le 1\}$ as

$$X^n(t)=\sum_{k=0}^{n-1}[X_k^n+(t-t_k^n)b(t_k^n,X_k^n))]\mathrm{I}_{[t_k^n,t_{k+1}^n[}(t)+X_n^n\mathrm{I}_{\{1\}}(t). \tag{4}$$

By defining $\mathcal{F}_t^n=\mathcal{F}_{[nt]}^n, t\in[0,1]$, X^n turns out to be adapted to $\mathbb{F}^n=\{\mathcal{F}_t^n;0\le t\le 1\}$, and for each $t\in[0,1]$ we have

$$X^n(t) \;=\; X^n(0)+\int_0^t b(c_n(u),X^n(c_n(u)))\,\mathrm{d}\,u+L_{[nt]}^n/n, \tag{5}$$

where $c_n(u)=[nu]/n$.

3 Main Results

In this section we state the main results and their proofs are deferred to the next section. We will make the following standing assumptions throughout the paper.

(A1) $\| b(t,x)-b(t,y)\|_d^2 < K_1 \| x-y\|_d^2$.

(A2) $\| b(t,x) \|_d^2 < K_2(1+ \| x \|_d^2)$.

(A3) $\| b(t,x) - b(s,x) \|_d^2 < K_3(1+ \| x \|_d^2)|s-t|^2$.

for all $x, y \in \mathbb{R}^d$, $s,t \in [0,1]$ where K_1, K_2 and K_3 are positive real constants, and $\| \cdot \|_d$ stands for the usual norm in $\mathbb{R}^d$.
The first theorem states existence of a bound for the global error when the Markov chain $X_0^n, X_1^n, \ldots, X_n^n$ is used in order to approximate the solution to (1).

Theorem 3.1. (Global error.) *There exists a positive constant D such that*

$$\mathbb{E}\left(\sup_{0\le t\le 1} ||X^n(t) - X(t)||_d^2\right) \le D/n.$$

The following result is concerned with the asymptotic behavior of the scheme used to approximate the solution to (1). Before stating it, let $\mathrm{D_x}(b(s,X))$ and $p_i(u)$ denote, respectively, the $d \times d$-matrix defined as

$$\mathrm{D_x}(b(s,X)) = \begin{pmatrix} \frac{\partial b_1}{\partial x_1}(s,X) & \cdots & \frac{\partial b_1}{\partial x_d}(s,X) \\ \vdots & \ddots & \vdots \\ \frac{\partial b_d}{\partial x_1}(s,X) & \cdots & \frac{\partial b_d}{\partial x_d}(s,X). \end{pmatrix},$$

and

$$p_i(s) = b_i(s, X(s)) - h_i(b_i(s, X(s)) \quad i = 1, \ldots, d.$$

Theorem 3.2. (Central Limit Theorem.) *Let $Z^n(t) = \sqrt{n}\,(X^n(t) - X(t))$ and suppose b continuously differentiable at the second variable. Then, the sequence $(Z^n; n \in \mathbb{N})$ converges in law to the solution Z of the following stochastic differential equation*

$$Z(t) = \int_0^t \mathrm{D_x}\left(b(s, X(s))\right) \cdot Z(s)\mathrm{d}s + M(t),$$

where $M = (M_1, \ldots, M_d)$ is a continuous Gaussian martingale starting at zero and having predictable quadratic variation given by the diagonal matrix $< M > (t)$ whose diagonal elements are respectively given by

$$< M_i > (t) = \int_0^t p_i(s)(1 - p_i(s))\mathrm{d}s, \quad i = 1, \ldots, d.$$

4 Proofs

In order to prove the theorems state in Section 3, we need the following five lemmas.

Lemma 4.1. *There exist a positive constants C and such that for each $n \in \mathbb{N}$,*

$$\mathbb{E}\left(\sup_{0\le t\le 1} ||X^n(t)||_d^2\right) \le C.$$

Proof: From conditions (A1) and (A2) the following inequality holds:

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le t\le 1}||X^n(t)||_d^2\right) &\le 3||X^n(0)||_d^2+3K_2+3K_2\mathbb{E}\left(\int_0^1\sup_{0\le u\le s}||X^n(u)||_d^2\mathrm{d}s\right)\\ &+\frac{3}{n^2}\mathbb{E}\left(\sup_{1\le r\le[nt]}||L_r^n||_d^2\right).\end{aligned}$$

Since conditional covariances between $m_k^n(u)$ and $m_k^n(v)$ are equal to zero for $u,v=1\dots d$ with $u\neq v$, we obtain

$$\begin{aligned}\mathbb{E}\left(\sup_{1\le r\le[nt]}||L_r^n||_d^2\right) &= \mathbb{E}\left(\sup_{1\le r\le[nt]}\sum_{u=1}^d|\sum_{k=1}^r m_k^n(u)|^2\right)\\ &\le \sum_{u=1}^d\mathbb{E}\left(\sup_{1\le r\le[nt]}|\sum_{k=1}^r m_k^n(u)|^2\right)\\ &\le 4\sum_{u=1}^d\mathbb{E}\left(\left\langle\sum_{k=1}^{[nt]}m_k^n(u)\right\rangle\right)\\ &= 4\sum_{u=1}^d\mathbb{E}\left(\sum_{k=1}^{[nt]}\mathbb{E}\left((m_k^n(u))^2/\mathcal{F}_{k-1}^n\right)\right)\\ &= 4\sum_{u=1}^d\left(\sum_{k=1}^{[nt]}p_u^n(t_k^n)(1-p_u^n(t_k^n))\right)\\ &\le dn. \end{aligned}\tag{6}$$

Then

$$\mathbb{E}\left(\sup_{0\le t\le 1}||X^n(t)||_d^2\right)\le 3||X^n(0)||_d^2+3K_2+\frac{3d}{n}+3K_2\mathbb{E}\left(\int_0^1\sup_{0\le u\le s}||X^n(u)||_d^2\mathrm{d}s\right),$$

and by the Gronwall inequality we have

$$\mathbb{E}\left(\sup_{0\le t\le 1}||X^n(t)||_d^2\right)\le C_3\left(1+||X^n(0)||_d^2\right)e^{C_4},$$

where $C_3=3(K_2+d)$, and $C_4=3K_2$. This completes the proof. □

Lemma 4.2. *There exists a positive constant C such that*

$$||X(t)-X(s)||_d^2\le C|t-s|^2.$$

Proof: Since $X(t)$ satisfies (1) for all $s<t$ we have

$$||X(t)-X(s)||_d^2\le (t-s)\int_s^t||b(u,X(u))||_d^2\mathrm{d}u.$$

This fact and (A2) imply that

$$||X(t)-X(s)||_d^2\le K_2(t-s)\int_s^t(1+||X(u)||_d^2)\mathrm{d}u$$

$$\leq C(t-s)^2,$$

where $C = K_2 \sup_{0\leq u\leq 1}(1+||X(u)||_d^2)$. Therefore, the proof is complete. $\square$

Proof of Theorem 3.1 Let

$$a_n(t) = \mathbb{E}\left(\sup_{0\leq u\leq t} \|X^n(u) - X(u)\|_d^2\right).$$

We have

$$a_n(t) \leq 4\left(A^n(t) + B^n(t) + C^n(t) + D^n(t)\right)$$

where

$$\begin{aligned}
A^n(t) &= \mathbb{E}\left(\sup_{0\leq u\leq t} || \int_0^u b(s, X(s)) - b(c_n(s), X(s))\mathrm{d}s||_{\mathrm{d}}^2\right) \\
B^n(t) &= \mathbb{E}\left(\sup_{0\leq u\leq t} || \int_0^u b(c_n(s), X(s)) - b(c_n(s), X(c_n(s)))\mathrm{d}s||_{\mathrm{d}}^2\right) \\
C^n(t) &= \mathbb{E}\left(\sup_{0\leq u\leq t} || \int_0^u b(c_n(s), X(c_n(s))) - b(c_n(s), X^n(c_n(s)))\mathrm{d}s||_{\mathrm{d}}^2\right) \\
D^n(t) &= \mathbb{E}\left(\sup_{1\leq r\leq [nt]} ||L_r^n||_d^2\right)/n^2.
\end{aligned}$$

From (A3) there exists $C_1 > 0$ such that

$$A^n(1) \leq C_1/n^2, \tag{7}$$

and by (A1) and Lemma 4.2, there exists $C_2 > 0$ such that

$$B^n(1) \leq C_2/n^2. \tag{8}$$

From condition (A1) and Jensen's inequality, we have

$$C^n(t) \leq K_1 \int_0^t a_n(s)\mathrm{d}s, \tag{9}$$

and from (6) we obtain

$$D^n(t) \quad \leq \quad d/n. \tag{10}$$

Consequently, (7), (8), (9) and (10) imply that

$$a_n(t) \leq \frac{1}{n}E + 4K_1 \int_0^t a_n(u)\mathrm{d}u, \tag{11}$$

where $E = 4(C_1 + C_2 + d))$, and by applying Gronwall's inequality to (11), we have $a_n(1) \leq E\exp(4K_1)/n$, which concludes the proof. $\square$

In the sequel, for each $n \in \mathbb{N}$, M^n stands for the martingale defined as $M^n(t) = L_{[nt]}^n/\sqrt{n}$, $t \in [0,1]$.

Lemma 4.3, Lemma 4.4 and Lemma 4.5 below are used to prove Theorem 3.2.

Lemma 4.3. *For each $t \in [0,1]$,*

$$< M^n > (t) \to < M > (t) \quad \textit{in probability as} \quad n \to \infty.$$

Proof: Since, the conditional covariance between $m_i^n(j)$ and $m_i^n(k)$ for $j,k = 1 \ldots d$ and $j \neq k$ is equal to zero, we need only prove that for each $i = 1, \ldots, d$,

$$< M_i^n > (t) \to < M_i > (t) \quad \text{in probability,} \tag{12}$$

where $M^n = (M_1^n, \ldots, M_d^n)$.
Let $f : [0,1) \to \mathbb{R}$ be and $g_i : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$ be the functions defined by $f(p) = p(1-p)$ and $g_i(u,x) = f(b_i(u,x) - [b_i(u,x)]), i = 1, \ldots, d$, respectively.
We have,

$$\begin{aligned} | < M_i^n > (t) - < M_i > (t)| &= \left| \int_0^t (f(p_i^n(u)) - f(p_i(u)))\mathrm{d}u \right| \\ &\leq \left| \int_{\{u \in [0,t] : p_i(u) = 0\}}^t g_i(u, X^n(u))\mathrm{d}u \right| \\ &+ \left| \int_{\{u \in [0,t] : p_i(u) > 0\}}^t (f(p_i^n(u)) - f(p_i(u)))\mathrm{d}u \right|. \end{aligned}$$

First and second terms on the right hand side of this inequality converges in probability to zero due to g_i is continuous on $\mathbb{R}$ and $f \circ (b_i - h_i \circ b_i)$ is continuous at (u,x) whenever $p_i(u) > 0$. Therefore the proof is complete. □

Lemma 4.4. *The sequence $(M^n; n \in \mathbb{N})$ converges in law to M.*

Proof: We use the criterion given by Rebolledo in [5] (Proposition 1). We have $\sup\limits_{0 \leq t \leq 1} \|\Delta M^n(t)\|_d \leq 1/\sqrt{n}$ and from Lemma 4.3, for each $t \geq 0$, $< M^n > (t)$ converges in probability to $< M > (t)$. Therefore, by the mentioned criterion, the proof is complete. □

Lemma 4.5. *There exists $C > 0$ such that*

$$\sup_{n \in \mathbb{N}} \mathbb{E}(\sup_{0 \leq t \leq 1} \|Z^n(t)\|_d^2) \leq C. \tag{13}$$

Proof: Since $X^n(c_n(u)) = X^n(u)$, from (1) and (5), we have

$$\begin{aligned} X^n(t) - X(t) &= \int_0^t [b(c_n(u), X^n(u)) - b(c_n(u), X(u))] \,\mathrm{d}\, u \\ &+ \int_0^t [b(c_n(u), X(u)) - b(u, X(u))] \,\mathrm{d}\, u \\ &+ M^n(t)/\sqrt{n}. \end{aligned} \tag{14}$$

For a bounded function x from $\mathbb{R}_+$ to $\mathbb{R}^d$, we define $x_*(t) = \sup\limits_{0 \leq u \leq t} \|x(u)\|_d^2$. With this notation, the following inequality is obtained from (14), (A1) and (A2):

$$Z_*^n(t) \leq 3K_1 \int_0^t Z_*^n(u) \,\mathrm{d}\, u + 3K_4 + 3M_*^n(t), \tag{15}$$

where $K_4 = K_3 \sup_{0 \leq u \leq t}(1 + \| X(u) \|_d^2)$.

From (6), $\sup_{n\in\mathbb{N}} \mathbb{E}(M_*^n(t)) \leq d$. Hence, (15) and Gronwall's inequality, imply that (13) holds. $\square$

Proof of the Theorem 3.2 Since $b : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$ is continuously differentiable at the second variable and $X^n(c_n(u)) = X^n(u)$, the Mean Value Theorem implies that

$$\begin{aligned} b(c_n(u), X^n(c_n(u))) - b(u, X(u)) &= \mathrm{D_x}(b(c_n(u), \eta^n(u))) \cdot (X^n(u) - X(u)) \\ &+ b(c_n(u), X(u)) - b(u, X(u)) \end{aligned} \tag{16}$$

where $\eta^n = (\eta_1, \ldots, \eta_d)$ and $\eta_i(u)$ lies between $X_i^n(u)$ and $X_i(u)$, for $i = 1, \ldots, d$.
By combining (14) and (16) we obtain,

$$Z^n(t) = \int_0^t \mathrm{D_x}\left(b(u, X(u))\right) \cdot Z^n(u)\,\mathrm{d}\,u + M^n(t) + R^n(t), \tag{17}$$

where

$$\begin{aligned} R^n(t) &= \int_0^t [\mathrm{D_x}(b(c_n(u), \eta^n(u))) - \mathrm{D_x}(b(u, X(u)))] \cdot Z^n(u)\mathrm{d}u \qquad (18) \\ &+ \sqrt{n}\int_0^t [b(c_n(u), X(u)) - b(u, X(u))]\mathrm{d}u. \end{aligned}$$

As usual, $\mathrm{D}([0,1], \mathbb{R})$ denotes the Skorohod space of right-continuous and left-hand limited functions from $[0,1]$ to $\mathbb{R}$. Let G be the function from $\mathrm{D}([0,1], \mathbb{R})$ into itself defined by

$$G(w)(t) = w(t) - \int_0^t \mathrm{D_x}(b(u, X(u)))w(u)\mathrm{d}u, \quad 0 \leq t \leq 1.$$

Then,

$$G(Z^n) = M^n + R^n. \tag{19}$$

From (18) we have

$$\begin{aligned} R_*^n(1)^{1/2} &\leq Z_*^n(1)^{1/2} \sum_{i=1}^d \int_0^1 \left(\sum_{j=1}^d \left(\frac{\partial b_i}{x_j}(c_n(u), \eta^n(u)) - \frac{\partial b_i}{x_j}(u, X(u)) \right)^2 \right)^{1/2} \mathrm{d}u \\ &+ \sqrt{n}\int_0^1 \|b(c_n(u), X(u)) - b(u, X(u))\|_d \mathrm{d}u. \end{aligned}$$

From the above inequality, Lemma 4.5 and (A3), there exist positive constants C and D such that

$$\begin{aligned} \mathbb{E}(R_*^n(1)) &\leq C \sum_{i=1}^d \sum_{j=1}^d \int_0^1 \left(\frac{\partial b_i}{x_j}(c_n(u), \eta^n(u)) - \frac{\partial b_i}{x_j}(u, X(u)) \right)^2 \mathrm{d}u \\ &+ D/n. \end{aligned} \tag{20}$$

Since b is continuously differentiable at the second variable, it follows from Theorem 3.1 and (20) that

$$\mathbb{E}(R_*^n(1)) \to 0 \quad \text{as} \quad n \to \infty. \tag{21}$$

Hence (19) and Lemma 4.4 imply that $(G(Z^n); n \in \mathbb{N})$ converges in law to M.
From (A1)-(A3), G is a continuous and injective function. Hence in order to conclude the proof it suffices to verify that $(Z^n; n \in \mathbb{N})$ is a tight sequence in the Skorohod topology. For any $\delta > 0$ and $z \in \mathrm{D}([0,1], \mathbb{R}^d)$, let

$$\omega(z,\delta) = \sup_{|t-s|<\delta} \{\|z(t) - z(s)\|_d : 0 \le s,t \le 1\}.$$

From (17), we obtain

$$\omega(Z^n,\delta) \le Z^n_*(1) + \omega(M^n,\delta) + \omega(R^n,\delta).$$

Consequently, it follows from Lemma 4.4, Lemma 4.5 and (21) that

$$\lim_{\delta\to 0} \sup_{n\in\mathbb{N}} \mathbb{P}(\omega(Z^n,\delta) > \epsilon) = 0.$$

This fact implies that $(Z^n; n \in \mathbb{N})$ is tight and therefore, the proof is complete. □

Acknowledgments:
The authors would like to thank the anonymous referee for helpful suggestions that improved this paper.

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