Elect. Comm. in Probab. **12** (2007), 106–119

ELECTRONIC COMMUNICATIONS in PROBABILITY

A GENERAL STOCHASTIC TARGET PROBLEM WITH JUMP DIFFUSION AND AN APPLICATION TO A HEDGING PROBLEM FOR LARGE INVESTORS

NICOLAS SAINTIER
Departamento de Matemática, FCEyN UBA (1428), Buenos Aires, Argentina
email: `nsaintie@dm.uba.ar`

Submitted April 19 2006, *accepted in final form* April 6 2007

AMS 2000 Subject classification: 49J20; 49L20; 60J60; 60J75; 35K55
Keywords: Stochastic control; jump diffusion; viscosity solutions; mathematical finance; large investor

Abstract
Let $Z^{\nu}_{t,z}$ be a $\mathbb{R}^d$-valued jump diffusion controlled by ν with initial condition $Z^{\nu}_{t,z}(t) = z$. The aim of this paper is to characterize the set $V(t)$ of initial conditions z such that $Z^{\nu}_{t,z}$ can be driven into a given target at a given time by proving that the function $u(,z) = 1 - 1_{V(t)}$ satisfies, in the viscosity sense, the equation (2) below. As an application, we study the problem of hedging in a financial market with a large investor.

1 Introduction and statement of the results

Let $Z^{\nu}_{t,z}$ be a $\mathbb{R}^d$-valued process controlled by ν with initial condition $Z^{\nu}_{t,z}(t) = z$. Given an horizon time $T > 0$, an initial time $t \geq 0$ and a target $\mathcal{C} \subset \mathbb{R}^d$, the general stochastic target problem consists in finding the set $V(t)$ of initial condition z such that there exists a control process ν, belonging to a well-defined set of admissible controls, for which $Z^{\nu}_{t,z}(T) \in \mathcal{C}$. The study of the set $V(t)$ is usually carried out by proving that the characteristic function $u(.,z)$ of its complement, i.e $u(t,z) = 1 - 1_{V(t)}(z)$, satisfies a partial differential equation (only in the viscosity sense since u is not even continuous).

Motivated by applications in finance, namely the super-replication problem, stochastic target problems were first considered by H.M. Soner and N. Touzi [13], [14] assuming the controlled process $Z^{\nu}_{t,z}$ follows a diffusion. Their proof relies on a direct dynamic programming principle (see theorem 3 below) that enables them to derive an equation for u similar to equation (2) below. Their result was then extended to the jump diffusion case by B. Bouchard [6] assuming the target is the epigraph of some function. The purpose of this paper is to extend both results to the jump diffusion case with an arbitrary target. Our proof follows the line of [13] by using tricks introduced by Bouchard to deal with jump process. The result obtained is then applied to a financial market with a large trader i.e. a financial market where the price of the risky asset depends on the strategy of some trader, a case that cannot be covered by Bouchard' result and that seems difficult to handle by forward-backward SDE or duality tecnics as was done up to now (see Cvitanic-Ma [10] or Bank-Baum [2]).

We now describe the model. Let $T > 0$ be a finite time horizon, $\mathcal{C} \subset \mathbb{R}^d$ a borel subset of $\mathbb{R}^d$ (the "target"), Σ a compact measurable space, called the mark space, and $v(dt, d\sigma) = \left(v^1(dt, d\sigma), \ldots, v^d(dt, d\sigma)\right)$ be a vector of d independant integer-valued Σ-marked right continuous point processes defined on a complete probability space $(\Omega, \mathcal{F}, P)$ (see Bremaud [7] for a detailed account on point-process). Let $W = (W_t)$ be a $\mathbb{R}^d$-valued standard Brownian motion defined on $(\Omega, \mathcal{F}, P)$ such that W and v are independant. We denote by $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ the P-completed filtration generated by W and $v(., d\sigma)$. We further assume that $\mathcal{F}_0$ is trivial and that $v(dt, d\sigma)$ has a predictable $(P, \mathbb{F})$-intensity kernel of the form $\lambda(d\sigma)dt$ satisfying $\int_\Sigma \lambda(d\sigma) < \infty$ (this assumption could probably be weaken by assuming that $\lambda(d\sigma)$ is a σ-finite measure such that $\int_\Sigma (1 \wedge |\sigma|^2)\lambda(d\sigma) < \infty$, see [8]). We then denote by $\tilde{v}(dt, d\sigma) = v(dt, d\sigma) - \lambda(d\sigma)dt$ the associated compensated measure. Since $P[v([0, T] \times (\Sigma - supp\ \lambda)) > 0] = 0$, we can also assume that $supp\ \lambda = \Sigma$. Eventually, we let U be a compact subset of $\mathbb{R}^d$ and denote by $\mathcal{U}$ the set of all $\mathbb{F}$-predictable process $\nu = (\nu_t)_{0 \leq t \leq T}$ valued in U; $\mathcal{U}$ is the set of admissible control. Given a control process $\nu \in \mathcal{U}$ and initial condition $(t, z) \in [0, T] \times \mathbb{R}^d$, we define the controlled process $Z^\nu_{t,z}$ as the solution on $[t, T]$ of the following stochastic differential equation:

$$\begin{cases} dZ(s) = \mu\left(s, Z(s), \nu_s\right) ds + \alpha\left(s, Z(s), \nu_s\right) dW_s + \int_\Sigma \beta\left(s, Z(s^-), \nu_s, \sigma\right) \ v(ds, d\sigma) \\ Z(t) = z \end{cases} \tag{1}$$

where $\mu : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}^d$, $\alpha : [0, T] \times \mathbb{R}^d \times U \to M_d(\mathbb{R})$ (where $M_d(\mathbb{R})$ is the set $d \times d$ real matrices) and $\beta : [0, T] \times \mathbb{R}^d \times U \times \Sigma \to \mathbb{R}^d$ are assumed to be continuous with respect to (s, ν, σ), lipschitz in t, lipschitz and linearly growing in z uniformly in (s, ν, σ) and bounded with respect to σ. This guarantees existence and uniqueness of a strong solution $Z^\nu_{t,z}$ for each control process $\nu \in \mathcal{U}$ (see Protter [12]).
Let $\mathcal{L}^\nu$ be the Dynkin operator associated to the controlled diffusion part of (1):

$$\mathcal{L}^\nu \phi(t, z) = \partial_t \phi(t, z) +^t \nabla\phi(t, z).\mu(t, z, \nu) + \frac{1}{2} tr\left({}^t\alpha(t, z, \nu).\nabla^2\phi(t, z).\alpha(t, z, \nu)\right),$$

where $\nabla\phi$ and $\nabla^2\phi$ are respectively the gradient and the hessian of ϕ with respect to z. Let also $V(t)$ be the reachability set defined by

$$V(t) = \left\{z \in \mathbb{R}^d \text{ such that } \exists \nu \in \mathcal{U}, Z^\nu_{t,z}(T) \in \mathcal{C} \text{ P-ps}\right\}.$$

We denote by $u(t, z)$ the characteristic function of the complement of $V(t)$:

$$u(t, z) = 1_{^c V(t)} = 1 - 1_{V(t)}(z).$$

The choice of using the characteristic function of the complement of $V(t)$ and not the one of $V(t)$ itself is only tecnical.

The main purpose of the paper is to prove that u satisfies in the viscosity sense the following equation

$$\sup_{\nu \in \mathcal{N}(t, z, \nabla u(t, z))} \min\left\{-\mathcal{L}^\nu u(t, z); \inf_{\sigma \in \Sigma} -\mathcal{G}^{\nu,\sigma} u(t, z)\right\} = 0, \tag{2}$$

where

$$\mathcal{G}^{\nu,\sigma} u(t, z) = u\left(t, z + \beta(t, z, \nu, \sigma)\right) - u(t, z)$$

is the jump of u at (t,z), and

$$\mathcal{N}(t,z,p)=\{\nu\in U \text{ such that } {}^t p.\alpha(t,z,\nu)=0\},\ (t,z,p)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d.$$

The set $\mathcal{N}$ appears naturally when one wants to control the brownian part of the diffusion. We make the two following assumptions on $\mathcal{N}$:
(H1) for all $(t,z,p)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d$, $\mathcal{N}(t,z,p)\neq\emptyset$, and
(H2) (existence of a continuous selection) for any $(t_0,z_0,p_0)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d$ and any $\nu_0\in\mathcal{N}(t_0,z_0,p_0)$, there exists a continuous function $\hat{\nu}:[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\to U$ locally lipschitz on $\{(t,z,p),p\neq 0\}$ satisfying $\hat{\nu}(t_0,z_0,p_0)=\nu_0$ and $\hat{\nu}(t,z,p)\in\mathcal{N}(t,z,p)$ for all $(t,z,p)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d$.

Our result is the following:

Theorem 1. *If $\mathcal{N}$ satisfies (H1) and (H2), then u satisfies (2) in the viscosity sense in $(0,T)\times\mathbb{R}^d$, i.e. the upper- and lower-semicontinuous envelopes u^* and u_* of u are respectively sub- and supersolution of (2) in the viscosity sense in $(0,T)\times\mathbb{R}^d$.*

We refer to Crandall-Ishii-Lions [9] or Barles [3] for the notion of viscosity solutions.

We now turn our attention to the terminal conditions satisfied by u. According to the definition of $V(T)$, we obviously have $u(T,z)=1_{^c\mathcal{C}}(z)$. However, u may be discontinuous at time T and we are thus led to introduce the functions $\underline{G},\overline{G}:\mathbb{R}^d\to\{0,1\}$ defined by

$$\underline{G}(z)=\liminf_{t\uparrow T,\ \tilde{z}\to z}u(t,\tilde{z}) \text{ and } \overline{G}(z)=\limsup_{t\uparrow T,\ \tilde{z}\to z}u(t,\tilde{z}).$$

Given $\phi:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$, let $\mathcal{M}_\phi$ be defined by

$$\mathcal{M}_\phi(t,z)=\sup_{\nu\in\mathcal{N}(t,z,\nabla\phi(t,z))}\ \inf_{\sigma\in\Sigma}-\mathcal{G}^{\nu,\sigma}\phi(t,z).$$

The result is then the following:

Theorem 2. *$\underline{G}$ is a viscosity supersolution of*

$$\min\{\phi-1_{^c\overline{\mathcal{C}}},\mathcal{M}_\phi(T,z)\}=0, \tag{3}$$

$\overline{G}$ is a viscosity subsolution of

$$\min\{\phi-1_{\overline{^c\mathcal{C}}},\mathcal{M}_\phi(T,z)\}=0. \tag{4}$$

Remark 1.1. $1_{^c\overline{\mathcal{C}}}=(1_{^c\mathcal{C}})_*$ *and* $1_{\overline{^c\mathcal{C}}}=(1_{^c\mathcal{C}})^*$.

The following section gives an application of these theorems to the hedging problem in a financial market with a large investor. Sections 2 and 3 are devoted to the proof of theorem 1 and 2 respectively.

2 Application to a large investor problem

In this section, we apply theorems 1 and 2 to a large investor problem. We consider a financial market consisting of one bank account and one risky asset and assume the existence of a large investor i.e. a trader whose trading strategy $\nu(t)$ influences significatively the evolution of the price process $S(t)$ of the risky asset (but not the price of the bank account). We indeed assume that $S(t)$ follows a diffusion with jump whose coefficients not only depend on $(t, S(t))$ but also on the strategy $\nu(t)$ of the large investor:

$$\frac{dS(t)}{S(t)} = \mu(t, S(t), \nu(t))dt + \alpha(t, S(t), \nu(t))dW(t) + \beta(t, S(t-), \nu(t))v(dt), \tag{5}$$

where μ, α, and β are as in equation (1). We moreover assume that $\beta > -1$, which implies, in view of the properties of the exponential (see Protter [12]), that $S(t) > 0$ as soon as $S(0) > 0$. Up to discounting, we can assume that the price of the bank account is constant equal to 1. The dynamic of the wealth process $X(t)$ of the large investor is then given by

$$\frac{dX(t)}{X(t)} = \nu(t)\frac{dS(t)}{S(t)}, \tag{6}$$

where $\nu(t) \in [-l, u]$, $l, u > 0$, denotes the proportion of his wealth the large investor invests in the risky asset. We assume, as in the previous section, that the process $(\nu(t))$ is predictable with respect to the augmented filtration generated by $(W(t))_{0\leq t\leq T}$ and $v(dt, d\sigma)$. We denote by $S^\nu_{t,s}$ and $X^\nu_{t,s,x}$ the solution of (5) and (6) respectively such that $S^\nu_{t,s}(t) = s$ and $X^\nu_{t,s,x}(t) = x$. Given a contingent claim whose maturity price may depend on $S(t)$ and thus on the behaviour of the large investor, the problem for him is to find the set $V(t)$ of initial values (s, t) such that he is able to hedge or super-replicate the claim at the maturity date. Such problems of hedging and super-replication in financial market with a large investor were investigated by Cvitanic-Ma [10] in the case of a diffusion by the study of forward-backward SDE, and by Bank-Baum [2] in the general case by means of duality technics. We refer more generally to Platen-Schweizer [11] for the study of the feedback effects of hedging strategy over the price of the underlying asset.

We apply here the results stated above to solve these problems. Given a target $\mathcal{C} \subset \mathbb{R}^2$ (e.g. the epigraph of a function $g : \mathbb{R} \to \mathbb{R}$) and a horizon time $T > 0$ (e.g. the maturity date of some claim of price $g(S^\nu_{t,s}(T))$ the large investor wants to hedge), we consider the reachability set $V(t)$ defined by

$$V(t) = \left\{(s, x) \in \mathbb{R}^2 \ s.t.\ (S^\nu_{t,s}(T), X^\nu_{t,s,x}(T)) \in \mathcal{C} \text{ for some strategy } \nu\right\},$$

and the function $u(t, (s, x)) = 1 - 1_{V(t)}(s, x)$. We also define as previously

$$\begin{aligned}\mathcal{G}^{\nu,\sigma}u(t, (s, x)) &= u\left(t, s + s\beta(t, s, \nu, \sigma), x + x\nu\beta(t, s, \nu, \sigma)\right) - u(t, (s, x)),\\ \mathcal{N}(t, (s, x), p) &= \{\nu \in [-l, u] \text{ s.t. } (sp_s + x\nu p_x)\alpha(t, (s, x), \nu) = 0\},\ p = (p_s, p_x) \in \mathbb{R}^2,\end{aligned}$$

and for $\phi \in C^2\left([0, T] \times \mathbb{R}^2\right)$,

$$\begin{aligned}\mathcal{L}^\nu\phi(t, (s, x)) &= \partial_t\phi + (s\partial_s\phi + x\nu\partial_x\phi)\mu\\ &\quad + \frac{1}{2}\alpha^2(s^2\partial^2_{ss}\phi + x^2\nu^2\partial^2_{xx}\phi + 2sx\nu\partial^2_{xs}\phi))(t, (s, x), \nu).\end{aligned}$$

Concerning the definition of $\mathcal{L}^\nu$, just remark that the brownian motion appearing in (5) and (6) are identic and thus are correlated. According to theorem 1, u is then a viscosity solution of

$$\sup_{\nu\in\mathcal{N}(t,(s,x),\nabla u(t,(s,x)))} \min\left\{-\mathcal{L}^\nu u(t, (s, x)); \inf_{\sigma\in\Sigma} -\mathcal{G}^{\nu,\sigma}u(t, (s, x))\right\} = 0 \tag{7}$$

satisfying the terminal conditions given by theorem 2.

3 Proof of theorem 1

The proof of theorem 1 relies mainly on the following direct dynamic programming principle proved by Soner-Touzi [13, proposition 6.1]:

Theorem 3. *Let $t \in [0,T]$. Then for any stopping time θ valued in $[t,T]$,*

$$V(t) = \left\{ z \in \mathbb{R}^d \text{ such that } \exists \nu \in \mathcal{U}, Z^{\nu}_{t,z}(\theta) \in V(\theta) \text{ a.s.} \right\}.$$

We also quote two lemmas which will be used in the proof of theorem 1. Their proofs rely on standard technics and can be found in Bouchard [6, lemmas 15, 16]. The first one is

Lemma 3.1. *Fix initial condition $(t,z) \in [0,T] \times \mathbb{R}^d$ and a sequence $(\nu_n) \subset \mathcal{U}$ of admissible controls. Then, for any sequence $(t_n, \tilde{t}_n, z_n) \subset [0,T] \times [0,T] \times \mathbb{R}^d$ such that $t_n \le \tilde{t}_n$and $(t_n, \tilde{t}_n, z_n) \to (t,t,z)$, we have*

$$\sup_{t_n \le s \le \tilde{t}_n} \left| Z^{\nu_n}_{t_n,z_n}(s) - z \right| \to 0 \text{ in } L^2.$$

The second one states as follow:

Lemma 3.2. *Let $\psi : [0,T] \times \mathbb{R}^d \times U \times \Sigma \to \mathbb{R}$ be locally lipschitz in (t,z) uniformly in $(\nu,\sigma) \in U \times \Sigma$. Then for any sequence $(t_n, z_n, h_n) \to (t_0, z_0, 0)$ and $(\nu_n) \subset \mathcal{U}$,*

$$\frac{1}{h_n} \int_{t_n}^{t_n+h_n} \int_{\Sigma} \left| \psi(s, Z^{\nu_n}_{z_n,t_n}(s), \nu_n(s), \sigma) - \psi(t_0, z_0, \nu_n(s), \sigma) \right| \lambda(d\sigma) ds \to 0$$

a.s. up to a subsequence.

3.1 Proof of the viscosity supersolution property

We need to prove that the lower-semicontinuous envelope u_* of u is a viscosity supersolution of (2). We thus fix a function $\phi \in C^\infty([0,T], \mathbb{R}^d)$ and a point $(t_0, z_0) \in (0,T) \times \mathbb{R}^d$ such that

$$\min_{[0,T]\times\mathbb{R}^d} (u_* - \phi) = (u_* - \phi)(t_0, z_0) = 0,$$

and are going to prove that

$$\sup_{\nu \in \mathcal{N}(t_0,z_0,\nabla\phi(t_0,z_0))} \min \left\{ -\mathcal{L}^\nu \phi(t_0,z_0); \inf_{\sigma\in\Sigma} -\mathcal{G}^{\nu,\sigma}\phi(t_0,z_0) \right\} \ge 0. \tag{8}$$

Let us first assume that $u_*(t_0,z_0) = 1$. Then $\phi(t_0,z_0) = 1$. Since $\phi \le u_* \le 1$ on $[0,T] \times \mathbb{R}^d$, we deduce that ϕ has a maximum at (t_0,z_0) and thus that $\partial_t \phi(t_0,z_0) = \nabla\phi(t_0,z_0) = 0$ and $\nabla^2\phi(t_0,z_0) \le 0$. Therefore $-\mathcal{L}^\nu\phi(t_0,z_0) \ge 0$ for any $\nu \in U$. On the other hand, $\nu \in U$ and $\sigma \in \Sigma$ being given, one has

$$\begin{aligned} -\mathcal{G}^{\nu,\sigma}\phi(t_0,z_0) &= \phi(t_0,z_0) - \phi(t_0, z_0 + \beta(t_0,z_0,\nu,\sigma)) \\ &= 1 - \phi(t_0, z_0 + \beta(t_0,z_0,\nu,\sigma)) \\ &\ge 0. \end{aligned}$$

Thus $\inf_{\sigma\in\Sigma} -\mathcal{G}^{\nu,\sigma}\phi(t_0,z_0) \ge 0$ for any $\nu \in U$. This proves (8). We are thus left with the case $u_*(t_0,z_0) = 0$. We split the proof of this case into several steps.

Step 1. Let $(t_n, z_n) \subset (0,T) \times \mathbb{R}^d$ be such that $(t_n, z_n) \to (t_0, z_0)$ and $u(t_n, z_n) = 0$ for any n i.e. $z_n \in V(t_n)$. Let also θ_n be a $[t_n, T]$-valued stopping time that we shall specify later. The dynamic programming principle (cf theorem 3) gives the existence for any n of an admissible control $\nu_n \in \mathcal{U}$ such that $Z_n(\theta_n) \in V(\theta_n)$ i.e. $u(\theta_n, Z_n(\theta_n)) = 0$ for any n. We let $Z_n = Z^{\nu_n}_{t_n,z_n}$ and $\beta_n = \phi(t_n, z_n)$. Itô's lemma gives

$$\begin{aligned} 0 \leq & -\beta_n - \int_{t_n}^{\theta_n} \mathcal{L}^{\nu_n}\phi(s, Z_n(s))dx - \int_{t_n}^{\theta_n} {}^t\nabla\phi(s, Z_n(s)).\alpha(s, Z_n(s), \nu_n(s))\ dW_s \\ & - \int_{t_n}^{\theta_n}\int_\Sigma \mathcal{G}^{\nu_n(s),\sigma}\phi(s, Z_n(s^-))\ v(ds, d\sigma). \end{aligned} \tag{9}$$

We now specify θ_n. Let $\eta \in (0, T-t_0)$ and γ be the largest jump of Z_n in the ball $B_{(t_0,z_0)}(\eta) \subset \mathbb{R}^{d+1}$:

$$\gamma = \sup_{(t,z,\nu,\sigma)\in B_{(t_0,z_0)}(\eta)\times U\times\Sigma} |\beta(t,z,\nu,\sigma)|.$$

The assumptions made on β implies that γ is finite. Let τ_n be the first exit time of $(s, Z_n(s))$ of the ball $B_{(t_0,z_0)}(\eta + 2\gamma)$:

$$\tau_n = \inf\left\{s \geq t_n \text{ such that } (s, Z_n(s)) \notin B_{(t_0,z_0)}(\eta+2\gamma)\right\}.$$

Then $\tau_n > t_n$ by definition of γ and lemma 3.1 gives

$$\liminf_{n\to+\infty}(\tau_n - t_n) > 0 \text{ ps.} \tag{10}$$

We define a sequence of positive real $h_n \to 0$ such that $\frac{\beta_n}{h_n} \to 0$ by:
(i) if the set $\{n \in \mathbb{N}; \beta_n = 0\}$ is finite then, up to a subsequence, we can assume that $\beta_n \neq 0$ for any n and we let $h_n = \sqrt{|\beta_n|}$;
(ii) if not, we can assume, up to a subsequence, that $\beta_n = 0$ for any n and we then define $h_n = 1/n$.
Let $\theta_n = \tau_n \wedge (t_n + h_n)$. Thanks to (10), $\theta_n = t_n + h_n$ for n large enough.

Step 2. We now define a family of equivalent probability measures $(Q^k_n)_{n\geq 0,k\geq 1}$ which will be used in the next step. For $(\nu, \sigma) \in U \times \Sigma$, let

$$\chi(\nu,\sigma) = 1_{\{-\mathcal{G}^{\nu,\sigma}\phi(t_0,z_0)<0\}}.$$

Given integers $n \geq 0$ and $k \geq 1$, we define as in Bouchard [6] a probability measure Q^k_n on $(\Omega, \mathcal{F})$ equivalent to P by $Q^k_n = M^k_n(T)P$ where

$$\begin{aligned} M^k_n(t) &= \mathcal{E}(k\int_{t_n}^{t\wedge\theta_n} {}^t\nabla\phi(s, Z_n(s), \nu_n(s)).\alpha(s, Z_n(s), \nu_n(s))\ dW_s \\ &\quad + \int_\Sigma (k\chi(\nu_n(s),\sigma) + k^{-1} - 1)\tilde{v}(ds, d\sigma)), \end{aligned}$$

and $\mathcal{E}$ denotes the Doleans-Dade exponential (see Protter [12]). Girsanov theorem (cf Bremaud [7]) then asserts that, under Q^k_n, $v(dt\times d\sigma)$ has $(k\chi(\nu_n(t),\sigma)+k^{-1})\lambda(d\sigma)dt$ as intensity kernel, and that

$$\int_{t_n}^{\cdot\wedge\theta_n} dW_s - k^t\nabla\phi(s, Z_n(s), \nu_n(s)).\alpha(s, Z_n(s), \nu_n(s))ds$$

is a stopped brownian motion. Moreover (see Bouchard [6]),

$$\lim_{n\to+\infty} M_n^k(T) \to 1 \text{ in } L^2(P) \tag{11}$$

for any k. We denote by E_n^k the $\mathcal{F}_{t_n}$-conditionnal expectation operator under Q_n^k.

Etape 3. Applying E_n^k to (9) yields

$$\begin{aligned} 0 \leq &- \beta_n - E_n^k \int_{t_n}^{\theta_n} \mathcal{L}^{\nu_n}\phi(s, Z_n(s))dx - kE_n^k \int_{t_n}^{\theta_n} |{}^t\nabla\phi(s, Z_n(s)).\alpha(s, Z_n(s), \nu_n(s))|^2 ds \\ &- E_n^k \int_{t_n}^{\theta_n} \int_\Sigma \left(k\chi(\nu_n(s), \sigma) + k^{-1}\right) \mathcal{G}^{\nu_n(s),\sigma}\phi(s, Z_n(s^-))\ \lambda(d\sigma)ds. \end{aligned}$$

Dividing the inequality by h_n and passing to the limit $n \to +\infty$ using (11), the dominated convergence theorem and the right-continuity of the filtration, we get (see Soner-Touzi [13] for details),

$$\begin{aligned} 0 \leq \liminf_{n\to+\infty} \big\{ &-h_n^{-1} \int_{t_n}^{\theta_n} \mathcal{L}^{\nu_n}\phi(s, Z_n(s))dx - kh_n^{-1} \int_{t_n}^{\theta_n} |{}^t\nabla\phi(s, Z_n(s)).\alpha(s, Z_n(s), \nu_n(s))|^2 ds \\ &- h_n^{-1} \int_{t_n}^{\theta_n} \int_\Sigma \left(k\chi(\nu_n(s), \sigma) + k^{-1}\right) \mathcal{G}^{\nu_n(s),\sigma}\phi(s, Z_n(s^-))\ \lambda(d\sigma)ds \big\}. \end{aligned}$$

We deduce from this inequality and lemma 3.2 that

$$0 \leq \liminf_{n\to+\infty} h_n^{-1} \int_{t_n}^{t_n+h_n} H_k(t_0, z_0, \nu_n(s))ds, \tag{12}$$

where

$$\begin{aligned} H_k(t_0, z_0, \nu) = &- \mathcal{L}^{\nu}\phi(t_0, z_0) - k|{}^t\nabla\phi(t_0, z_0).\alpha(t_0, z_0, \nu)|^2 \\ &- \int_\Sigma \left(k\chi(\nu, \sigma) + k^{-1}\right) \mathcal{G}^{\nu,\sigma}\phi(t_0, z_0)\lambda(d\sigma). \end{aligned}$$

Therefore

$$\sup_{\nu\in U} H_k(t_0, z_0, \nu) \geq 0.$$

According to the compacity of U and the continuity of H_k in ν, this sup is attained for each k by some $\nu_k \in U$:

$$0 \leq \sup_{\nu\in U} H_k(t_0, z_0, \nu) = H_k(t_0, z_0, \nu_k).$$

We can moreover assume that, up to a subsequence, $\nu_k \to \hat{\nu} \in U$. Passing to the limit in the previous inequality then gives

$$-\mathcal{L}^{\hat{\nu}}\phi(t_0, z_0) \geq 0,$$

$${}^t\nabla\phi(t_0, z_0).\alpha(t_0, z_0, \hat{\nu}) = 0 \text{ i.e } \hat{\nu} \in \mathcal{N}(t_0, z_0, \nabla\phi(t_0, z_0)), \text{ and}$$

$$-\mathcal{G}^{\hat{\nu},\sigma}\phi(t_0, z_0) \geq 0\ \lambda(d\sigma) - \text{ps}.$$

The function $\sigma \to -\phi(t_0, z_0 + \beta(t_0, z_0, \hat{\nu}, \sigma))$ being continuous and $\Sigma = supp\ \lambda$, this proves (8).

3.2 Proof of the viscosity subsolution property

Let $\phi \in C^\infty(\mathbb{R}^{d+1})$ and $(t_0, z_0) \in (0,T) \times \mathbb{R}^d$ be such that (t_0, z_0) is a strict maximum point of $u^* - \phi$ with $(u^* - \phi)(t_0, z_0) = 0$. We need to prove that

$$\sup_{\nu \in \mathcal{N}(t_0, z_0, \nabla\phi(t_0,z_0))} \min\left\{ -\mathcal{L}^\nu \phi(t_0, z_0); \inf_{\sigma \in \Sigma} -\mathcal{G}^{\nu,\sigma}\phi(t_0, z_0) \right\} \leq 0. \tag{13}$$

Let us first assume that $u^*(t_0, z_0) = 0$. Then $u \equiv 0$ in a neighborhood $\mathcal{V}$ of (t_0, z_0) and thus $\phi \geq 0$ in $\mathcal{V}$. Since $\phi(t_0, z_0) = u^*(t_0, z_0) = 0$, (t_0, z_0) is a local minimum point of ϕ and therefore $\partial_t \phi(t_0, z_0) = \nabla\phi(t_0, z_0) = 0$ and $\nabla^2\phi(t_0, z_0) \geq 0$. Hence $-\mathcal{L}^\nu\phi(t_0, z_0) \leq 0$ for any $\nu \in U$. This proves (13).

We now assume that $u^*(t_0, z_0) = 1$ and $\nabla\phi(t_0, z_0) \neq 0$, and prove (13) by contradiction. We thus suppose that there exists $\nu_0 \in \mathcal{N}(t_0, z_0, \nabla\phi(t_0, z_0))$ such that

$$\begin{aligned} &-\mathcal{L}^{\nu_0}\phi(t_0, z_0) > 0, \text{ and} \\ &\inf_{\sigma\in\Sigma} -\mathcal{G}^{\nu_0,\sigma}\phi(t_0, z_0) > 0. \end{aligned}$$

Since $\nabla\phi(t,z) \neq 0$ in a neighborhood of (t_0, z_0) and according to assumption (H2), the function $\tilde{\nu}(t,z) := \hat{\nu}(t, z, \nabla\phi(t,z))$, where $\hat{\nu}$ is given by (H2), is continuous in a neighborhood of (t_0, z_0). There thus exists $\delta > 0$ such that

$$\nabla\phi(t,z) \neq 0,$$

$$-\mathcal{L}^{\tilde{\nu}(t,z)}\phi(t,z) > 0, \text{ and} \tag{14}$$

$$\inf_{\sigma\in\Sigma} -\mathcal{G}^{\tilde{\nu}(t,z),\sigma}\phi(t,z) > 0 \tag{15}$$

in $U_\delta := \bar{B}_{z_0}(\delta) \times [t_0, t_0 + \delta]$. Independently, since $u^* - \phi$ has a strict maximum at (t_0, z_0), we can also assume that there exists $\beta > 0$ such that

$$u^*(t,z) < \phi(t,z) - \beta \tag{16}$$

on the parabolic boundary $\partial_p U_\delta := (\partial B_{z_0}(\delta) \times [t_0, t_0 + \delta)) \cup (\bar{B}_{z_0}(\delta) \times \{t_0 + \delta\})$ of U_δ. Let $(t_n, z_n) \in [0,T] \times \mathbb{R}^d$ be such that $(t_n, z_n) \to (t_0, z_0)$ and $u(t_n, z_n) \to u^*(t_0, z_0)$. Since $\nabla\phi(t,z) \neq 0$ in a neighborhood of (t_0, z_0), $\tilde{\nu}$ is lipschitz in this neighborhood. Therefore, there exists a process Z_n solution of

$$\begin{cases} dZ_n(s) = \mu\left(s, Z_n(s), \nu_n(s)\right) ds + \alpha\left(s, Z_n(s), \nu_n(s)\right) dW_s + \int_\Sigma \beta\left(s, Z_n(s), \nu_n(s), \sigma\right) v(ds, d\sigma) \\ Z_n(t_n) = z_n \end{cases}$$

for $|s - t_n|$ small, and where $\nu_n(s) := \tilde{\nu}(s, Z_n(s))$. We define stopping times θ_n^j and τ_n by

$$\begin{aligned} \theta_n^j &= T \wedge \inf\left\{s > t_n, \Delta Z_n(s) \neq 0\right\}, \\ \tau_n &= T \wedge \inf\left\{s > t_n, (s, Z_n(s)) \notin U_\delta\right\}, \text{ and} \\ \theta_n &= \theta_n^j \wedge \tau_n, \end{aligned}$$

i.e. θ_n^j are τ_n respectively the first jump time and exit time of U_δ of Z_n. Let $\mathcal{J}_n = \left\{\tau_n < \theta_n^j\right\}$ and denote by Z_n^c the continuous part of Z_n. On $\mathcal{J}_n$, $Z_n = Z_n^c$ is continuous on $[t_n, \tau_n] = [t_n, \theta_n]$ and thus $(\theta_n, Z_n(\theta_n)) \in \partial_p U_\delta$. According to (16), we get

$$u\left(\theta_n, Z_n(\theta_n)\right) \leq \phi\left(\theta_n, Z_n(\theta_n)\right) - \beta \text{ on } \mathcal{J}_n. \tag{17}$$

On the other hand, on ${}^c\mathcal{J}_n$, we have by Itô's lemma

$$\begin{aligned}\phi(\theta_n^j, Z_n(\theta_n^j)) &= \phi(t_n, z_n) + \int_{t_n}^{\theta_n^{j-}} \mathcal{L}^{\nu_n}\phi(s, Z_n(s))ds \\ &+ \int_{t_n}^{\theta_n^{j-}} {}^t\nabla\phi(s, Z_n(s)).\alpha(s, Z_n(s), \nu_n(s))dW_s \\ &+ \int_\Sigma \mathcal{G}^{\nu_n(\theta_n^{j-}),\sigma}\phi\left(\theta_n^{j-}, Z_n(\theta_n^{j-})\right)\ v(\{\theta_n^j\}, d\sigma).\end{aligned}$$

Since $\theta_n^j \le \tau_n$, $(s, Z_n(s)) \in U_\delta$ for any $t_n \le s \le \theta_n^{j-}$ and thus, by (14), (15) and recalling that $\nu_n(s) \in \mathcal{N}(s, Z_n(s), \nabla\phi(s, Z_n(s)))$, there exists $\varepsilon > 0$ such that

$$\phi(\theta_n^j, Z_n(\theta_n^j)) \le \phi(t_n, z_n) - \varepsilon \text{ on } {}^c\mathcal{J}_n. \tag{18}$$

We deduce from (17) and (18) the existence of $\eta > 0$ such that for any n,

$$u(\theta_n, Z_n(\theta_n)) \le \phi(\theta_n, Z_n(\theta_n)) \le \phi(t_n, z_n) - \eta \text{ a.e.}$$

Since $\phi(t_n, z_n) = \phi(t_0, z_0) + o(1) = 1 + o(1)$, we get for n large enough that $u(\theta_n, Z_n(\theta_n)) = 0$ i.e. $u(t_n, z_n) = 0$ according to the dynamic programming principle. Hence

$$1 = u^*(t_0, z_0) = \lim_{n\to+\infty} u(t_n, z_n) = 0$$

which is the desired contradiction.
We eventually assume that $u^*(t_0, z_0) = 1$ and $\nabla\phi(t_0, z_0) = 0$. This time, $\hat{\nu}$ being not a priori locally lipschitz, the process Z_n defined above may not exist and the previous method doesn't work. We treat this case following Soner-Touzi [13] by proving that if (13) is false, then $\nabla^2\phi(t_0, z_0)$ has a negative eigenvalue which allows us to construct a perturbation ϕ_ϵ of ϕ for which the previous case applies. Passing then to the limit gives the conclusion. We only sketch the proof of the existence of a negative eigenvalue for $\nabla^2\phi(t_0, z_0)$ and refer the reader to Soner-Touzi [13] for more details.
Assuming (13) false, we get $\nu_0 \in \mathcal{N}(t_0, z_0, \nabla\phi(t_0, z_0)) = \mathcal{N}(t_0, z_0, 0)$ such that

$$\begin{aligned}&-\mathcal{L}^{\nu_0}\phi(t_0, z_0) > 0, \text{ and} \\ &\inf_{\sigma\in\Sigma} -\mathcal{G}^{\nu_0,\sigma}\phi(t_0, z_0) > 0.\end{aligned} \tag{19}$$

According to (H2), there exists a continuous map $\hat{\nu} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to U$ such that

$$d -^t p.\mu(t, z, \hat{\nu}(t, z, p)) - \frac{1}{2}tr\left({}^t\alpha(t, z, \hat{\nu}(t, z, p)).B.\alpha(t, z, \hat{\nu}(t, z, p))\right) > 0, \tag{20}$$

$$\inf_{\sigma\in\Sigma} -\mathcal{G}^{\hat{\nu}(t,z,p),\sigma}\phi(t, z) > 0 \tag{21}$$

for (d, t, z, p, B) in a neighborhood of $(-\partial_t\phi(t_0, z_0), t_0, z_0, 0, \nabla^2\phi(t_0, z_0))$. Let us assume by contradiction that, in some orthonormal basis of $\mathbb{R}^n$,

$$\nabla^2\phi(t_0, z_0) = diag(\lambda_1, \dots, \lambda_n), \text{ with } \lambda_i \ge 0\ \forall\ i : 1 \dots n. \tag{22}$$

Choose for each i a real $m_i > \lambda_i/2$ and consider the function

$$\Psi(z) = \sum_i m_i (z^i - z_0^i)^2, \; z = (z^1, \ldots, z^n) \in \mathbb{R}^n.$$

Given $\delta > 0$, we define $B_\delta = \{z \in \mathbb{R}^n \; s.t. \; \Psi(z) < \delta\}$ and denote by $\bar{B}_\delta$ the closure of B_δ. We are going to prove that $u \equiv 0$ in $Q = (t_0, t_0 + \delta) \times (B_\delta - B_\epsilon)$. Since $\epsilon > 0$ is arbitrary, this will imply $u^*(t_0, z_0) = 0$, which is a contradiction. We thus fix $(\tilde{t}, \tilde{z}) \in Q$ such that $\Psi(\tilde{z}) \geq 4\epsilon$, and consider $\tilde{\nu}(t, z) = \hat{\nu}(t, z, \nabla\Psi(t, z))$, $(t, z) \in Q$. Since $\nabla\Psi(z) \neq 0$ in $B_\delta - B_\epsilon$, $\tilde{\nu}$ is locally lipshitz in Q and there thus exists $\tilde{Z} = Z^{\tilde{\nu}}_{\tilde{t}, \tilde{z}}$ solution of (1) with initial conditions $\tilde{Z}(\tilde{t}) = \tilde{z}$. We can assume that $\tilde{Z}$ exists on $[\tilde{t}, t_0 + \delta]$. Let τ, θ^j and θ be the stopping times defined by

$$\tau = \inf \left\{ t \geq \tilde{t} \; s.t. \; (t, \tilde{Z}(t)) \notin Q \right\},$$

$$\theta^j = \inf \left\{ t \geq \tilde{t} \; s.t. \; \Delta\tilde{Z}(t) \neq 0 \right\} \text{ and}$$

$$\theta = \tau \wedge \theta^j.$$

The same proof as in Soner-Touzi [13] gives that $u^*(\theta, \tilde{Z}(\theta)) = 0$ a.e. on $\{\tau < \theta^j\}$ for $\epsilon > 0$ small enough. In that case, $\tilde{Z}(\tau)$ belongs to the parabolic boundary of Q. The result then follows by using two estimates. The first one is based on the remark that (19) and (22) imply the existence of $b > 0$ such that $\partial_t\phi(t_0, z_0) < -b$. Taylor' formula then gives

$$\phi(t, z) \leq \tilde{\phi}(t, z) := 1 - b(t - t_0) + \Psi(z)$$

on $(t_0, t_0 + \delta] \times \bar{B}_\delta$ with some small $\delta > 0$. The other estimate is a minoration of $\sqrt{\Psi(\tilde{Z}(t))}$. On the other hand, on $\{\theta^j \leq \tau\}$, Itô's lemma yields

$$\begin{aligned} u^*(\theta^j, \tilde{Z}(\theta^j)) &\leq \phi(\theta^j, \tilde{Z}(\theta^j)) \\ &= \phi(\tilde{t}, \tilde{z}) + \int_{\tilde{t}}^{\theta^{j-}} \mathcal{L}^{\tilde{\nu}(t)}\phi(t, \tilde{Z}(t))dt \\ &\quad + \int_\Sigma \mathcal{G}^{\tilde{\nu}(\theta^{j-}), \sigma}\phi\left(\theta^{j-}, \tilde{Z}(\theta^{j-})\right) \; v(\{\theta^j\}, d\sigma). \end{aligned}$$

For $t \in [\tilde{t}, \theta^{j-}] \subset [\tilde{t}, \tau)$, $(t, \tilde{Z}(t)) \in Q$. Therefore, according to (20) and (21), and up to reduce δ, there exists some $\eta > 0$ such that

$$u^*(\theta^j, \tilde{Z}(\theta^j)) \leq \phi(\tilde{t}, \tilde{z}) - \eta.$$

Since $\phi(\tilde{t}, \tilde{z}) \to 1$ as $\epsilon \to 0$ and η is independent of ϵ, we deduce that, for ϵ small enough, $u^*(\theta^j, \tilde{Z}(\theta^j)) = 0$ a.e. on $\{\theta^j \leq \tau\}$. Hence, for small ϵ, $u^*(\theta, \tilde{Z}(\theta)) = 0$ a.e., and thus, by the dynamic programmming principle, $u^*(\tilde{t}, \tilde{z}) = 0$. As explained above, this ends the proof of the subsolution property.

4 Proof of theorem 2

4.1 Terminal conditions for $\underline{G}$

We first prove that $\underline{G} \geq 1_{c\overline{C}}$. Let $z_0 \in \mathbb{R}^d$ be such that $\underline{G}(z_0) = 0$. Consider a sequence $(t_n, z_n) \to (T, z_0)$ satisfying, for any n, $u(t_n, z_n) = 0$ i.e. $z_n \in V(t_n)$. There thus exists $\nu_n \in \mathcal{U}$

such that $Z^{\nu_n}_{t_n,z_n}(T) \in \mathcal{C}$ a.e. for any n. We can show as in the proof of lemme 3.1 that

$$\lim_{n\to+\infty} Z^{\nu_n}_{t_n,z_n}(T) = z_0$$

a.e. up to a subsequence. Hence $z_0 \in \overline{\mathcal{C}}$ i.e. $1_{^c\overline{\mathcal{C}}}(z_0) = 0$.
Let $\phi \in C^2(\mathbb{R}^d)$ and $z_0 \in \mathbb{R}^d$ be such that

$$(\underline{G} - \phi)(z_0) = \min_{z\in\mathbb{R}^d} (\underline{G} - \phi)(z) = 0.$$

We extend ϕ to $[0,T] \times \mathbb{R}^d$ by $\phi(t,z) = \phi(z)$. Let (s_n, ξ_n) be a sequence in $(0,T) \times \mathbb{R}^d$ such that

$$(s_n, \xi_n) \to (T, z_0) \text{ and } u_*(s_n, \xi_n) \to \underline{G}(z_0),$$

and ϕ^k_n be the function defined on $(s_n, T] \times \mathbb{R}^d$ for $n \in \mathbb{N}$, $k > 0$, by

$$\phi^k_n(t,z) = \phi(z) - \frac{k}{2}|z - z_0|^2 + k\frac{T-t}{T-s_n}.$$

Since β is bounded in σ and continuous in (t,z,ν) with U compact, we see that, for a given constant $C > 0$,

$$\eta := \sup \ \{|\beta(t,z,\nu,\sigma)|,\ \sigma\in\Sigma,\ \nu \in U,\ |t-t_0| + |z - z_0| \le C\} < \infty.$$

Let $B := B_{z_0}(C + \eta)$ and $\bar{B}$ denote the closure of B. We have

$$\lim_{k\to 0} \limsup_{n\to+\infty} \sup_{(t,z)\in[s_n,T]\times\bar{B}} |\phi^k_n(t,z) - \phi(z)| = 0, \text{ and} \tag{23}$$

$$\lim_{k\to 0} \limsup_{n\to+\infty} \sup_{(t,z)\in[s_n,T]\times\bar{B}} |\nabla\phi^k_n(t,z) - \nabla\phi(z)| = 0. \tag{24}$$

Let (t^k_n, z^k_n) be a minimum point of $u_* - \phi^k_n$ on $[s_n, T] \times \bar{B}$. We can prove as in Bouchard [6] that

$$\text{for all } k > 0,\ (t^k_n, z^k_n) \to (T, z_0), \tag{25}$$

$$\text{for all } k > 0,\ t^k_n < T \text{ for sufficiently large } n, \tag{26}$$

$$\lim_{k\to 0} \limsup_{n\to+\infty}\ u_*(t^k_n, z^k_n) \to \underline{G}(x_0). \tag{27}$$

Thanks to (25), we can assume that $x^k_n \in B$ for all n, k. By (26), theorem 1 then gives

$$\mathcal{M}_{\phi^k_n}(t^k_n, z^k_n) \ge 0. \tag{28}$$

For $\psi \in C([0,T] \times \mathbb{R}^d)$, define

$$\mathcal{F}\psi(t,z,\nu) = \inf_{\sigma\in\Sigma} -\mathcal{G}^{\nu,\sigma}\psi(t,z),\ (t,z,\nu) \in [0,T] \times \mathbb{R}^d \times U.$$

By Berge maximum theorem (see Berge [4], Aubin-Ekeland [1] or Border [5]), $\mathcal{F}\psi$ is continuous on $[0,T] \times \mathbb{R}^d \times U$. Fix $\epsilon > 0$. By (28), there exists $\nu^k_n(\epsilon) \in \mathcal{N}(t^k_n, z^k_n, \nabla\phi^k_n(t^k_n, z^k_n)) \subset U$ such that

$$\mathcal{F}\phi^k_n(t^k_n, z^k_n, \nu^k_n(\epsilon)) \ge -\epsilon. \tag{29}$$

Since U is compact, we can assume that

$$\nu(\epsilon) := \lim_{k\to 0} \lim_{n\to+\infty} \nu_n^k(\epsilon) \in U. \tag{30}$$

Moreover, by (24) and (25),

$$\nu(\epsilon) \in \mathcal{N}(T, z_0, \nabla\phi(T, z_0)). \tag{31}$$

We eventually have according to (23) that, for a given $\delta > 0$, there exist k_0 small and n_0 large such that for $k \le k_0$ and $n \ge n_0$,

$$\left|\mathcal{F}\phi_n^k(t, z, \nu) - \mathcal{F}\phi(t, z, \nu)\right| \le \delta \tag{32}$$

for any $\nu \in U$ and $(t, z) \in [s_n, T] \times \bar{B}$. Since $(t_n^k, z_n^k) \in [s_n, T] \times \bar{B}$, we deduce from (25), (29) - (32) and the continuity of $\mathcal{F}\phi$ that

$$\mathcal{M}_\phi(T, z_0) \ge \mathcal{F}\phi(T, z_0, \nu(\epsilon)) \ge -\epsilon$$

for any $\epsilon > 0$. Therefore

$$\mathcal{M}_\phi(T, z_0) \ge 0,$$

what we wanted to prove.

4.2 Terminal conditions for $\overline{G}$

Let $\phi \in C^2(\mathbb{R}^d)$ and $z_0 \in \mathbb{R}^d$ be such that

$$(\overline{G} - \phi)(z_0) = \max_{z\in\mathbb{R}^d}(\overline{G} - \phi)(z) = 0.$$

Suppose that (4) does not hold i.e. that there exists $\epsilon > 0$ satisfying

$$\sup_{\nu\in\mathcal{N}(T,z_0,\nabla\phi(z_0))} \inf_{\sigma\in\Sigma} -\mathcal{G}^{\nu,\sigma}\phi(z_0) > \epsilon, \text{ and} \tag{33}$$

$$\overline{G}(z_0) = \phi(z_0) > 1_{\overline{^c\mathcal{C}}}(z_0). \tag{34}$$

Consider the map $\psi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ defined for $\eta > 0$ by

$$\psi_\eta(t, z) = \phi(z) + \eta|z - z_0|^2 + \sqrt{T - t}.$$

Then $\lim_{t\uparrow T} \partial_t\psi_\eta(t, z) = -\infty$ uniformly in η and z. Let $C > 0$. By continuity of α, μ and ϕ,

$$\lim_{t\uparrow T} \mathcal{L}^\nu\psi_\eta(t, z) = -\infty$$

uniformly in $\nu \in U$, $z \in \bar{B}_{z_0}(C)$ and $\eta > 0$ in a bounded subset of $(0, +\infty)$. There thus exists $\eta > 0$ such that

$$-\mathcal{L}^\nu\psi_\eta(t, z) > 0 \tag{35}$$

for any $t \in (T - \eta, T)$, $\nu \in U$, and $z \in \bar{B}_{z_0}(C)$. Consider a sequence $(s_n, \xi_n) \in (T - \eta, T) \times \bar{B}_{z_0}(C)$ satisfying $(s_n, \xi_n) \to (T, z_0)$ and $u^*(s_n, \xi_n) \to \overline{G}(z_0)$, and denote by $(t_n, z_n) \in [s_n, T] \times \bar{B}_{z_0}(C)$ a maximum point on $[s_n, T] \times \bar{B}_{z_0}(C)$ of $u^* - \psi_\eta$. Using (34), we can show as in Bouchard [6] that, up to a subsequence,

$$(t_n, z_n) \to (T, z_0),\ t_n < T \ et\ u^*(t_n, z_n) \to \overline{G}(z_0). \tag{36}$$

Theorem 1 gives for any n that

$$\sup_{\nu\in\mathcal{N}(t_n,z_n,\nabla\psi_\eta(t_n,z_n))} \min\left\{-\mathcal{L}^\nu\psi_\eta(t_n,z_n); \inf_{\sigma\in\Sigma} -\mathcal{G}^{\nu,\sigma}\psi_\eta(t_n,z_n)\right\} \le 0.$$

Then, by (35),

$$\sup_{\nu\in\mathcal{N}(t_n,z_n,\nabla\psi_\eta(t_n,z_n))} \inf_{\sigma\in\Sigma} -\mathcal{G}^{\nu,\sigma}\psi_\eta(t_n,z_n) \le 0 \tag{37}$$

for all n. Consider the function $\Phi : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ defined by

$$\Phi(t,z) = \sup_{\nu\in\mathcal{N}(t,z,\nabla\psi_\eta(t,z))} \inf_{\sigma\in\Sigma} -\mathcal{G}^{\nu,\sigma}\psi_\eta(t,z).$$

Then

$$\begin{aligned}\Phi(T,z_0) &= \sup_{\nu\in\mathcal{N}(T,z_0,\nabla\phi(z_0))} \inf_{\sigma\in\Sigma} \left\{-\mathcal{G}^{\nu,\sigma}\phi(z_0) - \eta|\beta(T,z_0,\nu,\sigma)|^2\right\} \\ &> \epsilon - \eta \inf_{\nu\in\mathcal{N}(T,z_0,\nabla\phi(z_0))} \sup_{\sigma\in\Sigma} |\beta(T,z_0,\nu,\sigma)|^2.\end{aligned}$$

Since β is bounded in σ, we can thus assume, up to reduce η, that

$$\Phi(T,z_0) > \frac{\epsilon}{2}. \tag{38}$$

Independently, U being compact and α being continuous in (t,z,p), the correspondance $(t,z,p) \to \mathcal{N}(t,z,p)$ is continuous. Since $\nabla\psi_\eta$ is continuous, the correspondance $(t,z) \to \mathcal{N}(t,z,\nabla\psi_\eta(t,z))$ is in particular lower semicontinuous. We then deduce from the continuity of the map $(t,z,\nu) \to \inf_{\sigma\in\Sigma} -\mathcal{G}^{\nu,\sigma}\phi_\eta(t,z)$ and [4] that Φ is lower semicontinuous. Therefore, by (38),

$$\Phi(t,z) = \sup_{\nu\in\mathcal{N}(t,z,\nabla\psi_\eta(t,z))} \inf_{\sigma\in\Sigma} -\mathcal{G}^{\nu,\sigma}\psi_\eta(t,z) > \frac{\epsilon}{4}$$

for all (t,z) close enough to (T,z_0). For n large enough, this contradicts (37).

Acknowledgments. The author would like to thank Prof. B. Bouchard for his help and support.

References

[1] J.P. Aubin, I. Ekeland. *Applied non linear analysis.* Pure and applied mathematics (1984) Wiley. MR 0749753

[2] P. Bank, D. Baum. Hedging and portfolio optimization in financial markets with a large trader, *Mathematical Finance.* **14** (2004), 1–18. MR 2030833

[3] G. Barles. *Solutions de viscosité des équations de Hamilton-Jacobi.* Mathématiques et applications **17** (1994) Springer-Verlag. MR 1613876

[4] C. Berge. *Espaces topologiques: fonctions multivoques.* Collection universitaire de mathématiques **3** (1959) Dunod. MR 0105663

[5] K. C. Border. *Fixed points theorems with applications to economics and game theory.* (1985) Cambridge University Press. MR 0790845

[6] B. Bouchard. Stochastic targets with mixed diffusion processes and viscosity solutions. *Stochastic processes and their applications.* **101** (2002), 273–302. MR 1931270

[7] P. Bremaud. *Point processs and queues, martingale dynamics.* Springer series in statistics (1981) Springer-Verlag. MR 0636252

[8] R. Buckdahn, E. Pardoux. BSDE's with jumps and associated integro-partial differential equations. *Preprint.*

[9] M.G. Crandall, H. Ishii and P-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bulletin of the AMS* **1** (1992), 1–67. MR 1118699

[10] J. Cvitanic, J. Ma. Hedging options for a large investor and forward-backward SDE's. *The annals of applied probability* **6** (1996), 370–398. MR 1398050

[11] E. Platen, M. Schweizer. On feedback effects from hedging derivatives. *Mathematical Finance* **1** (1998), 67–84. MR 1613291

[12] P. Protter. *Stochastic integration and differential equations.* Applications of Mathematics (1990) Springer-Verlag. MR 1037262

[13] H.M. Soner, N. Touzi. Dynamic programming for stochastic target problems and geometric flows. *JEMS* **4** (2002), 201–236. MR 1924400

[14] H.M. Soner, N. Touzi. Stochastic target problems, dynamic programming and viscosity solutions. *SIAM J. Control Optim* **41** (2002), 404–424. MR 1920265