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LINEAR STOCHASTIC DIFFERENTIAL-ALGEBRAIC EQUATIONS WITH CONSTANT COEFFICIENTS

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Abstract
We consider linear stochastic differential-algebraic equations with constant coefficients and additive white noise. Due to the nature of this class of equations, the solution must be defined as a generalised process (in the sense of Dawson and Fernique). We provide sufficient conditions for the law of the variables of the solution process to be absolutely continuous with respect to Lebesgue measure.

1 Introduction

A Differential-Algebraic Equation (DAE) is, essentially, an Ordinary Differential Equation (ODE) $F(x, \dot{x}) = 0$ that cannot be solved for the derivative $\dot{x}$. The name comes from the fact that in some cases they can be reduced to a two-part system: A usual differential system plus a "nondifferential" one (hence "algebraic", with some abuse of language), that is

$$\begin{cases} \dot{x}_1 = f(x_1, x_2) \\ 0 = g(x_1, x_2) \end{cases} \tag{1}$$

for some partitioning of the vector x into vectors x_1 and x_2. In general, however, such a splitting need not exist.

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In comparison with ODE's, these equations present at least two major difficulties: the first lies in the fact that it is not possible to establish general existence and uniqueness results, due to their more complicate structure; the second one is that DAE's do not regularise the input (quite the contrary), since solving them typically involves differentiation in place of integration. At the same time, DAE's are very important objects, arising in many application fields; among them we mention the simulation of electrical circuits, the modelling of multibody mechanisms, the approximation of singular perturbation problems arising e.g. in fluid dynamics, the discretisation of partial differential equations, the analysis of chemical processes, and the problem of protein folding. We refer to Rabier and Rheinboldt [10] for a survey of applications.

The class of DAE's most treated in the literature is, not surprisingly, that of linear equations, which have the form

$$A(t)\dot{x}(t) + B(t)x(t) = f(t) \ ,$$

with $x, f\colon \mathbb{R}^+ \to \mathbb{R}^n$ and $A, B\colon \mathbb{R}^+ \to \mathbb{R}^{n\times n}$. When A and B are constant matrices the equation is said to have constant coefficients. Note that these equations cannot in general be split as in (1).

Recently, there has been some incipient work (Schein and Denk [12] and Winkler [14]) on Stochastic Differential-Algebraic Equations (SDAE). In view to incorporate to the model a random external perturbation, an additional term is attached to the differential-algebraic equation, in the form of an additive noise (white or coloured). The solution will then be a stochastic process instead of a single function.

Since the focus in [12] and [14] is on numerical solving and the particular applications, some interesting theoretical questions have been left aside in these papers. Our long-term purpose is to put SDAE into the mainstream of stochastic calculus, developing as far as possible a theory similar to that of stochastic differential equations. In this first paper our aim is to investigate the solution of linear SDAE with constant coefficients and an additive white noise, that means

$$A\dot{x}(t) + Bx(t) = f(t) + \Lambda\xi(t) \ ,$$

where ξ is a white noise and A, B, Λ are constant matrices of appropriate dimensions. We shall first reduce the equation to the so-called Kronecker Canonical Form (KCF), which is easy to analyse, and from whose solution one can recover immediately the solution to the original problem. Unfortunately, it is not possible to extend this approach to the case of linear SDAE with varying coefficients, just as happens in the deterministic case, where several different approaches have been proposed. Among these, the most promising in our opinion is that of Rabier and Rheinboldt [9].

Due to the simple structure of the equations considered here, it is not a hard task to establish the existence of a unique solution in the appropriate sense. However, as mentioned before, a DAE does not regularise the input $f(t)$ in general. If white noise, or a similarly irregular noise is used as input, then the solution process to a SDAE will not be a usual stochastic process, defined as a random vector at every time t, but instead a "generalised process", the random analogue of a Schwartz generalised function.

The paper is organised as follows: in the next section we shall provide a short introduction to linear DAE's and to generalised processes. In the third section we shall define what we mean by a solution to a linear SDAE and in Section 4 we shall provide a sufficient condition for the existence of a density of the law of the solution. In the final Section 5 we shall discuss a simple example arising in the modelling of electrical circuits.

Superscripts in parentheses mean order of derivation. The superscript $^\top$ stands for transposition. All function and vector norms throughout the paper will be L^2 norms, and the inner product will be denoted by $\langle \cdot, \cdot \rangle$ in both cases. Covariance matrices of random vectors will be denoted by $\mathrm{Cov}(\cdot)$. The Kronecker delta notation $\delta_{ij} := 1_{\{i=j\}}$ will be used throughout.

2 Preliminaries on DAE and generalised processes

In this section we briefly introduce two topics: the (deterministic) differential-algebraic equations and the generalised processes. An exhaustive introduction on the first topic can be found in Rabier and Rheinboldt [10], while the basic theory of generalised processes can be found in Dawson [1], Fernique [2], or Chapter 3 in Gel'fand and Vilenkin [3].

2.1 Differential-Algebraic Equations

Consider an implicit autonomous ODE,

$$F(x, \dot{x}) = 0 \ , \tag{2}$$

where $F := F(x,p) : \mathbb{R}^{n\times n} \to \mathbb{R}^n$ is a sufficiently smooth function. If the partial differential $D_pF(x,p)$ is invertible at every point (x_0, p_0), one can easily prove that the implicit ODE is locally reducible to an explicit ODE. If $D_pF(x_0,p_0)$ is not invertible, two cases are possible: either the total derivative $DF(x_0,p_0)$ is onto $\mathbb{R}^n$ or it is not. In the first case, and assuming that the rank of $D_pF(x,p)$ is constant in a neighbourhood of (x_0,p_0), (2) is called a *differential-algebraic equation*, while in the remaining cases one speaks of an ODE with a singularity at (x_0,p_0).

A *linear DAE* is a system of the form

$$A(t)\dot{x} + B(t)x = f(t) \ , \quad t \geq 0 \ , \tag{3}$$

where $A(t), B(t) \in \mathbb{R}^{n\times n}$ and $f(t) \in \mathbb{R}^n$. The matrix function $A(t)$ is assumed to have a constant (non-full) rank for any t in the interval of interest. (Clearly, if $A(t)$ has full rank for all t in an interval, then the DAE reduces locally to an ODE.) In the simplest case, when A and B do not depend on t, we have a *linear DAE with constant coefficients*, and an extensive study of these problems has been developed. Since we want to allow solutions of DAE in the distributional sense, let us make precise the definition of a solution.

Let $\mathcal{D}'$ be the space of distributions (generalised functions) on some open set $U \subset \mathbb{R}$, that is, the dual of the space $\mathcal{D} = \mathcal{C}_c^\infty(U)$ of smooth functions with compact support defined on U. An n-dimensional distribution is an element of $(\mathcal{D}')^n$, and, for $x = (x_1, \dots, x_n) \in (\mathcal{D}')^n$ and $\phi \in \mathcal{D}$, we denote by $\langle x, \phi \rangle$, the column vector $(\langle x_1, \phi\rangle, \dots, \langle x_n, \phi\rangle)^\top$, the action of x on ϕ. We will always assume, without loss of generality, $U =]0, +\infty[$.

Definition 2.1 *Let f be an n-dimensional distribution on U, and A, B two $n\times n$ constant matrices. A solution to the linear DAE with constant coefficients*

$$A\dot{x} + Bx = f \tag{4}$$

is an n-dimensional distribution x on U such that, for every test function $\phi \in \mathcal{D}$, the following equality holds:

$$A\langle \dot{x}, \phi\rangle + B\langle x, \phi \rangle = \langle f, \phi \rangle$$

The theory of linear DAE starts with the definition of a regular matrix pencil:

Definition 2.2 *Given two matrices $A, B \in \mathbb{R}^{n\times n}$, the* matrix pencil (A, B) *is the function $\lambda \mapsto \lambda A + B$, for $\lambda \in \mathbb{R}$. It is called a* regular matrix pencil *if* $\det(\lambda A + B) \neq 0$ *for some λ.*

If the matrices A and B in equation (4) form a regular matrix pencil, then a solution exists. This is a consequence of the following classical result due to Weierstrass and Kronecker, which states that A and B can be simultaneously transformed into a convenient *canonical form* (see e.g. Griepentrog and März [4] for the proof).

Lemma 2.3 *Given a regular matrix pencil (A, B), there exist nonsingular $n \times n$ matrices P and Q and integers $0 \leq d, q \leq n$, with $d + q = n$, such that*

$$PAQ = \begin{pmatrix} I_d & 0 \\ 0 & N \end{pmatrix} \quad \text{and} \quad PBQ = \begin{pmatrix} J & 0 \\ 0 & I_q \end{pmatrix}$$

where I_d, I_q are identities of dimensions d and q, $N = blockdiag(N_1, \ldots, N_r)$, with N_i the $q_i \times q_i$ matrix

$$N_i = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix},$$

and J is in Jordan canonical form.

Proposition 2.4 *If (A, B) is a regular pencil, x is a solution to the DAE (4) if and only if $y = Q^{-1}x$ is a solution to*

$$PAQ\dot{y} + PBQy = Pf \ , \tag{5}$$

where P and Q are the matrices of Proposition 2.3.

Proof: The result is obvious, since (5) is obtained from (4) multiplying from the left by the invertible matrix P. □

System (5) is said to be in *Kronecker Canonical Form* (KCF) and splits into two parts. The first one is a linear differential system of dimension d, and the second one is an "algebraic system" of dimension q. Denoting by u and v the variables in the first and the second part respectively, and by b and c the related partitioning of the vector distribution Pf we can write the two systems as follows:

$$\begin{pmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_d \end{pmatrix} + J \begin{pmatrix} u_1 \\ \vdots \\ u_d \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_d \end{pmatrix} , \tag{6}$$

$$N \begin{pmatrix} \dot{v}_1 \\ \vdots \\ \dot{v}_q \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_q \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_q \end{pmatrix} . \tag{7}$$

We refer to u as the *differential variables* and to v as the *algebraic variables.*
The differential system has a unique solution once an initial condition, i.e. the value of the solution at some suitable test function ϕ_0, is given. The function must have a nonvanishing integral (see Schwartz [13], p.51 and p.130). It can be assumed without any loss of generality that $\int_0^\infty \phi_0 = 1$.

On the other hand, system (7) consists of a number of decoupled blocks, which are easily and uniquely solved by backward substitution, without the need of any additional condition. For instance, for the first block,

$$N_1 \begin{pmatrix} \dot{v}_1 \\ \vdots \\ \dot{v}_{q_1} \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_{q_1} \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_{q_1} \end{pmatrix} , \tag{8}$$

a recursive calculation gives the following distributional solution:

$$\langle v_j, \phi \rangle = \sum_{k=j}^{q_1} \left\langle c_k, \phi^{(k-j)} \right\rangle , \quad j = 1, \dots, q_1 , \quad \phi \in \mathcal{D} \tag{9}$$

We can thus state the following proposition and corollary:

Proposition 2.5 *Assume (A, B) is a regular matrix pencil. Then, for every $u^0 = (u_1^0, \dots, u_d^0) \in \mathbb{R}^d$, and every fixed test function ϕ_0 with $\int_0^\infty \phi_0 = 1$, there exists a unique solution y to (5) such that*

$$\langle y, \phi_0 \rangle_i = u_i^0 , \quad i = 1, \dots, d .$$

Corollary 2.6 *Assume (A, B) is a regular matrix pencil. Then, for every $u^0 = (u_1^0, \dots, u_d^0) \in \mathbb{R}^d$, and every fixed test function ϕ_0 with $\int_0^\infty \phi_0 = 1$, there exists a unique solution x to (4) such that*

$$\langle Q^{-1} x, \phi_0 \rangle_i = u_i^0 , \quad i = 1, \dots, d .$$

Note that the matrix N is nilpotent, with nilpotency index given by the dimension of its largest block. The nilpotency index of N in this canonical form is a characteristic of the matrix pencil and we shall call it the *index* of the equation (4). The regularity of the solution depends directly on the index of the equation.

Remark 2.7 *Without the hypothesis of regularity of the pencil, a linear DAE may possess an infinity of solutions or no solution at all, depending on the right-hand side. This is the case, for instance, of*

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \dot{x}(t) + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} x(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$$

with any fixed initial condition.

2.2 Generalised processes

As before, let $\mathcal{D}'$ be the space of distributions on an open set U. A *random distribution* on U, defined in the probability space $(\Omega, \mathcal{F}, P)$, is a measurable mapping $X : (\Omega, \mathcal{F}) \to (\mathcal{D}', \mathcal{B}(\mathcal{D}'))$, where $\mathcal{B}(\mathcal{D}')$ denotes the Borel σ-field, relative to the weak-$\star$ topology (equivalently, the strong dual topology, see Fernique [2]). Denoting by $\langle X(\omega), \phi \rangle$ the action of the distribution $X(\omega) \in \mathcal{D}'$ on the test function $\phi \in \mathcal{D}$, it holds that the mapping $\omega \mapsto \langle X(\omega), \phi \rangle$ is measurable from $(\Omega, \mathcal{F})$ into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, hence a real random variable $\langle X, \phi \rangle$ on $(\Omega, \mathcal{F}, \mathcal{P})$. The law of X is determined by the law of the finite-dimensional vectors $(\langle X, \phi_1 \rangle, \dots, \langle X, \phi_n \rangle)$, $\phi_i \in \mathcal{D}$, $n \in \mathbb{N}$. The sum of random distributions X and Y on $(\Omega, \mathcal{F}, \mathcal{P})$, defined in the obvious manner, is again a random distribution. The product of a real random variable α and a random distribution,

defined by $\langle \alpha X, \phi \rangle := \alpha \langle X, \phi \rangle$, is also a random distribution. The derivative of a random distribution, defined by $\langle \dot{X}, \phi \rangle := -\langle X, \dot{\phi} \rangle$, is again a random distribution.
Given a random distribution X, the mapping $X : \mathcal{D} \to L^0(\Omega)$ defined by $\phi \mapsto \langle X, \phi \rangle$ is called a *generalised stochastic process.* This mapping is linear and continuous with the usual topologies in $\mathcal{D}$ and in the space of all random variables $L^0(\Omega)$. Note that we can safely overload the meaning of the symbol X.
The *mean functional* and the *correlation functional* of a random distribution are the deterministic distribution $\phi \mapsto \mathrm{E}[\langle X, \phi \rangle]$ and the bilinear form $(\phi, \psi) \mapsto \mathrm{E}[\langle X, \phi \rangle \langle X, \psi \rangle]$, respectively, provided they exist.
A simple example of random distribution is *white noise* ξ, characterised by the fact that $\langle \xi, \phi \rangle$ is centred Gaussian, with correlation functional $\mathrm{E}[\langle \xi, \phi \rangle \langle \xi, \psi \rangle] = \int_U \phi(s)\psi(s)\, ds$. In particular, $\langle \xi, \phi \rangle$ and $\langle \xi, \psi \rangle$ are independent if the supports of ϕ and ψ are disjoint. In this paper we will use as the base set the open half-line $U =]0, +\infty[$. White noise on U coincides with the Wiener integral with respect to a Brownian motion W: Indeed, if ϕ is a test function, then

$$\langle \xi, \phi \rangle = \int_0^\infty \phi(t)\, dW(t) \tag{10}$$

in the sense of equality in law. More precisely, the Wiener integral is defined as the extension to $L^2(\mathbb{R}^+)$ of white noise (see Kuo [7] for a construction of the Wiener integral as extension of white noise). Now, integrating by parts in (10), we can write

$$\langle \xi, \phi \rangle = -\int_0^\infty W(t)\dot{\phi}(t)\, dt = -\langle W, \dot{\phi} \rangle \ ,$$

so that ξ is the derivative of the Brownian motion W as random distributions. A random distribution is *Gaussian* if every finite-dimensional projection is a Gaussian random vector. This is the case of white noise and Brownian motion.
Further results on random distributions and generalised stochastic processes can be found for instance in the classical papers by Dawson [1] and Fernique [2]. We will also use in Section 3 the following facts about deterministic distributions, which apply as well to random distributions. The hyperplane $\mathcal{H}$ of $\mathcal{D}$ consisting of those functions whose integral on U is equal to zero coincides with the set of test functions which are derivatives of other test functions. Therefore, fixing a test function $\phi_0 \in \mathcal{D}$ such that $\int_U \phi_0(t)\, dt = 1$, every $\phi \in \mathcal{D}$ can be uniquely decomposed as $\phi = \lambda \phi_0 + \dot{\psi}$, for some $\psi \in \mathcal{D}$ and $\lambda = \int_U \phi(t)\, dt$.
If $f \in \mathcal{D}'$ is a distribution, the equation $\dot{T} = f$ has an infinite number of solutions (the primitives of f): T is completely determined on $\mathcal{H}$ by $\langle T, \dot{\psi} \rangle = -\langle f, \psi \rangle$ whereas $\langle T, \phi_0 \rangle$ can be arbitrarily chosen (for more details see Schwartz [13], II.4).

3 The generalised process solution

Consider the linear *stochastic differential-algebraic equation* (SDAE) with constant coefficients

$$A\dot{x} + Bx = f + \Lambda \xi \ , \tag{11}$$

where A and B are $n \times n$ real matrices, f is an n-dimensional distribution, Λ is an $n \times m$ constant matrix, and ξ is an m-dimensional white noise: $\xi = (\xi_1, \ldots, \xi_m)$, with ξ_i independent one-dimensional white noises. Recall that we will always take $U =]0, +\infty[$ as the base set for all distributions.

Definition 3.1 *A solution to the SDAE*

$$A\dot{x} + Bx = f + \Lambda\xi \tag{12}$$

is an n-dimensional random distribution x such that, for almost all $\omega \in \Omega$, $x(\omega)$ is a solution to the deterministic equation

$$A\dot{x}(\omega) + Bx(\omega) = f + \Lambda\xi(\omega) \ , \tag{13}$$

in the sense of Definition 2.1.

Theorem 3.2 *Assume (A, B) is a regular matrix pencil. Then, for every $u^0 = (u_1^0, \ldots, u_d^0) \in \mathbb{R}^d$, and every fixed test function ϕ_0, there exists an almost surely unique random distribution x, solution to (12), such that*

$$\langle Q^{-1}x, \phi_0\rangle_i = u_i^0 \ , \quad i = 1, \ldots, d \ ,$$

where Q is the matrix in the reduction to KCF. Furthermore, the solution is measurable with respect to the σ-field generated by ξ.

Proof: For every $\omega \in \Omega$, we have a linear DAE with constant coefficients, given by (13), and we know from Corollary 2.6 that there exists a unique solution $x(\omega) \in \mathcal{D}'$, satisfying

$$\langle Q^{-1}x(\omega), \phi_0\rangle_i = u_i^0 \ , \quad i = 1, \ldots, d \ .$$

In order to prove that the mapping $\omega \mapsto x(\omega)$ is measurable with respect to the σ-field generated by the white noise ξ, we will explicitly construct the solution as much as possible with a variation of constants argument.
Let P and Q be the invertible matrices of Lemma 2.3. Multiplying (12) from the left by P and setting $y = Qx$ we obtain the SDAE in Kronecker Canonical Form

$$\begin{pmatrix} I_d & 0 \\ 0 & N \end{pmatrix} \dot{y} + \begin{pmatrix} J & 0 \\ 0 & I_q \end{pmatrix} y = Pf + P\Lambda\xi \ , \tag{14}$$

System (14) splits into a stochastic differential system of dimension d and an "algebraic stochastic system" of dimension q, with $d + q = n$. Denoting by u and v the variables in the first and the second systems respectively, by b and c the related partitioning of the vector distribution $Pf = \begin{pmatrix} b \\ c \end{pmatrix}$, and by $S = (\sigma_{ij})$ and $R = (\rho_{ij})$ the corresponding splitting of $P\Lambda$ into matrices of dimensions $d \times m$ and $q \times m$, so that $P\Lambda = \begin{pmatrix} S \\ R \end{pmatrix}$, we can write the two systems as

$$\begin{pmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_d \end{pmatrix} + J \begin{pmatrix} u_1 \\ \vdots \\ u_d \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_d \end{pmatrix} + S \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} \ , \tag{15}$$

$$N \begin{pmatrix} \dot{v}_1 \\ \vdots \\ \dot{v}_q \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_q \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_q \end{pmatrix} + R \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} \ . \tag{16}$$

Fixing a test function ϕ_0 with $\int_0^\infty \phi_0 = 1$ and a vector $u^0 \in \mathbb{R}^d$, we have for the first one the distributional stochastic initial value problem

$$\left.\begin{aligned} \dot{u} + Ju &= \eta \\ \langle u, \phi_0 \rangle &= u^0 \end{aligned}\right\} \tag{17}$$

where $\eta := b + S\xi$.
Consider the matrix system

$$\left.\begin{aligned} \dot{\Phi} + J\Phi &= 0 \\ \int_0^\infty \Phi(t) \cdot \phi_0(t)\, dt &= I_d \end{aligned}\right\}\ , \tag{18}$$

whose distributional solution exists and is unique, and it is a $\mathcal{C}^\infty$ matrix function $\Phi\colon \mathbb{R} \to \mathbb{R}^{d\times d}$ (see Schwartz [13], V.6). Define $T := \Phi^{-1}u$. From (17), it follows that $\eta - Ju = \dot{u} = \dot{\Phi}T + \Phi\dot{T} = -J\Phi T + \Phi\dot{T} = -Ju + \Phi\dot{T}$, and therefore $\dot{T} = \Phi^{-1}\eta$. Let

$$\Phi_{ij}(t)\phi(t) = \lambda_{ij}\phi_0(t) + \dot{\psi}_{ij}(t)$$

be the unique decomposition of the function $\Phi_{ij} \cdot \phi \in \mathcal{D}$ into a multiple of ϕ_0 and an element of the hyperplane of derivatives $\mathcal{H}$ (see Subsection 2.2).
Then,

$$\begin{aligned} \langle u_i, \phi \rangle &= \Big\langle \sum_{j=1}^d \Phi_{ij} T_j, \phi \Big\rangle = \Big\langle \sum_{j=1}^d T_j, \Phi_{ij}\phi \Big\rangle = \sum_{j=1}^d \left[\lambda_{ij} \langle T_j, \phi_0 \rangle - \langle (\Phi^{-1}\eta)_j, \psi_{ij} \rangle \right] \\ &= \sum_{j=1}^d \Big[\lambda_{ij} \langle T_j, \phi_0 \rangle - \Big\langle \sum_{k=1}^d \Phi^{-1}_{jk} \eta_k, \psi_{ij} \Big\rangle \Big] = \sum_{j=1}^d \lambda_{ij} \langle T_j, \phi_0 \rangle - \sum_{k=1}^d \Big\langle \eta_k, \sum_{j=1}^d \psi_{ij} \Phi^{-1}_{jk} \Big\rangle \ . \end{aligned} \tag{19}$$

The terms $\langle T_j, \phi_0 \rangle$ should be defined in order to fulfil the initial condition. Using the decomposition

$$\Phi_{ij}(t)\phi_0(t) = \delta_{ij}\phi_0(t) + \dot{\psi}^0_{ij}(t)$$

and applying formula (19) to $\phi = \phi_0$, it is easily found that we must define

$$\langle T_j, \phi_0 \rangle = u^0_j + \sum_{k=1}^d \Big\langle \eta_k, \sum_{\ell=1}^d \psi^0_{j\ell} \Phi^{-1}_{\ell k} \Big\rangle \ .$$

Therefore,

$$\begin{aligned} \langle u_i, \phi \rangle &= \sum_{j=1}^d \lambda_{ij} \Big[u^0_j + \sum_{k=1}^d \Big\langle \eta_k, \sum_{\ell=1}^d \psi^0_{j\ell} \Phi^{-1}_{\ell k} \Big\rangle \Big] - \sum_{k=1}^d \Big\langle \eta_k, \sum_{j=1}^d \psi_{ij} \Phi^{-1}_{jk} \Big\rangle \\ &= \sum_{j=1}^d \lambda_{ij} u^0_j + \sum_{k=1}^d \Big\langle \eta_k, \sum_{j=1}^d \lambda_{ij} \sum_{\ell=1}^d \psi^0_{j\ell} \Phi^{-1}_{\ell k} - \sum_{j=1}^d \psi_{ij} \Phi^{-1}_{jk} \Big\rangle \\ &= \sum_{j=1}^d \lambda_{ij} u^0_j + \sum_{k=1}^d \Big\langle \eta_k, \sum_{\ell=1}^d \Big(\sum_{j=1}^d \lambda_{ij} \psi^0_{j\ell} - \psi_{i\ell} \Big) \Phi^{-1}_{\ell k} \Big\rangle \ . \end{aligned}$$

Taking into account that

$$\psi^0_{j\ell}(t) = \int_0^t \big(\Phi_{j\ell}(s)\phi_0(s) - \delta_{j\ell}\phi_0(s)\big)\, ds \ ,$$
$$\psi_{i\ell}(t) = \int_0^t \big(\Phi_{i\ell}(s)\phi(s) - \lambda_{i\ell}\phi_0(s)\big)\, ds \ ,$$

we obtain finally

$$\begin{aligned}
\langle u_i, \phi\rangle &= \sum_{j=1}^d \lambda_{ij} u_j^0 + \sum_{k=1}^d \Big\langle \eta_k, \sum_{\ell=1}^d \Big(\int_0^t \Big(\sum_{j=1}^d \lambda_{ij}\Phi_{j\ell}(s)\phi_0(s) - \Phi_{i\ell}(s)\phi(s)\Big)\, ds\Big)\Phi^{-1}_{\ell k}\Big\rangle \\
&= \sum_{j=1}^d \lambda_{ij} u_j^0 + \sum_{k=1}^d \Big\langle \eta_k, \sum_{\ell=1}^d \Big(\int_0^t (\lambda\Phi\phi_0 - \Phi\phi)(s)\, ds\Big)_{i\ell}\Phi^{-1}_{\ell k}\Big\rangle \\
&= \sum_{j=1}^d \lambda_{ij} u_j^0 + \sum_{k=1}^d \Big\langle \eta_k, \Big(\int_0^t (\lambda\Phi\phi_0 - \Phi\phi)(s)\, ds \cdot \Phi^{-1}\Big)_{ik}\Big\rangle \ . \qquad (20)
\end{aligned}$$

On the other hand, the algebraic part (16) consists of a number of decoupled blocks, which are easily solved by backward substitution. Any given block can be solved independently of the others and a recursive calculation gives, e.g. for a first block of dimension q_1, the following generalised process solution

$$\langle v_j, \phi\rangle = \sum_{k=j}^{q_1} \Big\langle c_k + \sum_{\ell=1}^m \rho_{k\ell}\xi_\ell, \phi^{(k-j)}\Big\rangle \ , \quad j = 1, \dots q_1 \ , \quad \phi \in \mathcal{D} \ . \qquad (21)$$

By (20) and (21), we have $(u, v) = G(\xi)$, for some deterministic function $G\colon (\mathcal{D}')^m \to (\mathcal{D}')^n$. Given generalized sequence $\{\eta_\alpha\}_\alpha \subset (\mathcal{D}')^m$ converging to η in the product of weak-$\star$ topologies, it is immediate to see that $G(\eta_\alpha)$ converges to $G(\eta)$, again in the product of weak-$\star$ topologies. This implies that the mapping G is continuous and therefore measurable with respect to the Borel σ-fields. Thus, the solution process x is measurable with respect to the σ-field generated by ξ. □

Remark 3.3 *In the case where $b = 0$ (or even if b is a function), so that the right hand side in (17) is simply $S\xi$, it is well known that the solution of the differential system is a classical stochastic process which can be expressed as a functional of a standard m-dimensional Wiener process. Indeed, we have, in the sense of equality in law, from (20),*

$$\langle u_i, \phi\rangle = \sum_{j=1}^d \lambda_{ij} u_j^0 + \sum_{k=1}^d \sum_{\ell=1}^m \int_0^\infty \Big(\int_0^t \big(\lambda\Phi\phi_0 - \Phi\phi\big)(s)\, ds \cdot \Phi^{-1}(t)\Big)_{ik} \sigma_{k\ell}\, dW_\ell(t) \ .$$

Fix an initial time $t_0 \in]0, \infty[$. Take a sequence $\{\phi_0^n\}_n \subset \mathcal{D}$ converging in $\mathcal{D}'$ to the Dirac delta δ_{t_0}, and with $\operatorname{supp}\phi_0^n \subset [t_0 - \frac{1}{n}, t_0 + \frac{1}{n}]$, and let $\{\Phi^n\}_n$ be the corresponding sequence of solutions to the matrix system (18). Then, $\lim_{n\to\infty} \int_0^t \Phi^n\phi_0^n = I_d \cdot \mathbf{1}_{[t_0,\infty[}(t)$ a.e. and we get

$$\begin{aligned}
\langle u_i, \phi\rangle = \sum_{j=1}^d \lambda_{ij} u_j^0 &+ \sum_{k=1}^d \sum_{\ell=1}^m \int_{t_0}^\infty \big(\lambda\Phi^{-1}\big)_{ik}(s)\sigma_{k\ell}\, dW_\ell(s) \\
&- \sum_{k=1}^d \sum_{\ell=1}^m \int_0^\infty \Big(\int_0^s \big(\Phi\cdot\phi\big)(u)\, du \cdot \Phi^{-1}(s)\Big)_{ik} \sigma_{k\ell}\, dW_\ell(s) \ ,
\end{aligned}$$

where $\Phi = \lim_{n\to\infty} \Phi^n$ *and* $\lambda = \lim_{n\to\infty} \int_0^\infty \Phi^n \phi = \int_0^\infty \Phi\phi$.
Now collapsing in the same way ϕ *to* δ_t*, with* $t \in \mathbb{R}$ *fixed,* λ *converges to* $\Phi(t)$ *and* $\int_0^s \Phi\phi$ *converges to* $\Phi(t) \cdot \mathbf{1}_{[t,\infty[}(s)$ *a.e. We arrive at*

$$u_i(t) = \sum_{j=1}^{d} \Phi_{ij} u_j^0 + \sum_{k=1}^{d} \sum_{\ell=1}^{m} \int_{t_0}^{t} \left(\Phi(t)\Phi^{-1}(s)\right)_{ik} \sigma_{k\ell}\, dW_\ell(s) \ .$$

Finally, using that the solution to (18) with δ_{t_0} *in place of* ϕ_0 *is known to be* $\Phi(t) = e^{-J(t-t_0)}$*, we obtain*

$$u(t) = e^{-J(t-t_0)} \Big[u_0 + \int_{t_0}^{t} e^{-J(s-t_0)} S\, dW(s) \Big] \ .$$

In a similar way we can express the first block of the algebraic part, if $c = 0$*, as*

$$\langle v_j, \phi \rangle = \sum_{k=j}^{q_1} \sum_{\ell=1}^{m} \rho_{k\ell} \int_0^\infty \phi^{(k-j)}(t)\, dW_\ell(t) \ , \quad j = 1, \dots q_1 \ , \tag{22}$$

and analogously for any other block. □

4 The law of the solution

In the previous section we have seen that the solution to a linear SDAE with regular pencil and additive white noise can be explicitly given as a functional of the input noise. From the modelling viewpoint, the law of the solution is the important output of the model. Using the explicit form of the solution, one can try to investigate the features of the law in which one might be interested.

To illustrate this point, we shall study the absolute continuity properties of the joint law of the vector solution evaluated at a fixed arbitrary test function ϕ. We will assume throughout this section that the base probability space is the canonical space of white noise: $\Omega = \mathcal{D}'$, $\mathcal{F} = \mathcal{B}(\mathcal{D}')$, and P is the law of white noise. This will be used in Theorem 4.5, to ensure the existence of conditional probabilities (see Dawson [1], Theorem 2.12). The main assumptions in Theorem 4.5 are that the dimensions in (11) satisfy $m \geq n$ and that the rank of the matrix Λ is equal to the number of rows n.

Let us start by considering separately the solutions to the decoupled equations (15) and (16). From the explicit calculation in the previous section (equation (20) for the differential part and equation (21) for the first algebraic block), we get that for any given test function ϕ the random vectors $\langle u, \phi \rangle$ and $\langle v, \phi \rangle$ have a Gaussian distribution with expectations

$$E[\langle u_i, \phi \rangle] = \sum_{j=1}^{d} \lambda_{ij} u_j^0 + \sum_{k=1}^{d} \Big\langle b_k, \Big(\int_0^t (\lambda\Phi\phi_0 - \Phi\phi)(s)\, ds \cdot \Phi^{-1}(t) \Big)_{ik} \Big\rangle \ , \quad i = 1, \dots, d \ ,$$

$$E[\langle v_i, \phi \rangle] = \sum_{k=i}^{q_1} \langle c_k, \phi^{(k-i)} \rangle \ , \quad i = 1, \dots, q_1 \ ,$$

and covariances

$$\mathrm{Cov}\big(\langle u,\phi\rangle\big)_{ij} = \sum_{\ell=1}^{m} \int_0^{\infty} \Big[\sum_{k=1}^{d} \int_0^t (\lambda\Phi\phi_0 - \Phi\phi)(s)\,ds\cdot\Phi^{-1}(t))_{ik}\sigma_{kl}\Big]^2 \tag{23}$$

$$\times \Big[\sum_{k=1}^{d} \int_0^t (\lambda\Phi\phi_0 - \Phi\phi)(s)\,ds\cdot\Phi^{-1}(t))_{jk}\sigma_{kl}\Big]^2 ds\ , \quad i,j=1,\ldots,d\ ,$$

and

$$\mathrm{Cov}\big(\langle v,\phi\rangle\big) = \begin{pmatrix} \rho_1 & \rho_2 & \cdots & \rho_{q_1-1} & \rho_{q_1} \\ \rho_2 & \rho_3 & \cdots & \rho_{q_1} & 0 \\ \vdots & & & & \vdots \\ \rho_{q_1} & 0 & \cdots & 0 & 0 \end{pmatrix}$$

$$\times\,\mathrm{Cov}\Big(\langle\xi,\phi\rangle,\langle\xi,\dot\phi\rangle,\ldots,\langle\xi,\phi^{(q_1-1)}\rangle\Big) \begin{pmatrix} \rho_1 & \rho_2 & \cdots & \rho_{q_1-1} & \rho_{q_1} \\ \rho_2 & \rho_3 & \cdots & \rho_{q_1} & 0 \\ \vdots & & & & \vdots \\ \rho_{q_1} & 0 & \cdots & 0 & 0 \end{pmatrix}^{\top}\ , \tag{24}$$

where ρ_i denotes the i-th row of the matrix R and $\mathrm{Cov}\big(\langle\xi,\phi\rangle,\ldots,\langle\xi,\phi^{(q_1-1)}\rangle\big)$ is a square matrix of dimension mq_1. We refer the reader to [6] for a comprehensive study of multidimensional Gaussian laws.

For the differential variables u alone, we are faced with a usual linear stochastic differential equation (see Remark 3.3), and there are well-known results on sufficient conditions for its absolute continuity, involving the matrices S and J (see e.g. Nualart [8], Section 2.3).

For the algebraic variables v, their absolute continuity depends in part on the invertibility of the covariance matrix of the white noise and its derivatives that appear in (24). We will use the following auxiliary result concerning the joint distribution of a one-dimensional white noise and its first k derivatives. This is a vector distribution with a centred Gaussian law and a covariance that can be expressed in full generality as (cf. Subsection 2.2)

$$\mathrm{Cov}\big(\langle\xi,\phi\rangle,\ldots,\langle\xi^{(k)},\phi\rangle\big)_{ij} = \mathrm{R{\small E}}\Big[(-1)^{\frac{|i-j|}{2}}\Big]\|\phi^{((i+j)/2)}\|^2\ , \tag{25}$$

where Re means the real part. We can prove the absolute continuity of this vector for $k\le 3$.

Lemma 4.1 *For all $\phi\in\mathcal{D}-\{0\}$, and a one-dimensional white noise ξ, the vector $\langle(\xi,\dot\xi,\ddot\xi,\dddot\xi),\phi\rangle$ is absolutely continuous.*

Proof: The covariance matrix of the vector $\langle(\xi,\dot\xi,\ddot\xi,\dddot\xi),\phi\rangle$ is

$$\begin{pmatrix} \|\phi\|^2 & 0 & -\|\dot\phi\|^2 & 0 \\ 0 & \|\dot\phi\|^2 & 0 & -\|\ddot\phi\|^2 \\ -\|\dot\phi\|^2 & 0 & \|\ddot\phi\|^2 & 0 \\ 0 & -\|\ddot\phi\|^2 & 0 & \|\dddot\phi\|^2 \end{pmatrix}$$

whose determinant is equal to

$$\det\begin{pmatrix} \|\phi\|^2 & -\|\dot\phi\|^2 \\ -\|\dot\phi\|^2 & \|\ddot\phi\|^2 \end{pmatrix}\cdot\det\begin{pmatrix} \|\dot\phi\|^2 & -\|\ddot\phi\|^2 \\ -\|\ddot\phi\|^2 & \|\dddot\phi\|^2 \end{pmatrix}\ .$$

Both factors are strictly positive, in view of the chain of strict inequalities

$$\frac{\|\phi\|}{\|\dot\phi\|} > \frac{\|\dot\phi\|}{\|\ddot\phi\|} > \frac{\|\ddot\phi\|}{\|\dddot\phi\|} > \cdots\ , \qquad \phi\in\mathcal{D},\ \phi\not\equiv 0 \tag{26}$$

These follow from integration by parts and Cauchy-Schwarz inequality, e.g.

$$\|\dot\phi\|^2 = \int_0^\infty \dot\phi\cdot\dot\phi = -\int_0^\infty \phi\cdot\ddot\phi \le \|\phi\|\cdot\|\ddot\phi\|\ ,$$

and the inequality is strict unless $\ddot\phi = K\phi$ for some K, which implies $\phi\equiv 0$. □

Remark 4.2 *The proof above does not work for higher order derivatives and we do not know if the result is true or false.*

Consider, as in the previous section, only the first algebraic block, and assume momentarily that its dimension is $q_1 = 2$. From (24), the covariance matrix of the random vector $\langle(v_1, v_2), \phi\rangle$ is

$$\begin{pmatrix} \|\phi\|^2\|\rho_1\|^2 + \|\dot\phi\|^2\|\rho_2\|^2 & \|\phi\|^2\langle\rho_1,\rho_2\rangle \\ \|\phi\|^2\langle\rho_1,\rho_2\rangle & \|\phi\|^2\|\rho_2\|^2 \end{pmatrix}\ ,$$

with determinant

$$\|\phi\|^4\big(\|\rho_1\|^2\|\rho_2\|^2 - \langle\rho_1,\rho_2\rangle^2\big) + \|\phi\|^2\|\dot\phi\|^2\|\rho_2\|^4\ .$$

Hence, assuming $\phi\not\equiv 0$, we see that the joint law of $\langle v_1,\phi\rangle$ and $\langle v_2,\phi\rangle$ is absolutely continuous with respect to Lebesgue measure in $\mathbb{R}^2$ if ρ_2 is not the zero vector. When $\|\rho_2\| = 0$ but $\|\rho_1\|\neq 0$, then $\langle v_2,\phi\rangle$ is degenerate and $\langle v_1,\phi\rangle$ is absolutely continuous, whereas $\|\rho_2\| = \|\rho_1\| = 0$ makes the joint law degenerate to a point.
This sort of elementary analysis, with validity for any test function ϕ, can be carried out for algebraic blocks of any nilpotency index, as it is proved in the next proposition. Let us denote by $\mathcal{E}(k)$ the subset of test functions ϕ such that the covariance $\mathrm{Cov}\big(\langle\xi,\phi\rangle,\ldots,\langle\xi^{(k-1)},\phi\rangle\big)$ is nonsingular. With an m-dimensional white noise, the covariance is a matrix with $(k+1)^2$ square $m\times m$ blocks, where the block (i,j) is $\mathrm{Re}\Big[(-1)^{\frac{|i-j|}{2}}\Big]\|\phi^{((i+j)/2)}\|^2$ times the identity I_m.

Proposition 4.3 *Let $(v_1,\ldots,v_{q_1})$ be the generalised process solution to the first block of the algebraic system (16) and r the greatest row index such that $\|\rho_r\|\neq 0$, and fix $\phi\in\mathcal{E}(q_1)$. Then $\langle(v_1,\ldots,v_r),\phi\rangle$ is a Gaussian absolutely continuous random vector and $\langle(v_{r+1},\ldots,v_{q_1}),\phi\rangle$ degenerates to a point.*

Proof: We can assume that $c = 0$, since the terms $\sum_{k=j}^{q_1}\langle c_k,\phi^{(k-j)}\rangle$ in (21) only contribute as additive constants. Then we can write

$$\begin{pmatrix} \langle v_1,\phi\rangle \\ \vdots \\ \langle v_{q_1},\phi\rangle \end{pmatrix} = \begin{pmatrix} \rho_1 & \rho_2 & \cdots & \rho_{q_1-1} & \rho_{q_1} \\ \rho_2 & \rho_3 & \cdots & \rho_{q_1} & 0 \\ \vdots & & & & \vdots \\ \rho_{q_1} & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} \langle\xi,\phi\rangle \\ \langle\xi,\dot\phi\rangle \\ \vdots \\ \langle\xi,\phi^{(q_1-1)}\rangle \end{pmatrix}\ .$$

If r is the greatest row index with $\|\rho_r\| \neq 0$, it is clear that the $q_1 \times mq_1$ matrix

$$\begin{pmatrix} \rho_1 & \rho_2 & \cdots & \rho_{q_1-1} & \rho_{q_1} \\ \rho_2 & \rho_3 & \cdots & \rho_{q_1} & 0 \\ \vdots & & & & \\ \rho_{q_1} & 0 & \cdots & 0 & 0 \end{pmatrix} ,$$

has rank r. The linear transformation given by this matrix is onto $\mathbb{R}^r \times \{0\}^{q_1-r}$. From this fact and the absolute continuity of the vector $(\langle \xi, \phi\rangle, \ldots, \langle \xi^{(q_1-1)}, \phi\rangle)$, it is immediate that the vector $(\langle v_1, \phi\rangle, \ldots, \langle v_r, \phi\rangle)$ is absolutely continuous, while $(\langle v_{r+1}, \phi\rangle, \ldots, \langle v_{q_1}, \phi\rangle)$ degenerates to a point. □

Let us now consider the solution x to the whole SDAE (12). We will state a sufficient condition for the absolute continuity of $\langle x, \phi\rangle$, $\phi \in \mathcal{D}$. The following standard result in linear algebra will be used (see e.g. Horn and Johnson [5], page 21).

Lemma 4.4 *Let the real matrix M read blockwise*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A \in \mathbb{R}^{d\times d}$, $B \in \mathbb{R}^{d\times q}$, $C \in \mathbb{R}^{q\times d}$, $D \in \mathbb{R}^{q\times q}$ and D is invertible. Then the $d \times d$ matrix

$$A - BD^{-1}C$$

is called the Schur complement *of D in M and it holds that*

$$\det M = \det D \cdot \det(A - BD^{-1}C) \ .$$

□

A natural application of this lemma is in solving a system of linear equations:

$$\left.\begin{aligned} Ax + By &= u \\ Cx + Dy &= v \end{aligned}\right\}$$

where $x, u \in \mathbb{R}^d$, $y, v \in \mathbb{R}^q$. We have

$$u = (A - BD^{-1}C)x + BD^{-1}v \tag{27}$$

and, if M is in addition invertible, the solution to the linear system is given by

$$\begin{aligned} x &= (A - BD^{-1}C)^{-1}(u - BD^{-1}v) \\ y &= D^{-1}(v - C(A - BD^{-1}C)^{-1}(u - BD^{-1})) \ . \end{aligned}$$

We now state and prove the main result of this section.

Theorem 4.5 *Assume (A, B) is a regular matrix pencil and that the $n \times m$ matrix Λ of equation (11) has full rank equal to n, and call r the nilpotency index of the SDAE (11). Then the law of the unique solution to the SDAE (11) at any test function $\phi \in \mathcal{E}(r)$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^n$.*

Corollary 4.6 *Under the assumptions of Theorem 4.5, if the nilpotency index is $r \leq 4$, then the law is absolutely continuous at every test function $\phi \in \mathcal{D} - \{0\}$.*

Proof of Theorem 4.5: It will be enough to prove that the random vector $\langle (u,v), \phi \rangle$, solution to (15) and (16), admits an absolutely continuous law at any test function $\phi \in \mathcal{E}$, since the solution to the original system is then obtained through the non-singular transformation Q.
We shall proceed in two steps: first we shall prove that $\langle v, \phi \rangle$ admits an absolutely continuous law, and then that the conditional law of $\langle u, \phi \rangle$, given $\langle v, \phi \rangle$, is also absolutely continuous, almost surely with respect to the law of $\langle v, \phi \rangle$. Note that the assumptions that Λ has full rank with $m \geq n$, implies that both submatrices of $P\Lambda$, S and R, have full rank equal to their respective number of rows.
Step 1: We can assume $c = 0$ in (16). By Proposition 4.3, the solution to any algebraic block is separately absolutely continuous. Assume now that there are exactly two blocks of dimensions q_1 and q_2, with $q_2 \leq q_1$, and $q_1 + q_2 = q$; the case with an arbitrary number of blocks does not pose additional difficulties.
As in Proposition 4.3, we have

$$\begin{pmatrix} \langle v_1, \phi \rangle \\ \vdots \\ \langle v_{q_1}, \phi \rangle \\ \langle v_{q_1+1}, \phi \rangle \\ \vdots \\ \langle v_{q_1+q_2}, \phi \rangle \end{pmatrix} = \begin{pmatrix} \rho_1 & \rho_2 & \cdots & \cdots & \rho_{q_1-1} & \rho_{q_1} \\ \rho_2 & \rho_3 & \cdots & \cdots & \rho_{q_1} & 0 \\ \vdots & & & & & \vdots \\ \rho_{q_1} & 0 & \cdots & 0 & 0 & 0 \\ \rho_{q_1+1} & \rho_{q_1+2} & \cdots & \cdots & \rho_{q_1+q_2} & 0 \\ \rho_{q_1+2} & \rho_{q_1+3} & \cdots & \rho_{q_1+q_2} & 0 & 0 \\ \vdots & & & & & \vdots \\ \rho_{q_1+q_2} & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \langle \xi, \phi \rangle \\ \langle \xi, \dot{\phi} \rangle \\ \vdots \\ \langle \xi, \phi^{(q_1-1)} \rangle \end{pmatrix} \tag{28}$$

Since the $(q_1+q_2) \times m$ matrix $R = (\rho_1, \dots, \rho_{q_1+q_2})^\top$ has, by the hypothesis on Λ, full rank equal to $q_1 + q_2$, the transformation defined by (28) is onto $\mathbb{R}^q$. From the absolute continuity of the vector $(\langle \xi, \phi \rangle, \dots, \langle \xi^{(q_1-1)}, \phi \rangle)$, we deduce that of $\langle v, \phi \rangle$.
Step 2: Since the $n \times m$ matrix $P\Lambda$ has full rank we can assume, reordering columns if necessary, that the submatrix made of its first n rows and columns is invertible, and that

$$P\Lambda = \begin{pmatrix} S \\ R \end{pmatrix} = \begin{pmatrix} A & B & E \\ C & D & F \end{pmatrix} ,$$

where $A \in \mathbb{R}^{d \times d}$, $B \in \mathbb{R}^{d \times q}$, $C \in \mathbb{R}^{q \times d}$, $D \in \mathbb{R}^{q \times q}$, $E \in \mathbb{R}^{d \times (m-n)}$, $F \in \mathbb{R}^{q \times (m-n)}$, with invertible D. Let us define

$$w := (v_1, \dots, v_q, v_1 + \dot{v}_2, \dots, v_{q_1-1} + \dot{v}_{q_1}, v_{q_1+1} + \dot{v}_{q_1+2}, \dots, v_{q_1+q_2-1} + \dot{v}_{q_1+q_2}, \xi_{n+1}, \dots, \xi_m) .$$

We can write then

$$\langle (w, \xi_1, \dots, \xi_d), \phi \rangle^\top = \begin{pmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \\ G_5 \end{pmatrix} \begin{pmatrix} \langle \xi, \phi \rangle \\ \langle \xi, \dot{\phi} \rangle \\ \vdots \\ \langle \xi, \phi^{(q_1-1)} \rangle) \end{pmatrix} , \tag{29}$$

where G_1 is the matrix in (28),

$$G_2 := \begin{pmatrix} \rho_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \rho_{q_1-1} & 0 & \cdots & 0 \end{pmatrix}, \quad G_3 := \begin{pmatrix} \rho_{q_1+1} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \rho_{q_1+q_2-1} & 0 & \cdots & 0 \end{pmatrix},$$

$$G_4 := \begin{pmatrix} e_{d+q+1} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ e_m & 0 & \cdots & 0 \end{pmatrix}, \quad G_5 := \begin{pmatrix} e_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ e_d & 0 & \cdots & 0 \end{pmatrix},$$

and $e_i \in \mathbb{R}^{1\times m}$, with $(e_i)_j = \delta_{ij}$.
By the invertibility of D and the fact that the rows ρ_{q_1} and $\rho_{q_1+q_2}$ have at least one element different from zero, it is easy to see that the matrix in (29) has itself full rank. Indeed, reordering its rows and columns, we can get a matrix with the first $q_1 + q_2 - 2$ rows given by G_1, without its q_1-th and $(q_1 + q_2)$-th rows. The lower part of this matrix has a block of zeros in the last $(q_1 - 1)m$ columns, while the first m columns can be reordered as a block lower triangular matrix, with diagonal blocks given by I_{m-q} and D.
Since (28) has full rank, the vector $\langle (w, \xi_1, \dots, \xi_d), \phi\rangle$ has itself an absolutely continuous Gaussian law.
Applying Lemma 4.4 and (27) to the system (15)-(16), with $\dot{u} + Ju$ in place of u, $N\dot{v} + v$ in place of v, (A, E) instead of A and (C, F) instead of C, we obtain

$$\begin{pmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_d \end{pmatrix} + J \begin{pmatrix} u_1 \\ \vdots \\ u_d \end{pmatrix} = (A - BD^{-1}C) \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_d \end{pmatrix} + BD^{-1} \begin{pmatrix} v_1 + \dot{v}_2 \\ v_2 + \dot{v}_3 \\ \vdots \\ v_{q_1} \\ v_{q_1+1} + \dot{v}_{q_1+2} \\ v_{q_1+2} + \dot{v}_{q_1+3} \\ \vdots \\ v_{q_1+q_2} \end{pmatrix} + (E - BD^{-1}F) \begin{pmatrix} \xi_{n+1} \\ \vdots \\ \xi_m \end{pmatrix}. \tag{30}$$

It is obvious that both Definition 3.1 and Theorem 3.2 continue to hold with any generalised process θ in place of the white noise ξ in the right-hand side. From Theorem 3.2 we have in particular that the solution u to the differential system (30) is a measurable function $G\colon (\mathcal{D}')^m \to (\mathcal{D}')^d$ of its right-hand side θ, that is, $u = G(\theta)$. Let

$$p : \mathcal{B}(\mathcal{D}') \times \mathcal{D}' \longrightarrow [0,1] \quad \text{and} \quad q : \mathcal{B}(\mathcal{D}') \times \mathcal{D}' \longrightarrow [0,1]$$

be conditional laws of u given w, and of θ given w, respectively. That means that

$$P\big(\{u \in B\} \cap \{w \in C\}\big) = \int_C p(B, w)\, \mu(dw)$$

and

$$P\big(\{\theta \in B\} \cap \{w \in C\}\big) = \int_C q(B, w)\, \mu(dw),$$

for any $B, C \in \mathcal{B}(\mathcal{D}')$, where μ is the law of w.

For every $x \in \mathcal{D}'$, let Z_x be a random distribution $Z_x \colon \Omega \to \mathcal{D}'$ with law $P\{Z_x \in B\} = q(B,x)$. Then

$$\begin{aligned}
&\int_C P\{G(Z_x) \in B\}\,\mu(dx) = \int_C P\{Z_x \in G^{-1}(B)\}\,\mu(dx) \\
&= \int_C q(G^{-1}(B),x)\,\mu(dx) = P\big(\{\theta \in G^{-1}(B)\} \cap \{x \in C\}\big) \\
&= P\big(\{G(\theta) \in B\} \cap \{x \in C\}\big) = P\big(\{u \in B\} \cap \{x \in C\}\big) \\
&= \int_C p(B,x)\,\mu(dx)
\end{aligned}$$

Therefore $P\{G(Z_x) \in B\} = p(B,x)$ almost surely with respect to the law of x, for all $B \in \mathcal{B}(\mathcal{D}')$. We have proved that if the right-hand side of the differential system has the law of θ conditioned to x, then its solution has the law of u conditioned to x. It remains to show that this conditional law is absolutely continuous, almost surely with respect to the law of x.
Now, for each x, we can take Z_x to be

$$Z_x = (A - BD^{-1}C)\eta_x + a_x$$

where a_x is a constant d-dimensional distribution, and $\langle \eta_x, \phi\rangle$ is, for each $\phi \in \mathcal{D}$, a Gaussian d-dimensional vector. This random vector is absolutely continuous: Indeed, its law is that of the d first components of the m-dimensional white noise $(\xi_1,\dots,\xi_m)$ conditioned to lie in an $(m-d)$-dimensional linear submanifold. Let $L_{x,\phi}$ be its covariance matrix. Then $\langle \eta_x, \phi\rangle = L_{x,\phi}^{1/2}\langle \zeta, \phi\rangle$, for some d-dimensional white noise $\zeta = (\zeta_1,\dots,\zeta_d)$.
Consider now the (ordinary) stochastic differential equation

$$\begin{pmatrix} \langle \dot{\underline{u}}_1, \phi\rangle \\ \vdots \\ \langle \dot{\underline{u}}_d, \phi\rangle \end{pmatrix} + J \begin{pmatrix} \langle \underline{u}_1, \phi\rangle \\ \vdots \\ \langle \underline{u}_d, \phi\rangle \end{pmatrix} = a_x + (A - BD^{-1}C)L_{x,\phi}^{1/2} \begin{pmatrix} \langle \zeta_1, \phi\rangle \\ \vdots \\ \langle \zeta_d, \phi\rangle \end{pmatrix}. \tag{31}$$

By hypothesis, the Schur complement $A - BD^{-1}C$ is non-singular, and therefore the matrix $(A - BD^{-1}C)L_{x,\phi}^{1/2}$ is itself non-singular. But in the situation of (31), it is well-know that the solution $\langle (\underline{u}_1,\dots,\underline{u}_d), \phi\rangle$ is a stochastic process with absolutely continuous law for any test function $\phi \not\equiv 0$.
We conclude that the law of $\langle u, \phi\rangle$ conditioned to $\langle w, \phi\rangle$, which coincides with the law of $\langle \underline{u}, \phi\rangle$, is absolutely continuous almost surely with respect to the law of w. This is sufficient to conclude that $\langle (u_1,\dots,u_d,v_1,\dots,v_q), \phi\rangle$ has an absolutely continuous law, which completes the proof. □

5 Example: An electrical circuit

In this last section we shall present an example of linear SDAE's arising from a problem of electrical circuit simulation.
An electrical circuit is a set of devices connected by wires. Each device has two or more connection *ports*. A wire connects two devices at specific ports. Between any two ports of a device there is a *flow* (current) and a *tension* (voltage drop). Flow and tension are supposed to be the same at both ends of a wire; thus wires are just physical media for putting together two ports and they play no other role.

The circuit topology can be conveniently represented by a network, i.e. a set of nodes and a set of directed arcs between nodes, in the following way: Each port is a node (taking into account that two ports connected by a wire collapse to the same node), and any two ports of a device are joined by an arc. Therefore, flow and tension will be two quantities circulating through the arcs of the network.

It is well known that a network can be univocally described by an incidence matrix $A = (a_{ij})$. If we have n nodes and m arcs, A is the $m \times n$ matrix defined by

$$a_{ij} = \begin{cases} +1, & \text{if arc } j \text{ has node } i \text{ as origin} \\ -1, & \text{if arc } j \text{ has node } i \text{ as destiny} \\ 0, & \text{in any other case.} \end{cases}$$

At every node i, a quantity d_i (positive, negative or null) of flow may be supplied from the outside. This quantity, added to the total flow through the arcs leaving the node, must equal the total flow arriving to the node. This conservation law leads to the system of equations $Ax = d$, where x_j, $j = 1, \dots, n$, is the flow through arc j.

A second conservation law relates to tensions and the cycles formed by the flows. A *cycle* is a set of arcs carrying nonzero flow when all external supplies are set to zero. The *cycle space* is thus $\ker A \subset \mathbb{R}^n$. Let B be a matrix whose columns form a basis of the cycle space, and let $c \in \mathbb{R}^n$ be the vector of externally supplied tensions to the cycles of the chosen basis. Then we must impose the equalities $B^\top u = c$, where u_j, $j = 1, \dots, n$, is the tension through arc j.

Once we have the topology described by a network, we can put into play the last element of the circuit modelling. Every device has a specific behaviour, which is described by an equation $\varphi(x, u, \dot{x}, \dot{u}) = 0$ involving in general flows, tensions, and their derivatives. The system $\Phi(x, u, \dot{x}, \dot{u}) = 0$ consisting of all of these equations is called the *network characteristic*. For instance, typical simple two-port (linear) devices are the *resistor*, the *inductor* and the *capacitor*, whose characteristic (noiseless) equations, which involve only their own arc j, are $u_j = Rx_j$, $u_j = L\dot{x}_j$, and $x_j = C\dot{u}_j$, respectively, for some constants R, L, C. Also, the *current source* (x_j constant) and the *voltage source* (u_j constant) are common devices.

Solving an electrical circuit therefore means finding the currents x and voltage drops u determined by the system

$$\begin{cases} Ax = d \\ B^\top u = c \\ \Phi(x, u, \dot{x}, \dot{u}) = 0 \end{cases}$$

Example 5.1 *Let us write down the equations corresponding to the circuit called* LL-cutset *(see [11], pag. 60), formed by two inductors and one resistor, which we assume submitted to random perturbations, independently for each device. This situation can be modelled, following the standard procedure described above, by the stochastic system*

$$\begin{cases} x_1 = -x_2 = x_3 \\ u_1 - u_2 + u_3 = 0 \\ u_1 = L_1\dot{x}_1 + \tau_1\xi_1 \\ u_2 = L_2\dot{x}_2 + \tau_2\xi_2 \\ u_3 = Rx_3 + \tau_3\xi_3 \end{cases} \tag{32}$$

where ξ_1, ξ_2, ξ_3 are independent white noises, and τ_1, τ_2, τ_3 are non-zero constants which measure the magnitude of the perturbations. With a slight obvious simplification, we obtain from

(32) the following linear SDAE:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & L_1 & 0 \\ 0 & 0 & 0 & L_2 \end{pmatrix} \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} + \begin{pmatrix} R^{-1} & -R^{-1} & 1 & 0 \\ -R^{-1} & R^{-1} & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\tau_3 R^{-1} \\ 0 & 0 & \tau_3 R^{-1} \\ -\tau_1 & 0 & 0 \\ 0 & -\tau_2 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} , \tag{33}$$

Let us now reduce the equation to KCF. To simplify the exposition, we shall fix to 1 the values of τ_i, R and L_i. (A physically meaningful magnitude for R and L_i would be of order 10^{-6} for the first and of order 10^4 for the latter. Nevertheless the structure of the problem does not change with different constants.) The matrices P and Q, providing the desired reduction (see Lemma 2.3), are

$$P = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 1 & -1 \\ 0 & -1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad , \quad Q = \begin{pmatrix} -\frac{1}{4} & -\frac{1}{2} & -\frac{3}{4} & -1 \\ \frac{1}{4} & -\frac{1}{2} & -\frac{1}{4} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix} .$$

Indeed, multiplying (33) by P from the left and defining $y = Q^{-1}x$, we arrive to

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \dot{y}(t) + \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} y(t) = \begin{pmatrix} -\tau_1 & \tau_2 & -\tau_3 \\ -\tau_1 & -\tau_2 & -\tau_3 \\ 0 & 0 & 0 \\ 0 & 0 & \tau_3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} , \tag{34}$$

We see that the matrix N of Section 3 has here two blocks: A single zero in the last position ($\dot{y}_4$) and a 2-nilpotent block affecting $\dot{y}_2$ and $\dot{y}_3$. We have therefore an index 2 SDAE. From Propositions 4.3 and Theorem 4.5, we can already say that, when applied to any test function $\phi \neq 0$, the variables y_4, y_2 and y_1, as well as the vectors (y_1, y_2) and (y_1, y_4), will be absolutely continuous, whereas y_3 degenerates to a point.
In fact, in this case, we can of course solve completely the system: The differential part is the one-dimensional classical SDE

$$\dot{y}_1 + \frac{1}{2} y_1 = -\tau_1 \xi_1 + \tau_2 \xi_2 - \tau_3 \xi_3 , \tag{35}$$

and the algebraic part reads simply

$$\begin{cases} \dot{y}_3 + y_2 = -\tau_1 \xi_1 - \tau_2 \xi_2 - \tau_3 \xi_3 \\ y_3 = 0 \\ y_4 = \tau_3 \xi_3 . \end{cases} \tag{36}$$

The solution to (34) can thus be written as

$$\begin{aligned} y_1(t) &= e^{-(t-t_0)/2} \Big[y(t_0) + \int_{t_0}^{t} e^{-(s-t_0)/2} (-\tau_1 dW_1 + \tau_2 dW_2 - \tau_3 dW_3)(s) \Big] \\ y_2 &= -\tau_1 \xi_1 - \tau_2 \xi_2 - \tau_3 \xi_3 \\ y_3 &= 0 \\ y_4 &= \tau_3 \xi_3 , \end{aligned}$$

where W_1*,* W_2*,* W_3 *are independent Wiener processes whose generalised derivatives are* ξ_1*,* ξ_2 *and* ξ_3*. Multiplying by the matrix* Q *we finally obtain the value of the original variables:*

$$\begin{aligned} x_1(t) = -x_2(t) &= -\tfrac{1}{2} y_1(t) \\ u_1 &= -\tfrac{1}{4} y_1 - \tfrac{1}{2} y_2 - \tfrac{3}{4} y_4 \\ u_2 &= \tfrac{1}{4} y_1 - \tfrac{1}{2} y_2 \ , \end{aligned}$$

with $x_1(t_0) = -\frac{1}{4} y_1(t_0)$ *a given intensity at time* t_0*.*
It is clear that the current intensities, which have almost surely continuous paths, are much more regular than the voltage drops, which are only random distributions.

References

[1] Donald A. Dawson. Generalized stochastic integrals and equations. *Trans. Amer. Math. Soc.*, 147:473–506, 1970. MR0261719

[2] Xavier Fernique. Processus linéaires, processus généralisés. *Ann. Inst. Fourier (Grenoble)*, 17(fasc. 1):1–92, 1967. MR0221576

[3] I. M. Gel′fand and N. Ya. Vilenkin. *Generalized functions. Vol. 4.* Academic Press, New York, 1964.

[4] E. Griepentrog and R. März. Basic properties of some differential-algebraic equations. *Z. Anal. Anwendungen*, 8(1):25–41, 1989. MR0997289

[5] Roger A. Horn and Charles R. Johnson. *Matrix Analysis.* Cambridge University Press, Cambridge, 1990. MR1084815

[6] Jean Jacod and Philip Protter. *Probability essentials.* Universitext. Springer-Verlag, Berlin, second edition, 2003. MR1956867

[7] Hui Hsiung Kuo. *Gaussian measures in Banach spaces.* Springer-Verlag, Berlin, 1975. MR0461643

[8] David Nualart. *The Malliavin calculus and related topics.* Probability and its Applications (New York). Springer-Verlag, Berlin, second edition, 2006. MR2200233

[9] Patrick J. Rabier and Werner C. Rheinboldt. Classical and generalized solutions of time-dependent linear differential-algebraic equations. *Linear Algebra Appl.*, 245:259–293, 1996. MR1404179

[10] Patrick J. Rabier and Werner C. Rheinboldt. Theoretical and numerical analysis of differential-algebraic equations. In *Handbook of numerical analysis, Vol. VIII*, pages 183–540. North-Holland, Amsterdam, 2002. MR1893418

[11] O. Schein. *Stochastic differential-algebraic equations in circuit simulation.* PhD thesis, Technische Universität Darmstadt, 1999.

[12] O. Schein and G. Denk. Numerical solution of stochastic differential-algebraic equations with applications to transient noise simulation of microelectronic circuits. *J. Comput. Appl. Math.*, 100(1):77–92, 1998. MR1658742

[13] Laurent Schwartz. *Théorie des distributions.* Publications de l'Institut de Mathématique de l'Université de Strasbourg, No. IX-X. Nouvelle édition, entiérement corrigée, refondue et augmentée. Hermann, Paris, 1966. MR0209834

[14] Renate Winkler. Stochastic differential algebraic equations of index 1 and applications in circuit simulation. *J. Comput. Appl. Math.*, 157(2):477–505, 2003. MR1998346