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AN ORIENTED COMPETITION MODEL ON Z_+^2

GEORGE KORDZAKHIA
University of California, Department of Statistics, Berkeley CA 94720
email: kordzakh@stat.berkeley.edu

STEVEN P LALLEY[1]
University of Chicago, Department of Statistics, 5734 University Avenue, Chicago IL 60637
email: lalley@galton.uchicago.edu

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Abstract
We consider a two-type oriented competition model on the first quadrant of the two-dimensional integer lattice. Each vertex of the space may contain only one particle, either Red or Blue. A vertex flips to the color of a randomly chosen southwest nearest neighbor at exponential rate equal to the number of occupied south-west neighbors. At time zero there is one Red particle located at $(1,0)$ and one Blue particle located at $(0,1)$. The main result is a partial shape theorem for the red and blue regions $R(t)$ and $B(t)$ as time $t \to \infty$. In particular, we prove that (i) the upper-left half of the unit square is asymptotically blue, and the lower-right half is asymptotically red; and (ii) with positive probability there are angular sectors rooted at $(1,1)$ that are eventually either red or blue. The second result is contingent on the uniform curvature of the boundary of the corresponding Richardson shape.

1 Introduction.

In this paper we study a model where two species Red and Blue compete for space on the first quadrant of $\mathbb{Z}^2$. At time $t > 0$ every vertex of $\mathbb{Z}^2$ is in one of the three possible states: vacant, occupied by a Red particle, or occupied by a Blue particle. An unoccupied vertex $z = (x, y)$ may be colonized from either $(x, y-1)$ or $(x-1, y)$ at rate equal to the number of occupied south-west neighbors; at the instant of first colonization, the vertex flips to the color of a randomly chosen occupied south-west neighbor. Once occupied, a vertex remains occupied forever, but its color may flip: the flip rate is equal to the number of south-west neighbors occupied by particles of the opposite color. The state of the system at any time t is given by the pair $(R(t), B(t))$, where $R(t)$ and $B(t)$ denote the set of sites occupied by Red and Blue particles respectively. The set $R(t) \cup B(t)$ evolves precisely as the occupied set in the oriented Richardson model, and thus, for any initial configuration with only finitely many occupied sites, the growth of this set is governed

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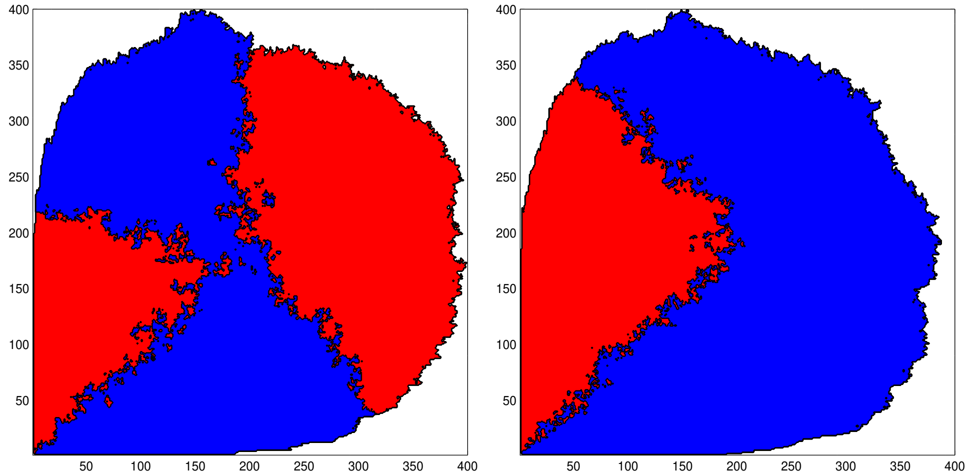

Figure 1: Two Realizations of Oriented Competition.

by the *Shape Theorem*, which states that the set of occupied vertices scaled by time converges to a deterministic set $\mathscr{S}$ (see for example [2]). A rigorous construction and more detailed description of the oriented competition model is given in Section 2.2.

The simplest interesting initial configuration has a single Red particle at the vertex $(1,0)$, a single Blue particle at $(0,1)$, and all other sites unoccupied. We shall refer to this as the *default* initial configuration. When the oriented competition process is started in the default initial configuration, the red and blue particles at $(1,0)$ and $(0,1)$ are protected: their colors can never be flipped. Thus, both colors survive forever w.p.1. Computer simulations for the oriented competition model started in the default and other finite initial configurations suggest that the shapes of the regions occupied by the Red and Blue types stabilize as times goes to infinity – see Figure 1 for snapshots of two different realizations of the model, each started from the default initial configuration. A peculiar feature of the stabilization is that the limit shapes of the red and blue regions are partly deterministic and partly random: The southeast corner of the occupied region is always equally divided between the red and blue populations, with boundary lying along the line $y = x$. However, the outer section seems to stabilize in a random union of angular wedges rooted at a point near the center of the Richardson shape. Although the location of the root appears to be deterministic, both the number and angles of the outer red and blue regions vary quite dramatically from one simulation to the next.

The purpose of this paper is to prove that stabilization of Red and Blue zones occurs with positive probability. (We conjecture that in fact it occurs with probability 1, but we have been unable to prove this.) To state our result precisely, we shall need several facts about the limit shape $\mathscr{S}$ of the oriented Richardson model $Z(t) := R(t) \cup B(t)$. (Note: Henceforth, the term *Richardson model* will refer the oriented version of the process, that is, the version in which infection can only travel north and east. The Shape Theorem for this model can be proved by the same arguments as used in the unoriented case – see [2] for details.) First, $\mathscr{S}$ is a compact, convex subset of the first quadrant of $\mathbb{R}^2$. It is generally believed – but has not been proved – that the outer boundary $\partial^o \mathscr{S}$ of $\mathscr{S}$ (the portion of $\partial \mathscr{S}$ that lies in the interior of the first quadrant) is *uniformly curved* in the following sense: If K_ε is the cone

$$K_\varepsilon = \{z \in \mathbb{R}^2 : \arg\{z - (1,1)\} \in (-\pi/2 + \varepsilon, \pi - \varepsilon)\},$$

then for every $\varepsilon > 0$ there exists $\varrho(\varepsilon) < \infty$ such that for every $x \in \partial^o \mathscr{S} \cap K_\varepsilon$ there is a circle of radius $\varrho(\varepsilon)$ passing through x that contains $\mathscr{S}$ in its interior.

Note: Throughout the paper we will use the term *cone* to mean a set $U \subset \mathbb{R}^2$ with a distinguished point u_*, the *root*, such that $U \setminus \{u_*\} = \{z : \arg(z - u_*) \in J\}$ for some interval J. An *angular sector* A of $\mathscr{S}$ is the intersection of $\mathscr{S}$ with a cone $A \subset K_0$ rooted at $a = (1,1)$.

In section 3 we shall prove the following.

Lemma 1. *The Richardson shape $\mathscr{S}$ intersects the coordinate axes in the line segments connecting the origin to the points $(1,0)$ and $(0,1)$, respectively. Furthermore the point $(1,1)$ lies in the* interior *of $\mathscr{S}$.*

It will follow by the convexity of $\mathscr{S}$ that the unit square $\mathscr{Q} = [0,1]^2$ lies entirely in $\mathscr{S}$. Define $\mathscr{Q}_1$ and $\mathscr{Q}_2$ to be the open subsets of $\mathscr{Q}$ that lie below and above the main diagonal $x = y$. For any set $U \subset \mathbb{R}^2$ and $\varepsilon > 0$, define

$$U^\varepsilon = \{u \in U : \operatorname{dist}(u, \partial U) \geq \varepsilon\} \quad \text{and}$$
$$\hat{U} = \{x \in \mathbb{R}^2 : \operatorname{dist}(x, U) \leq 1/2\}.$$

where dist denotes the L^∞−metric on $\mathbb{R}^2$. For any set $U \subset \mathbb{R}^2$ and any scalar $s > 0$, let $U/s = \{y/s : y \in U\}$. The main result of the paper is the following.

Theorem 1. *For every $\varepsilon > 0$, with probability one, for all large t*

$$\mathscr{Q}_1^\varepsilon \subset \hat{R}(t)/t \quad \textit{and} \quad \mathscr{Q}_2^\varepsilon \subset \hat{B}(t)/t. \tag{1}$$

Furthermore, if the outer boundary $\partial^o \mathscr{S}$ of the Richardson limit shape $\mathscr{S}$ is uniformly curved, then for every $\varepsilon > 0$ the following holds with positive probability: There exist random open angular sectors $A_1, .., A_n \subset \mathscr{S}$ rooted at $(1,1)$ that do not intersect the unit square $\mathscr{Q}$ such that for every $\varepsilon > 0$ and each $i = 1, 2, \ldots, n$,

(a) either $A_i^\varepsilon \subset \hat{R}(t)/t$ or $A_i^\varepsilon \subset \hat{B}(t)/t$ eventually, and

(b) the complement of $\bigcup A_i$ in $\mathscr{S} \setminus \mathscr{Q}$ has angular measure less than ε.

A number of similar results concerning interfaces in competition models have appeared in the literature. Ferrari and Pimentel [4] considered competition between two-growing clusters in the last-passage percolation model in $\mathbb{Z}^2$. They showed that the competition interface converges almost surely to an asymptotic random direction. Another competition model on $\mathbb{Z}^d$ (non-oriented version) was studied in [6]. It was shown that if the process starts with finitely many particles of both types (Red and Blue), then the two types coexist with positive probability under the condition that the shape set of the corresponding non-oriented Richardson model is uniformly curved. The behavior of the oriented model differs from that of the model considered in [6] in that the limit shape contains the deterministic component (1).

2 Preliminaries

2.1 Graphical Constructions

The competition model, the Richardson model, and the competition model in a hostile environment (see sec. 3 below) may be built using the same *percolation structure* Π. For details on percolation structures and their use in constructing interacting particle systems see [3]. Here we briefly describe the construction of the percolation structure Π appropriate for the *oriented* models described in sec. 1. To each directed edge xy such that $x \in \mathbb{Z}_+^2 \setminus \{(0,0)\}$, and $y = x + (0,1)$ or $y = x + (1,0)$ is assigned a rate-1 Poisson process. The Poisson processes are mutually independent. Above each vertex x is drawn a timeline, on which are placed marks at the occurrence times T_i^{xy} of the Poisson processes attached to directed edges emanating from x; at each such mark, an arrow is drawn from x to y. A *directed path* through the percolation structure Π may travel upward, at speed 1, along any timeline, and may (but does not have to) jump across any outward-pointing arrow that it encounters. A *reverse path* is a directed path run backward in time: thus, it moves downward along timelines and jumps across inward-pointing arrows. A *voter-admissible* [2] path is a reverse path that crosses every inward-pointing arrow that it encounters. Note that for every site x and time $t > 0$ there is a unique voter-admissible path starting at (x,t). Observe that the voter-admissible path starting at (x,t) evolves as a (reverse-time) random walk with jumps downward or leftward occurring independently at rate 1 as long as the random walk is still in the interior of the positive quadrant; upon reaching the boundary, jumps that would take the random walk outside the quadrant are suppressed.

Path Precedence: For each (z,t) denote by $\Gamma(z,t)$ the collection of reverse paths in Π originating at (z,t) and terminating in $(\mathbb{Z}_+^2, 0)$. (We may also use $\Gamma(z,t)$ to denote the set of endpoints of all paths in the collection – the proper meaning should be clear from the context.) Assume that reverse paths are parametrized by reverse time: thus, for a reverse path $\gamma \in \Gamma(z,t)$ and $s \in [0,t]$ denote by $\gamma(s)$ the location of the path in $\mathbb{Z}_+^2$ at time $t - s$ in the natural time scale of Π, i.e. $\gamma(s) = z'$ if $(z', t-s) \in \gamma$. There is a natural order relation $\prec$ on $\Gamma(z,t)$ defined as follows: For two reverse paths $\gamma_1, \gamma_2 \in \Gamma(z,t)$, let $\tau = \inf\{s > 0 : \gamma_1(s) \neq \gamma_2(s)\}$, $i = 1,2$, and set $\gamma_1 \prec \gamma_2$ if and only if γ_2 jumps across an inward-pointing edge at time $t - \tau$. (Note that exactly one of the two paths γ_1, γ_2 must make such a jump.) The unique voter-admissible path $\tilde{\gamma}$ in $\Gamma(z,t)$ is *maximal* with respect to the order $\prec$.

Richardson Model: In the Richardson model, sites are at any time either vacant or occupied. If $Z(0)$ is the set of sites occupied at time zero, then the set $Z(t)$ of sites occupied at time t, then $z \in Z(t)$ if and only if there is a reverse path $\gamma \in \Gamma(z,t)$ that terminates in $Z(0) \times \{0\}$.

Competition Model: Let $B(0)$ and $R(0)$ be the (nonoverlapping) sets of sites occupied by Blue and Red particles, respectively, at time 0, and let $W(0) = (B(0) \cup R(0))$ be the set of initially vacant sites. As in the Richardson model, site z is occupied at time t if and only if there is a reverse path $\gamma \in \Gamma(z,t)$ that terminates in $(B(0) \cup R(0)) \times \{0\}$. Call such paths *attached*, and call their endpoints the *potential ancestors* of (z,t). Since there may be several attached paths, some ending in $B(0) \times \{0\}$ and others in $R(0) \times \{0\}$, priority must be established to determine the *ancestor* of (z,t). This is done using the order relation $\prec$: Among all attached paths $\gamma \in \Gamma(z,t)$ there is a unique maximal path γ^*. If γ^* terminates in $B(0) \times \{0\}$, then $z \in R(t)$; if γ^* terminates in $R(0) \times \{0\}$ then $z \in R(t)$. Note that if $Z(0) = B(0) \cup R(0)$ then $Z(t) = R(t) \cup B(t)$.

[2]The term *voter-admissible* is used for such paths because if the voter-admissible path starting at (x,t) ends at $(y,0)$ then the color of $(y,0)$ is the state of site x at time t in the standard *voter model*.

2.2 Large deviation estimates for the shape theorem

Let $Z(t)$ the set of vertices occupied at time t in the Richardson model, and fix a *finite* initial configuration $Z(0)$. For $z \in \mathbb{R}^2_+$ let $T(z) = \inf\{t : z \in \hat{Z}(t)\}$, and let $\mu(z) = \lim_{n\to\infty} n^{-1} T(nz)$. The limit exists almost surely by subadditivity. The growth of $\hat{Z}(t)$ is governed by a Shape Theorem [7].[3] The following large deviations result for the Richardson model follows by the same arguments as used by Kesten [5] and Alexander [1] in the non-oriented case:

Theorem 2. *There exists a non-random compact convex subset $\mathscr{S}$ of $\mathbb{R}^2_+$ such that for $\alpha \in (1/2, 1)$, constants $c_1, c_2 > 0$ (depending on α) and all $t > 0$*

$$P\{(t - t^\alpha)\mathscr{S} \subset \hat{Z}(t) \subset (t + t^\alpha)\mathscr{S}\} > 1 - c_1 t^2 \exp\{-c_2 t^{(\alpha - 1/2)}\}.$$

Note: For the remainder of the paper, the letters $c, c_1, c_2, \dots$ will denote constants whose values may change from one line to the next.

The dual of the oriented Richardson process is the *South-West oriented Richardson model*, which behaves in exactly the same manner as the oriented Richardson model except that infection travels only south and west. Let $\tilde{\mathscr{S}}$ be the limit set of the South-West oriented Richardson model. Started from the default initial configuration with particles at the vertices $(-1, 0)$ and $(0, -1)$, the process lives in the third quadrant of $\mathbb{Z}^2$ and has limit shape $\tilde{\mathscr{S}} = -\mathscr{S}$. The next lemma follows by an elementary geometric argument, as in the proof of Lemma 4 in [8]. Denote by $D(z, r)$ the Euclidean disk in $\mathbb{R}^2$ with center at z and radius $r > 0$.

Lemma 2. *Suppose that $\partial^o \mathscr{S}$ is uniformly curved. If $z \in \partial^o \mathscr{S}$, then for all $t_1, t_2 > 0$, the regions $t_1 \mathscr{S}$ and $(t_1 + t_2)z + t_2 \tilde{\mathscr{S}}$ are contained in disks that intersect in the single point $t_1 z$ (see Figure 2). Furthermore, for every $\varepsilon > 0$ and $\alpha \in (1/2, 1)$ there exists $c > 0$ so that if $z \in \partial \mathscr{S} \cap K_\varepsilon$, then for all $t_1, t_2 > 0$ we have*

$$(t_1 + t_1^\alpha)\mathscr{S} \cap ((t_1 + t_2)z + (t_2 + t_2^\alpha)\tilde{\mathscr{S}}) \subset D(t_1 z, c(t_1 + t_2)^{(\alpha+1)/2}).$$

3 Growth and competition in a hostile environment.

We will deduce the assertion (1) by comparison with a related process, the *competition model in a hostile environment*. This evolves in exactly the same manner as the competition model of Theorem 1 started from the default initial configuration *except* that vacant sites ("White") may now kill Red or Blue particles at northeast neighboring sites. The origin $(0, 0)$ does not count as a site: it can't kill the Red or Blue particles next to it. This process is more easily analyzed than the competition model of Theorem 1, because the state of any site x at time $t > 0$ is determined by the voter-admissible path γ starting at (x, t). In particular, if γ ends at $(1, 0)$, then site x will be occupied by a Red particle at time t; if γ ends at $(0, 1)$ then x will be occupied by a Blue particle at time t; and if γ ends at any other point then x will be vacant at time t. Define

$$\begin{aligned} R_H(t) &= \{\text{red sites at time } t\}; \\ B_H(t) &= \{\text{blue sites at time } t\}; \text{and} \\ Q(t) &= R_H(t) \cup B_H(t). \end{aligned}$$

[3] Subadditivity arguments show the existence of the limit $\mu(z)$ for each direction z, but do not imply continuity of μ on the boundaries of $\mathbb{R}^2_+$. Martin [7] showed that $\mu(z)$ is continuous on all of $\mathbb{R}^2_+$.

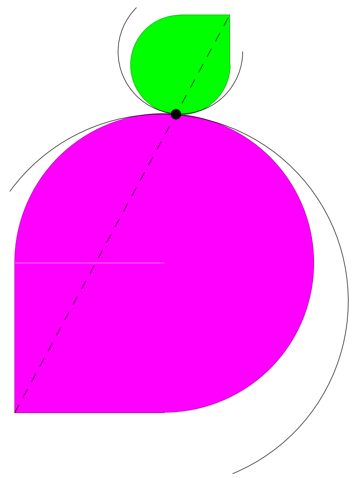

Figure 2: Uniform Curvature

Note that the set $Q(t)$ ("Black") consisting of all occupied sites at time t is just the (oriented) voter model with initial configuration $(1,0)$ and $(0,1)$ Black, all other sites White.

Observe that every vertex $(z_1,0)$ on the horizontal coordinate axes and every vertex $(0,z_2)$ on the vertical coordinate axes eventually flips to Black and then remains Black forever. Thus, almost surely for all large t, vertex (z_1,z_2) is Black. By subadditivity, the Black region $Q(t)$ obeys a shape theorem. Below it is shown that the limit shape is exactly $\mathscr{Q}$. (See Figure 3 for a simulation.)

Proposition 1. *For every $\alpha \in (1/2,1)$ there exist c_1, c_2 such that for all $t > 0$*

$$P[(t-t^{\alpha})\mathscr{Q} \subset \hat{Q}(t) \subset (t+t^{\alpha})\mathscr{Q}] > 1 - c_1 t^2 \exp\{-c_2 t^{(\alpha-1/2)}\}.$$

Proof. Recall that for every $t > 0$ and $z \in \mathbb{Z}_+^2$, there exists a unique reverse voter-admissible path $\tilde{\gamma}_{(z,t)}$ starting at (z,t). The path travel downward, at rate 1, and jumps across all inward-pointing arrows. Until the path hits the horizontal (vertical) axis the number of horizontal (vertical) jumps is distributed as Poisson process with parameter 1. Thus, there exist constants c_1 and c_2 such that for every $z \in (t-t^{\alpha})\mathscr{Q}$

$$P(\tilde{\gamma}_{(z,t)} \text{ terminates in } \{(1,0),(0,1)\}) \geq 1 - c_1 \exp\{-c_2 t^{(\alpha-1/2)}\}.$$

For the same reason, there exist constants c_1 and c_2 such that for every $z \in (t+t^{\alpha})\mathscr{Q}^c$,

$$P(\tilde{\gamma}_{(z,t)} \text{ terminates in } \{(1,0),(0,1)\}) \leq c_1 \exp\{-c_2 t^{(\alpha-1/2)}\}.$$

The proposition follows from the fact that the number of vertices in $(t-t^{\alpha})\mathscr{Q}$ is of order at most $O(t^2)$ and the number of vertices on the boundary of $(t+t^{\alpha})\mathscr{Q}$ is of order at most $O(t)$. $\square$

If the growth models $Q(t)$ and $S(t)$ are coupled on the same percolation structure Π, then clearly $Q(t) \subseteq S(t)$, and thus $\mathscr{Q} \subseteq \mathscr{S}$. Lemma 1 asserts that $\mathscr{S}$ is strictly larger than $\mathscr{Q}$.

Proof of Lemma 1. The first asserton of the lemma, that $\mathscr{S}$ intersects the coordinate axes in line segments of length 1, is fairly obvious: If T_n is the time that (say) $(n,0)$ is invaded, then the random variables $T_{n+1} - T_n$ are independent and exponentially distributed with mean 1. Thus, it remains to prove that $(1,1)$ is in the interior of the shape set. The following argument was communicated to the authors by Yuval Peres. Consider a representation of the Richardson model as a first passage percolation model. To each edge of the lattice associate a mean one exponential random variable, also called a passage time of the edge. The variables are mutually independent. For every pair of vertices $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ such that $x_1 \le x_2$ and $y_1 \le y_2$ define the passage time $T(z_1, z_2)$ from z_1 to z_2 as the infimum over traversal times of all North-East oriented paths from z_1 to z_2. The traversal time of an oriented path is the sum of the passage times of its edges. In the first passage percolation description of the Richardson model, let

$$Z(t) = \{z \in \mathbb{Z}_+^2 : T((1,0),z) \le t \text{ or } T((0,1),z) \le t\}.$$

It is enough to show that for some $\varepsilon > 0$, the vertex $(1,1)$ is in $(1-\varepsilon)\mathscr{S}$. Consider a sequence of vertices $z_n = (n,n)$ on the main diagonal of the first quadrant of $\mathbb{Z}^2$. By the shape theorem, it suffices to prove that almost surely for infinitely many n's the occupation times of z_n satisfy $T(z_n) \le n(1-\varepsilon)$.

Consider vertices $(0,2), (2,0)$, and $(1,1)$. There are exactly four oriented distinct paths from the origin to these vertices. Each such path has two edges and expected passage time 2. Let $\gamma_{(1)}$ be the path with the smallest passage time among these four paths. Denote by X_1 the terminal point of $\gamma_{(1)}$, and denote its passage time by T_1. By symmetry, $P(X_1 = (0,2)) = P(X_1 = (2,0)) = 1/4$ and $P(X_1 = (1,2)) = 1/2$. It easy to see that $ET_1 < 1$. Indeed, let γ_0 be the path obtained by the following procedure. Start at the origin and make two oriented steps each time moving in the direction of the edge with minimal passage time (either north or east). Clearly,

$$ET_1 < E\tau(\gamma_0) = 1$$

where $\tau(\gamma_0)$ is the total passage time of γ_0. Restart at X_1 and repeat the procedure. Denote by X_2 the displacement on the second step and by T_2 the passage time of the time minimizing path from X_1 to $X_1 + X_2$. Note that $W_k = \sum_{k=1}^{\infty} X_k$ is a random walk on $\mathbb{Z}_+^2$. The random walk visits the main diagonal infinitely often in such a way that $W_k = (k,k)$. Furthermore, if $S_k = \sum_{k=1}^{\infty} T_k$, then by SLLN for some $\varepsilon > 0$ almost surely for all large k we have $S_k \le (1-\varepsilon)k$. This finishes the proof. ☐

Next, we consider the evolution of the Red and Blue regions $R_H(t)$ and $B_H(t)$. Recall that these are defined to be the sets of all vertices z such that the unique voter-admissible path beginning at (z,t) terminates respectively at $(1,0)$ and $(0,1)$. For $t > 0$ define

$$K_1(t^\alpha) = \{z = (z_1, z_2) \in \mathbb{Z}_+^2 : z_1 - z_2 > t^\alpha\},$$

$$K_2(t^\alpha) = \{z = (z_1, z_2) \in \mathbb{Z}_+^2 : z_2 - z_1 > t^\alpha\}.$$

Proposition 2. *For every $\alpha \in (1/2, 1)$ there exist $c_1,\ c_2 > 0$ such that for all $t > 0$,*

$$P[(t - t^\alpha)\mathscr{Q} \cap K_1(t^\alpha) \subset \hat{R}_H(t)] > 1 - c_1 t^2 \exp\{-c_2 t^{(\alpha - 1/2)}\} \quad \textit{and} \tag{2}$$

$$P[(t - t^\alpha)\mathscr{Q} \cap K_2(t^\alpha) \subset \hat{B}_H(t)] > 1 - c_1 t^2 \exp\{-c_2 t^{(\alpha - 1/2)}\}. \tag{3}$$

Proof. We only prove (2), as the proof of (3) is virtually identical. First, observe that by Proposition 1 there exist $c_1, c_2 > 0$

$$P[(t - t^\alpha)\mathscr{Q} \subset \hat{R}_H(t) \cup \hat{B}_H(t)] > 1 - c_1 t^2 \exp\{-c_2 t^{(\alpha - 1/2)}\}.$$

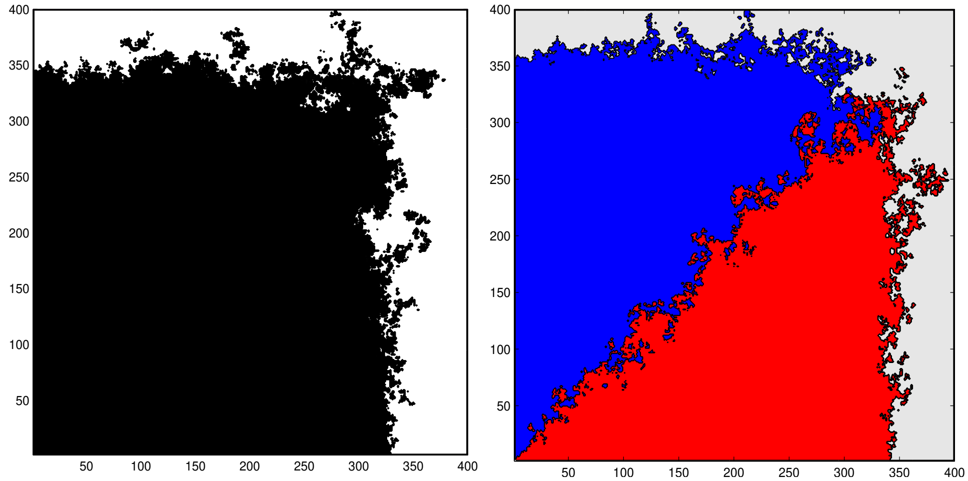

Figure 3: Growth and Competition Models in Hostile Environment.

Second, note that with probability exponentially close to one voter admissible paths of all vertices $z \in (t - t^\alpha)\mathscr{Q} \cap K_1(t^\alpha) \cap \mathbb{Z}_+^2$ terminate below the main diagonal. That is, there exist $c_1, c_2 > 0$ such that

$$P\left(\tilde{\gamma}_{(z,t)}(t) \in K_1(0)\right) \geq 1 - c_1 \exp\{-c_2 t^{(\alpha-1/2)}\}.$$

The result (2) follows immediately from these two observations. □

If the competition model and the competition model in hostile environment are constructed on the same percolation structure Π, then almost surely, for all $t \geq 0$,

$$R_H(t) \subseteq R(t) \quad \text{and} \quad B_H(t) \subseteq B(t).$$

Hence it follows from Proposition 2 that almost surely, for all large t,

$$(t - t^\alpha)\mathscr{Q} \cap K_1(t^\alpha) \subset \hat{R}(t) \quad \text{and}$$
$$(t - t^\alpha)\mathscr{Q} \cap K_2(t^\alpha) \subset \hat{B}(t).$$

Thus, asymptotically (as t goes to infinity) the square $\mathscr{Q} \subset \mathscr{S}$ is colored deterministically. In particular, the region below the main diagonal is red, and the region above the diagonal is blue. This proves the first part of Theorem 1.

4 Stabilization in angular sectors.

Next we consider how the oriented competition model evolves in the region $\mathscr{S} \setminus \mathscr{Q}$. By the Shape Theorem for the Richardson model, the occupied set $\hat{R}(t) \cup \hat{B}(t)$, after rescaling by $1/t$, converges to the deterministic Richardson shape $\mathscr{S}$. By the arguments of the preceding section, the Richardson shape contains the unit square $\mathscr{Q}$, and the upper corner $a = (1, 1)$ of this square lies in the *interior* of $\mathscr{S}$. In this section, we will show that once an angular sector $A \subset \mathscr{S}$ rooted at a has been occupied by one of the species (Red or Blue), it is very unlikely (exponentially in powers of the distances involved) that the opposite species will make a large incursion into this sector for some time afterward. Since there is positive (albeit possibly small) probability that in finite time the Blue and Red species will each colonize large angular sectors rooted at $a = (1,1)$, the second part of Theorem 1 will follow.

The stabilization argument has two parts. First, the Small Incursion Lemma following will show that once a species (say Blue) has been eliminated from a region tU of diameter proportional to t, it is exponentially unlikely that it will penetrate farther than $C\delta t$ into the interior of tU at any time in the next $t\delta$ time units. Second, the Stabilization Lemma below will imply that if $A = U$ is an angular sector of $\mathscr{S}$, then the probability that at the end of the next $t\delta$ time units there are Blue particles in $t(1+\delta)A$ farther than $C(\delta t)^\alpha$ from the boundary, for $1/2 < \alpha < 1$, is exponentially small. It will then follow routinely from the Borel-Cantelli lemma that with positive (conditional) probability, once an angular sector has been taken over by the Red species, the Blue species will never re-enter a slightly smaller angular sector.

Small Incursion Lemma. *Let Δ be the diameter (in L^∞) of the Richardson shape $\mathscr{S}$. There is a constant $c > 0$ such that the following is true, for every bounded region $U \subset \mathbb{R}^2_+$ and all $t \geq 1$ and $0 < \delta < 1$: If at time 0 the integer points in tU are all occupied by Red particles, then for every $\varepsilon > 2\Delta\delta$*

$$P\{\textit{there is a Blue particle in } tU^\varepsilon \textit{at some time } s \leq \delta t\} \leq \exp\{c\sqrt{\delta t}\}. \tag{4}$$

Proof. This is an easy consequence of the large deviations estimates in Theorem 2. In brief, any Blue incursion into tU^ε would have to originate at an integer point z outside of tU, and thus would have to travel distance at least $2\Delta\delta t + \mathrm{dist}(z, tU)$ in time δt. But regardless of the initial configuration of Red and Blue particles away from z, the region colonized by Blue particles with ancestry tracing back to z is dominated by the Richardson process initiated by a single particle at z. Since the Richardson process grows monotonically with time, Theorem 2 implies that for any integer point $z \notin tU$, the probability of a Blue incursion from z into tU^ε by time δt is exponentially small, in particular, bounded by $\exp\{-c\sqrt{\delta t + \mathrm{dist}(z, tU)}\}$. Since the number of integer points at given distance from tU grows only quadratically, the estimate (4) follows. □

Recall that

$$K = K_0 = \{y \in \mathbb{R}^2 : \ \arg\{y - (1,1)\} \in (-\pi/2, \pi)\}.$$

Denote by $\pi : K \to \partial^o \mathscr{S}$ the natural projection onto the outer boundary of the Richardson shape: in particular, for any $y \in K$, $\pi(y)$ is the the unique point where the line through $a = (1,1)$ and y intersects $\partial^o\mathscr{S}$. For a cone A rooted at $(1,1)$, define the *center* and *aperture* to be the point z and the positive number ϱ such that

$$A = A(z;\varrho) := \{y \in K \ : \ |\pi y - z| < \varrho\}.$$

Fix $\varepsilon > 0$ and $\alpha, \alpha_*, \beta \in (1/2, 1)$ such that $\alpha < \alpha_*$ and $(\alpha_* + 1)/2 < \beta$. For $\varrho, c > 0$ and $t = t_0 \geq 1$, let $A_1 \subset A_0$ be cones rooted at a with common center z and apertures $\varrho < \varrho + t^{\beta - 1}$, respectively.

Assume that $A_0 \subset K_{\varepsilon/3}$. Fix $\delta \in (0,1)$, and set

$$\begin{aligned}
t_1 &= (1+\delta)t_0 = (1+\delta)t, \\
W_i &= (A_i \cap \mathscr{S}) - a, \quad \text{for } i = 0,1, \\
\mathscr{R}_0 &= (t_0 - t_0^{\alpha})W_0 + t_0 a, \\
\mathscr{R}_0^* &= (t_0 - t_0^{\alpha_*})W_1 + t_0 a \\
\mathscr{R}_1^{\pm} &= (t_1 \pm t_1^{\alpha_*})W_1 + t_1 a, \\
\mathscr{B}_0 &= (t_0 + t_0^{\alpha})\mathscr{S} \setminus t_0 A_0, \\
\mathscr{B}_1 &= (t_1 + t_1^{\alpha_*})\mathscr{S} \setminus t_1 A_1, \quad \text{and} \\
\mathscr{D} &= D(at_1, (\delta t)^{\alpha}).
\end{aligned}$$

Stabilization Lemma. *There exists a constant $c > 0$ such that the following is true, for all sufficiently large $t > 0$: If the initial configuration is such that $\hat{R}(0) \supset \mathscr{R}_0$ and $\hat{B}(0) \subset \mathscr{B}_0$, then*

$$P[\hat{B}(\delta t) \not\subset \mathscr{B}_1 \cup \mathscr{D} \text{ or } \hat{R}(\delta t) \not\supset \mathscr{R}_1^- \setminus \mathscr{D}] \le t^2 \exp\{-c(\delta t)^{\alpha - 1/2}\}. \tag{5}$$

Under the hypothesis of uniform curvature of the outer boundary of $\mathscr{S}$, the constants do not depend on the angular measure and position of A_0 as long as the angular sector $A_0 \subset K_\varepsilon/3$.

Proof of the Stabilization Lemma. By monotonicity, we may assume that $\hat{R}(0) = \mathscr{R}_0^-$ and $\hat{B}(0) = \mathscr{B}_0$; this is the worst scenario for the Red. The proof of the lemma will proceed by tracing the ancestries of paths in $\Gamma(z, \delta t)$, where z is an integer point. Recall that if there is an attached path in $\Gamma(z, \delta t)$ then there is a unique *maximal* such path γ^* relative to the precedence order $\prec$; this path determines the color of z at time δt. Since the voter-admissible path is maximal in the precedence order, if it is attached then it will determine the color of z; however, it is possible that there are attached paths, but that the voter-admissible path is not attached, as in Claims 2–3 below. If there is no attached path in $\Gamma(z, \delta t)$ then site z is vacant at time δt. By the large deviations estimates for the Richardson model (Theorem 2), the region colonized by Blue at time t will be contained in $\mathscr{R}_1^+ \cup \mathscr{B}_1$, except with exponentially small probability, so it suffices to consider sites $z \in \mathscr{R}_1^+ \cup \mathscr{B}_1$. The following three claims will therefore imply the lemma, since there are at most $O(t^2)$ integer points in this region. □

Claim 1. *If $z \in \mathscr{R}_0^* + \delta t a$ and $z \notin \mathscr{D}$ then except with exponentially small probability the voter-admissible path in $\Gamma(z, \delta t)$ is attached, and its endpoint lies in $\mathscr{R}_0 \times \{0\}$.*

Claim 2. *If $z \in \mathscr{R}_1^- \setminus (\mathscr{R}_0^* + \delta t a)$ then except with exponentially small probability there is an attached path, and the maximal attached path γ^* has endpoint in $\mathscr{R}_0 \times \{0\}$.*

Claim 3. *If $z \in \mathscr{R}_1^+ \setminus \mathscr{R}_1^-$ then except with exponentially small probability, if there is an attached path in $\Gamma(z, \delta t)$ then the maximal attached path γ^* has endpoint in $\mathscr{R}_0 \times \{0\}$.*

In all cases the term *exponentially small* means that the probability is bounded by

$$\exp\{-c(\delta t)^{\alpha - 1/2}\}$$

for a constant $c > 0$ independent of the site z chosen.

Proof of Claim 1. The voter-admissible path $\tilde{\gamma}$ is a continuous time random walk with exponential waiting times between jumps and drift $-a$. For any initial point z in the region $\mathscr{R}_0^* + \delta t$, the

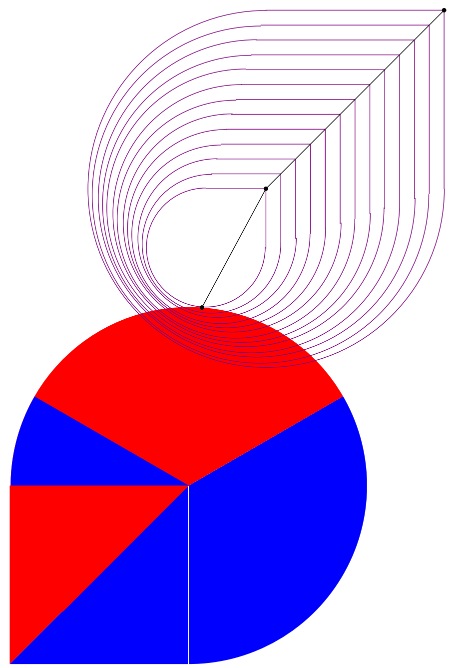

Figure 4: Maximal Path

mean displacement $-(\delta t)a$ of the voter-admissible path moves the point into the region $\mathscr{R}_0^*$. Furthermore, if $z \notin \mathscr{D}$, then $z - (\delta t)a$ will be at distance at least $O(t^\beta)$ from the bounding rays of the cone $\mathscr{R}_0$, and at distance at least $O(t^{\alpha_*} - t^\alpha)$ from the outer boundary of $\mathscr{R}_0^*$. Since $\beta > 1/2$, standard moderate deviations results for the simple continuous-time random walk imply that, except with exponentially small probability, the endpoint of the path $\tilde{\gamma}$ lies in $\mathscr{R}_0 \times \{0\}$. □

Proof of Claim 2. For sites $z \in \mathscr{R}_1^- \setminus (\mathscr{R}_0^* + \delta t a)$ the voter-admissible path will not necessarily be attached with high probability, because the mean displacement $-(\delta t)a$ moves z to a point outside $\mathscr{R}_0 \cup \mathscr{B}_0$. Nevertheless, for such z it is highly likely that there is an attached path in $\Gamma(z, \delta t)$, because the reverse Richardson shape started at z and run for time δt will intersect the *interior* of $\mathscr{R}_0$. (See Figure 4 above; z is the point at the upper right corner of the figure.) This follows from the definition of the region $\mathscr{R}_1^-$, and the fact that $\alpha_* > \alpha$.

Now consider the *maximal* attached path: This will follow the voter-admissible path as far as possible, to a neighborhood of a point w_* along the ray L^+ of slope -1 originating at z, and thereafter will follow a line of slope $\neq -1$ to a point near the outer boundary of $\mathscr{R}_0^*$. (In Figure 4, the point w_* is the endpoint of the line segment of slope -1 emanating from z.) To see this, consider points w along the ray L^+ between z and w_*. For each such point w, the voter-admissible path starting at z travels approximately at speed 1, and hence reaches a neighborhood of w in time $T_w = \mathrm{dist}(z, w)$. (Note: The metric is the L^∞ metric.) Now consider the reverse Richardson shape that starts at w at time T_w: if this is run for time $\delta t - T_w$, it will terminate in the interior of $\mathscr{R}_0^*$ (see Figure 4). Thus, by Theorem 2, with probability exponentially close to 1, there will be a path in $\Gamma(z, \delta t)$ that follows the voter-admissible path from z to a point near w, then follows another straight line in the reverse Richardson shape into the interior of $\mathscr{R}_0$. This path will necessarily be attached. Since this is true for each $w \neq w_*$ along the line segment from z to w_*, it follows that with high probability there is an attached path that reaches a neighborhood of w_*. A similar argument shows that, except with exponentially small probability, there is no attached path that follows the voter-admissible path much farther than w_*. Consequently, the *maximal* attached path will deviate from the voter-admissible path near w_*, and then will follow approximately the straight line path to the marked point on the boundary of $\mathscr{R}_0^*$ shown in Figure 4.

Following is a more detailed explanation. For $r_1, r_2 \in \mathbb{R}^2$, denote by $I(r_1, r_2)$ the interval with endpoints r_1 and r_2 and by $L(r_1, r_2)$ the half-line starting at r_1 and passing through r_2. For convenience, define $r = z/(1+\delta)t \in \mathscr{S}$. Draw a half-line $L(a, r)$ and let $b \in L(a, r)$ be the point on the outer boundary of $\mathscr{S}$. Let $b'(\delta)$ be a point in the interval $I(a, b)$ such that

$$|b' - a| = |b - a|/(1+\delta)$$

where $|\cdot|$ is an Euclidean norm. First consider the case where $r \in I(b', b)$. Define

$$\kappa = \frac{|r-b|}{|b'-b|}.$$

Let the voter admissible path run from $(z, \delta t)$ for $\kappa\delta t$ time units going backward in time. With probability exponentially close to one, at time $\delta t - \kappa\delta t$ the reverse voter admissible path will be in the $(\kappa\delta t)^\alpha$ neighborhood of $w^* = z - (\kappa\delta t)a$. From elementary geometry it immediately follows that $w^* \in I(bt, b(1+\delta)t)$ and

$$\kappa = |w^* - b(1+\delta)t|/|bt - b(1+\delta t)|.$$

Secondly, observe that bt is on the boundary of $w^* + (1-\kappa)(\delta t)\tilde{\mathscr{S}}$. Actually,

$$t\mathscr{S} \cap \{w^* + (1-\kappa)(\delta t)\tilde{\mathscr{S}}\} = bt.$$

Thus, if the reverse Richardson process starts from a vertex near w^* it should hit bt in approximately $(1-\kappa)(\delta t)$ units of time. Furthermore, by Lemma 2 the intersection of the set $\mathscr{R}_0 \cup \mathscr{B}_0$ with the set occupied by the Richardson process should lie in $D(bt, c(\delta t)^\beta)$ (unless the the intersection is empty). Since $z \in \mathscr{R}_1^- \setminus (\mathscr{R}_0^* + \delta t a)$, it is easy to see that $D(bt, c(\delta t)^\beta) \cap \mathscr{B}_0 = \emptyset$. To guarantee that with probability exponentially close to one the selected set of reverse paths contains at least one attached path, we follow the reverse voter admissible path for slightly less than $\kappa\delta t$, specifically, for $t_1 = (\kappa\delta t - (\delta t)^{\alpha_\star})_+$ units of time. Let $\Gamma_1(z, \delta t)$ be a subset of $\Gamma(z, \delta t)$ containing all reversed paths that coincide with the voter admissible path $\tilde{\gamma}$ for t_1 units of time. That is, for every $\gamma \in \Gamma_1(z, \delta t)$, for all $0 < s < t_1$, $\gamma(s) = \tilde{\gamma}(s)$ The set of ends of $\Gamma_1(z, \delta t)$ is obtained by constructing the reverse oriented Richardson process that starts with one occupied vertex at $\tilde{\gamma}(t_1)$, and runs backward in time for $t_2 = \delta t - t_1$ units of time. By Lemma 2, for some $c > 0$

$$P[\Gamma(z, \delta t) \cap (\mathscr{R}_0 \cup \mathscr{B}_0) \not\subset D(bt, c(\delta t)^\beta)] < \exp\{-c(\delta t)^{(\alpha - \frac{1}{2})}\}.$$

Thus, with probability exponentially close to one the end of maximal attached path γ^* belongs to a disk with center at bt and radius $c(\delta t)^\beta$. Since $D(bt, c(\delta t)^\beta) \cap \mathscr{B}_0 = \emptyset$ the result follows.
If the location of z is such that $z/(1+\delta)t \in I(a, b')$, take $\kappa = 1$ and use a similar argument. □

Proof of Claim 3. The proof is similar to that of Claim 2. If the ancestor of $(z, \delta t)$ exists, it is located in the set of ends of $\Gamma(z, \delta t)$. The set of ends of $\Gamma(z, \delta t)$ is obtained by constructing reverse oriented Richardson process starting at $(z,\delta t)$, and running the process on the subset $\mathbb{Z}_+^2 \times (0,\delta t)$ of the percolation structure backward in time for δt units of time. Then by Theorem 2 and Lemma 2,

$$P[\Gamma(z,\delta t) \cap (\mathscr{R}_0 \cup \mathscr{B}_0) \not\subset D(bt, c(\delta t)^\beta)] \leq \exp\{-c(\delta t)^{(\alpha-\frac{1}{2})}\}.$$

□

Proof of Theorem 1. It remains to be shown that with positive probability, Red and Blue eventually occupy disjoint angular sectors of $\mathscr{S}$ rooted at $a = (1,1)$ that fill all but a small part of $\mathscr{S} \setminus \mathscr{Q}$. Let L_ε^+ and L_ε^- be the rays emanating from a with slopes $1 \pm \varepsilon$, respectively; it suffices to show that, with positive probability, the region above $L_{2\varepsilon}^+$ will eventually be Red and the region below $L_{2\varepsilon}^-$ will eventually be Blue.

Fix T large. There is positive (although small) probability that at time T Red and Blue will occupy the regions above and below the line of slope 1 through the origin, and that their union will be precisely the scaled Richardson shape $T\mathscr{S} \cap \mathbb{Z}^2$. Consider the state of the system at times $T_n := (1+\delta)^n T$ for $n = 1, 2, \dots$. By the Stabilization Lemma and the Borel-Cantelli lemma, if T is sufficiently large, the Red and Blue regions will contain the sectors above and below the lines L_ε^+ and L_ε^- at all times T_n with positive probability. (Note: Here we use the fact that $\beta < \alpha$ [see discussion preceding the Stabilization Lemma], as this guarantees that $\sum_n T_n^{\beta-\alpha} < \varepsilon$ if T is large.) Finally, the Small Incursion Lemma ensures that, if δ is small and T large, then the Red and Blue populations will not enter the sectors above and below the lines $L_{2\varepsilon}^+$ and $L_{2\varepsilon}^-$ between successive time T_n and T_{n+1} with any appreciable probability. □

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References

[1] Alexander, K. S. (1993). *A Note on Some Rates of Convergence in First-passage Percolation.* Annals of Applied Probability Vol. 3, 81–90 MR1202516

[2] Cox, J. T. and Durrett, R (1981). *Some limit theorems for percolation processes with necessary and sufficient conditions.* The Annals of Probability Vol. 9, 583-603 MR0624685

[3] Durrett, R. (1988) *Lecture Notes on Particle Systems and Percolation.* Wadsworth, 1-325 MR0940469

[4] Ferrari, P.A. and Pimentel, L. P. R. (2005) *Competition interfaces and second class particles.* The Annals of Probability Vol. 33, 1235–1254 MR2150188

[5] Kesten, H. (1993). *On the Speed of Convergence in First-passage Percolation.* Annals of Applied Probability Vol. 3, 296–338 MR1221154

[6] Kordzakhia, G., and Lalley, S. (2005). *A two-species competition model on $\mathbb{Z}^d$.* Stochastic Processes and their Applications Vol. 115, 781-796. MR2132598

[7] Martin, J. (2004) *Limiting Shape for Directed Percolation Models.* The Annals of Probability Vol. 32, 2908–2937 MR2094434

[8] Newman, C.M. and Piza, M. S. (1995) *Divergence of Shape Fluctuations in Two Dimensions.* The Annals of Probability Vol. 23, 977–1005 MR1349159

[9] Richardson, D. (1973). *Random growth in a tessellation.* Proceedings of Cambridge Philosophical Society 74, 515-528 MR0329079