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REFLECTED BROWNIAN MOTION IN A WEDGE: SUM-OF-EXPONENTIAL STATIONARY DENSITIES

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Abstract

We give necessary and sufficient conditions for the stationary density of semimartingale reflected Brownian motion in a wedge to be written as a finite sum of terms of exponential product form. Relying on geometric ideas reminiscent of the reflection principle, we give an explicit formula for the density in such cases.

1 Introduction

It is well-known that Brownian motion on the positive half-line with negative drift and reflection at zero has a stationary density which is exponential. One derivation of this fact uses time reversal to relate the distribution of the corresponding transitory process at time t in the driftless case to the distribution of the maximum of a standard Brownian motion over $[0,t]$, which can be found using the reflection principle. A Cameron-Martin-Girsanov change of measure then introduces the drift, and letting $t \to \infty$ gives the required stationary distribution. In the d-dimensional setting, analogous arguments show that in certain situations, semimartingale reflected Brownian motion (SRBM) in a polyhedral cone has a stationary density which can be written as a finite sum of terms of exponential product form—that is, terms of the form $x \mapsto ae^{-\langle \lambda, x\rangle}$ for some $a \in \mathbb{R}, \lambda \in \mathbb{R}^d$. We call such a density a *sum of exponentials*. The aim of this paper is to give necessary and sufficient conditions under which the stationary density of SRBM in a two-dimensional wedge can be written as a sum of exponentials.

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In order to motivate our study, we describe the above argument in a multidimensional setting in Section 2 below. The invariant measure for SRBM with special pushing directions at the boundary of a polyhedral cone then relates to exit probabilities for the corresponding free Brownian motion. For some cones, in analogy with the one-dimensional case, these exit probabilities can be obtained explicitly using the reflection principle. If the reflected process is positive recurrent, then upon differentiating we obtain a stationary density which can be written as a sum of exponentials. Since the latter can thus be viewed as a natural multidimensional extension of the exponential stationary density of one-dimensional SRBM, this raises the question under what circumstances the stationary distribution of a given SRBM in a polyhedral cone has a sum-of-exponential density. This paper shows that, for SRBM in a two-dimensional polyhedral cone, sum-of-exponential stationary densities arise well beyond those cases to which the above arguments can be applied.

The stationary distribution of multidimensional SRBM in a polyhedral cone is only known in relatively few cases. It has a density consisting of a single exponential term under a skew-symmetry condition on the pushing directions on the faces [17, 25]. We also mention an apparently isolated example due to Harrison [13] and work of Foschini [11]; we discuss the latter in more detail below. In the absence of further explicit results, numerical techniques have been developed [6] and logarithmic tail asymptotics have been investigated [3, 9, 14].

Before describing our results in more detail, we introduce some notation which is summarised in Figure 1. Define the wedge as

$$S = \{x \in \mathbb{R}^2 : 0 \le \arg(x) \le \xi\}.$$

Throughout this paper we write $w_\theta = (\cos\theta, \sin\theta)$ for $\theta \in \mathbb{R}$. Given some $0 < \delta, \epsilon, \xi < \pi$, we set $v_1 = \|v_1\| w_\delta$, $v_2 = \|v_2\| w_{\xi-\epsilon}$. Let F_1 and F_2 be the two faces of the wedge. As is customary, we normalise v_1 and v_2 such that $\langle v_1, n_1\rangle = 1$ and $\langle v_2, n_2\rangle = 1$, where n_1 and n_2 are the unit normal vectors on F_1 and F_2, respectively. Let some vector $\mu \in \mathbb{R}^2$ also be given, and write $\theta_\mu = \arg(\mu) \in (-\pi, \pi]$. A key role in this paper is played by α, which is introduced by Varadhan and Williams [22] as

$$\alpha = \frac{\delta + \epsilon - \pi}{\xi}.$$

Our results concern the case $\alpha < 1$, and it is well-known (e.g., [22, 24]) that under this condition there exists a continuous semimartingale Markov process with properies that can intuitively be summarised as follows (we do not give the mathematically rigorous definition here, as it can be found in, e.g., [3, 10, 24]). The process behaves like a standard Brownian motion with drift $-\mu$ in the interior of the wedge S, at the boundary of the wedge it is pushed in some specified direction, and the time it spends at the vertex of the wedge has Lebesgue measure zero. The pushing directions are constant along each of the faces, and are given by v_i for the i-th face. We call this process SRBM in a wedge and seek its stationary distribution, which is absolutely continuous with respect to Lebesgue measure [5, 16].

We prove that the stationary density of this process is a sum of exponentials if and only if $\alpha = -\ell$ for some nonnegative integer ℓ. Moreover, it follows from our results that the number of exponential terms is $2\ell + 1$ when $\alpha = -\ell$. Note that this condition reduces to $\delta = \pi - \epsilon$ if $\ell = 0$, i.e., the skew-symmetry condition for product forms [17]. In fact, we show that the stationary density can be written as an $(\ell+1) \times (\ell+1)$ determinant with a special structure. A corollary is that the density behaves near the origin as in the driftless case studied earlier by Williams [23].

We next discuss some work related to our additive generalisations of the product form exponential stationary density. To our knowledge, Foschini's study of symmetric SRBM in the wedge [11] was the first to consider sum-of-exponential stationary densities. Our paper is a continuation and extension of Foschini's work, in the sense that we couple key ideas from his algorithm with geometric

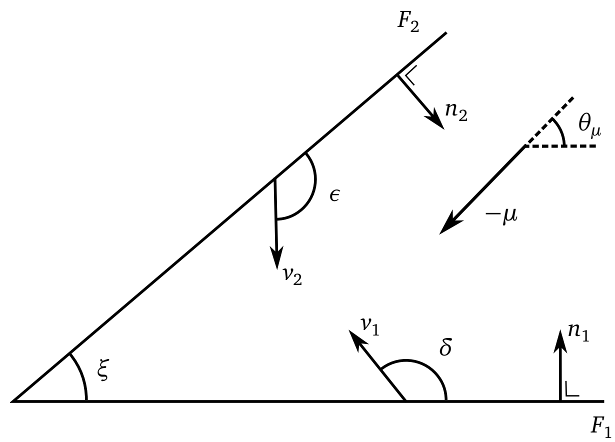

Figure 1: The wedge S.

arguments suggested by the reflection principle, and consequently obtain an explicit formula for the stationary density. A discrete-state space version of Foschini's algorithm, the *compensation method*, has been successfully applied to a variety of queueing problems [1, 2]. Another body of work loosely related to the present paper studies (driftless) two-dimensional SRBM with $\alpha = -1$ [8, 19], motivated by a connection with Schramm-Loewner evolutions.

This paper is organised as follows. Section 2 gives some background and discusses the special sum-of-exponential stationary densities mentioned above. Section 3 contains our main result and gives a geometric construction of the stationary density. In Sections 4–6, we prove the main result.

2 Survival probabilities, reflection groups, and Weyl chambers

This section describes a special class of sum-of-exponential stationary densities which can be obtained directly using time reversal. These densities motivate our main result, since they typify the general structure of a sum-of-exponential stationary density.

Consider an SRBM as in Figure 1, with drift $-\mu$ and $\delta = \epsilon = \xi$. Suppose that $\mu \in S^o$ (the interior of S) to ensure positive recurrence, and write Π for the stationary measure of the SRBM. Time reversal gives

$$\Pi(\{y \in S : \langle y, n_1 \rangle \leq \langle x, n_1 \rangle, \langle y, n_2 \rangle \leq \langle x, n_2 \rangle\}) = P_{-x}(T = \infty), \tag{1}$$

where P_y is the law of the corresponding free Brownian motion B starting in y and T is the first exit time from $-S$, i.e., $T = \inf\{t \geq 0 : B(t) \notin -S\}$. To see this, we use the wedge $-S$ to define a partial order $<$ on $\mathbb{R}^2$ (see for example [20]) and then if $\tilde{B_s} = B_{t-s} - B_t$,

$$\begin{aligned} \sup\{B_s : 0 \leq s \leq t\} - B_t &< x \text{ if and only if} && (2)\\ \sup\{-x + \tilde{B}_s : 0 \leq s \leq t\} &< 0, && (3) \end{aligned}$$

where sup denotes the supremum with respect to the partial order. On the left hand side of (3) is the supremum over $[0, t]$ of the free Brownian motion starting in $-x$, while on the left hand side of (2) we have an SRBM in S as defined above (starting in 0; for details see [20]). Applying Wiener measure to (2)–(3) and letting $t \to \infty$ gives (1).

Biane *et al.* [4] have recently shown that if the wedge angle is of the form $\xi = \pi/m$ for some integer $m \geq 2$ then

$$P_{-x}(T=\infty) = \sum_{w \in G} \mathrm{sgn}\,(w) e^{-\langle \mu, (I-w)x \rangle} \tag{4}$$

for any $x \in S^o$ and any $\mu \in S^o$, where the sum is taken over the finite reflection group G with associated signature function $\mathrm{sgn}(\cdot)$, as detailed below. We remark that this formula gives the probability that a standard Brownian motion with drift never exits a so-called Weyl chamber—in particular, it is not restricted to a two-dimensional setting.

We next describe the index of summation in (4), i.e., the elements of the reflection group G. Throughout, we represent any element of $\mathbb{R}^2$ as a column vector. Write R_θ for the reflection matrix across the line with argument θ, i.e.,

$$R_\theta = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

and ρ_θ for the rotation matrix over an angle θ, i.e.,

$$\rho_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The group G consists of the $2m$ matrices $I, R_\xi, \rho_{2\xi}, \rho_{2\xi}R_\xi, \ldots, \rho_{2(m-1)\xi}, \rho_{2(m-1)\xi}R_\xi$. The signature of the matrices $\rho_{2k\xi}R_\xi$ is -1, while the signature of the others is $+1$.

On combining (1) and (4), we find that the stationary *density* of SRBM with $\xi = \delta = \epsilon = \pi/m$ is proportional to

$$\sum_{w \in G} \mathrm{sgn}\,(w) \langle \mu, (I-w)v_2 \rangle \langle \mu, (I-w)v_1 \rangle e^{-\langle \mu, (I-w)x \rangle}. \tag{5}$$

This is a sum of $2m-3$ nonzero terms, since the prefactor vanishes when $w = I$, $w = R_\xi$, and $w = \rho_{2(m-1)\xi}R_\xi$. It is the aim of this paper to put the explicit formula (5) into the context of more general SRBMs. Specifically, we show that every sum-of-exponential stationary density has a representation reminiscent of (5).

3 Main results

This section presents our main result on sum-of-exponential stationary densities, and explains its geometric interpretation. Our result shows that sum-of-exponential stationary densities can be written as determinants with a special structure. The proofs are deferred to Sections 4–6.

The matrices

$$\mathrm{Rot}_k = \rho_{2\delta+2k\xi} \quad \text{and} \quad \mathrm{Ref}_k = \rho_{2\delta+2(k-1)\xi}R_\xi \tag{6}$$

for $k \geq 0$ play an important role in our main theorem. Write

$$\Theta_\ell = \{\theta \in (\xi-\epsilon, \delta) : \sin(\theta - 2\delta - k\xi) \neq 0 \text{ for } k = 0, \ldots, 2\ell\}.$$

We know from Hobson and Rogers [18] and from Dupuis and Williams [10] that there exists a unique stationary distribution of the SRBM if $\xi - \epsilon < \theta_\mu < \delta$. Let p^μ be its density with respect to Lebesgue measure, and write e_1 for the vector $(1,0)$. For integers j, we also define the function $\pi_j^\mu : S \to \mathbb{R}$ through

$$\pi_j^\mu(x) = \frac{\langle \mu, (I-\mathrm{Rot}_j)v_1 \rangle e^{-\langle \mu, (I-\mathrm{Rot}_j)x \rangle} - \langle \mu, (I-\mathrm{Ref}_j)v_1 \rangle e^{-\langle \mu, (I-\mathrm{Ref}_j)x \rangle}}{\langle \mu, (\mathrm{Ref}_j - \mathrm{Rot}_j)v_1 \rangle}.$$

Observe that both p^μ and π_j^μ depend on δ, ϵ and ξ, which is suppressed in the notation.

Theorem 1. *The following are equivalent:*

(i) $\alpha = -\ell$ *for some integer* $\ell \geq 0$*;*

(ii) $\alpha = -\ell$ *for some integer* $\ell \geq 0$ *and for all* μ *with* $\theta_\mu \in \Theta_\ell$*, the functions* p^μ *and* π^μ *are equal up to a multiplicative constant, where for* $x \in S$*,*

$$\pi^\mu(x) = \begin{vmatrix} \pi_0^\mu(x) & \pi_1^\mu(x) & \cdots & \pi_\ell^\mu(x) \\ \langle \mu, \mathrm{Rot}_0 e_1\rangle^{\ell-1} & \langle \mu, \mathrm{Rot}_1 e_1\rangle^{\ell-1} & \cdots & \langle \mu, \mathrm{Rot}_\ell e_1\rangle^{\ell-1} \\ \vdots & \vdots & & \vdots \\ \langle \mu, \mathrm{Rot}_0 e_1\rangle^{2} & \langle \mu, \mathrm{Rot}_1 e_1\rangle^{2} & \cdots & \langle \mu, \mathrm{Rot}_\ell e_1\rangle^{2} \\ \langle \mu, \mathrm{Rot}_0 e_1\rangle & \langle \mu, \mathrm{Rot}_1 e_1\rangle & \cdots & \langle \mu, \mathrm{Rot}_\ell e_1\rangle \\ 1 & 1 & \cdots & 1 \end{vmatrix}; \tag{7}$$

(iii) $\alpha < 1$ *and for some* μ *with* $\xi - \epsilon < \theta_\mu < \delta$*, there exist* $K < \infty$*, coefficients* $a_1, \dots, a_K$*, and vectors* $d_1, \dots, d_K$ *such that* p^μ *admits the representation*

$$p^\mu(x) = \sum_{i=1}^{K} a_i e^{-\langle d_i, x\rangle}.$$

In part (ii) of this theorem, the condition $\theta_\mu \in \Theta_\ell$ prevents a certain degeneracy which is fundamental to the problem. Indeed, it guarantees the linear independence of the exponential functions in π^μ. We remark that the determinant in (ii) has a different form than the ones recently studied in connection with transition probabilities for certain Markov chains [7].

A straightforward calculation using Vandermonde matrices shows that the function π^μ defined by (7) may be written as

$$\pi^\mu(x) = \sum_{k=0}^{\ell} c_k \left[\langle \mu, (I - \mathrm{Rot}_k) v_1 \rangle e^{-\langle \mu, (I - \mathrm{Rot}_k) x\rangle} - \langle \mu, (I - \mathrm{Ref}_k) v_1 \rangle e^{-\langle \mu, (I - \mathrm{Ref}_k) x\rangle} \right], \tag{8}$$

where

$$c_k = (-1)^k \frac{\prod_{0 \leq i < j \leq \ell; i,j \neq k} \langle \mu, (\mathrm{Rot}_i - \mathrm{Rot}_j) e_1 \rangle}{\langle \mu, (\mathrm{Ref}_k - \mathrm{Rot}_k) v_1 \rangle}. \tag{9}$$

This representation for π^μ is used later in the proof of Theorem 1; note that each exponential term is characterised by a rotation or a reflection matrix, $\mathrm{Rot}_0, \mathrm{Ref}_1, \mathrm{Rot}_1, \dots, \mathrm{Ref}_\ell, \mathrm{Rot}_\ell$.

Although there is not necessarily an underlying reflection group in the general setting of Theorem 1, the rotation and reflection matrices in Theorem 1 suggest a connection with the reflection-group framework. We illustrate this for $\alpha = -2$ (hence $\ell = 2$) in the leftmost diagram of Figure 2, where we depict the five points $\mathrm{Rot}_0 x, \mathrm{Ref}_1 x, \mathrm{Rot}_1 x, \mathrm{Ref}_2 x, \mathrm{Rot}_2 x$ for an arbitrarily chosen $x \in S$. By construction, $\mathrm{Rot}_0 x$ lies in the wedge $\rho_{2\delta} S$, which we call the 'initial wedge'. The other points $\mathrm{Ref}_0 x, \mathrm{Rot}_1 x, \dots$ are constructed by successive reflections, reminiscent of the orbit of $\mathrm{Rot}_0 x$ under the action of a reflection group. In particular

- Each point lies in one of the wedges constructed by rotating the initial wedge anticlockwise over multiples of the wedge angle ξ.

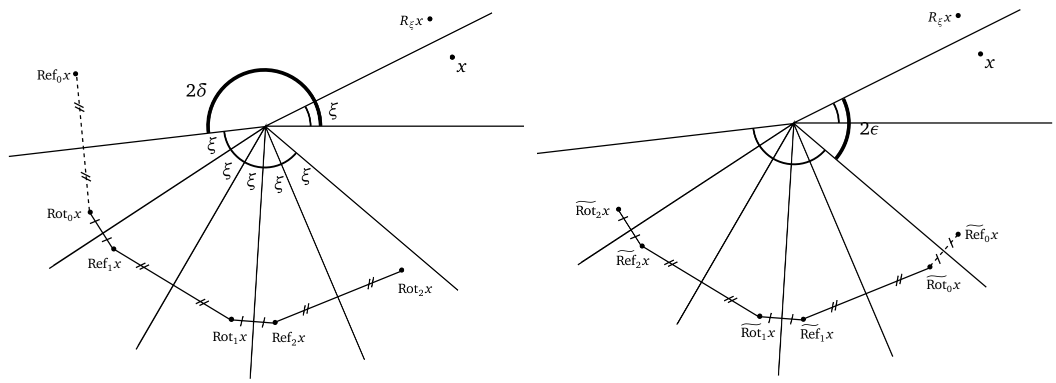

Figure 2: An anticlockwise construction of π^μ (left) and a clockwise construction of $\tilde{\pi}^\mu$ (right).

- Two points lying in adjacent wedges—that is, wedges which share a common boundary line—are reflections of each other in that line.

Note that $\mathrm{Ref}_0 x$, although not contributing to π^μ since $(I - \mathrm{Ref}_0)v_1 = 0$, is also obtained by reflection from $\mathrm{Rot}_0 x$. We indicate this by a dashed line in Figure 2.
If $\alpha = -\ell$ then the last point lies in the wedge $\rho_{2\delta+2\ell\xi}S$, which is the same as $\rho_{-2\epsilon}S$. On comparing this with the initial wedge $\rho_{2\delta}S$, it transpires that the *last* wedge in the anticlockwise construction given above is the *first* wedge in the following clockwise construction. For $k \geq 0$ we introduce the matrices

$$\widetilde{\mathrm{Rot}}_k = \rho_{-2k\xi-2\epsilon} \quad \text{and} \quad \widetilde{\mathrm{Ref}}_k = \rho_{-2(k-1)\xi-2\epsilon}R_0.$$

The rightmost diagram of Figure 2 illustrates a clockwise construction for the stationary density, starting with the wedge $\rho_{-2\epsilon}S$ and labelling the points using the matrices $\widetilde{\mathrm{Rot}}_k$ and $\widetilde{\mathrm{Ref}}_k$. The problem is exactly the same as in the leftmost diagram, so by uniqueness the corresponding sum-of-exponential densities must agree on S. We use this observation in the proof of Theorem 1.

We close this section by stating some properties of the function π^μ defined in (7), which play an important role in our proof of Theorem 1. These properties are proved in Section 4. We start with a result for the coefficients $\{c_k\}$ defined in (9).

Lemma 1. *Let $-\alpha = \ell \in \{1, 2, \ldots\}$ and $\theta_\mu \in \Theta_\ell$. The coefficients c_k defined by* (9) *satisfy, for $1 \leq k \leq \ell$,*

$$c_k \langle \mu, (I - \mathrm{Ref}_k)v_1 \rangle \langle \mu, (I - \mathrm{Rot}_{k-1})v_2 \rangle = c_{k-1} \langle \mu, (I - \mathrm{Ref}_k)v_2 \rangle \langle \mu, (I - \mathrm{Rot}_{k-1})v_1 \rangle. \qquad (10)$$

Using this lemma, it is readily checked that (5) is recovered from Theorem 1 by setting $\delta = \epsilon = \xi = \pi/m$. Indeed, we then have $\langle \mu, (I - \mathrm{Rot}_{k-1})v_1 \rangle = \langle \mu, (I - \mathrm{Ref}_k)v_1 \rangle$, so that (10) reduces to

$$c_k \langle \mu, (I - \mathrm{Rot}_{k-1})v_2 \rangle = c_{k-1} \langle \mu, (I - \mathrm{Ref}_k)v_2 \rangle.$$

In this special case we may therefore set

$$c_k = \langle \mu, (I - \mathrm{Rot}_k)v_2 \rangle = \langle \mu, (I - \mathrm{Ref}_k)v_2 \rangle,$$

and we obtain (5).
The limiting behaviour of π^μ given in the following proposition should be compared with the invariant measure found by Williams [23] for reflected Brownian motion *without* drift. We abbreviate $\lim_{r\to 0} f(r)/g(r) = 1$ by $f(r) \sim g(r)$ as $r \to 0$.

Proposition 1. *If $-\alpha = \ell \in \{1, 2, \ldots\}$, then for any $\theta \in [0, \xi]$ and any μ with $\theta_\mu \in \Theta_\ell$, we have $\pi^\mu(rw_\theta) \sim C^\mu r^\ell \sin(\ell\theta + \delta)$ as $r \to 0$, where C^μ is a finite nonzero constant independent of r and θ.*

Our next result is that π^μ defined by (7) does not change sign on S. Note that this resolves Conjecture 1 in Dai and Harrison [6] for the special class of SRBMs studied in this paper.

Proposition 2. *The function π^μ does not change sign on S.*

4 Properties of π^μ

In this section, we prove the properties of π^μ claimed in Section 3. The proof of the main result, Theorem 1, is deferred to Sections 5 and 6.

4.1 Proof of Lemma 1

We first divide (10) by $\sin(\theta_\mu - \delta - k\xi)\sin(\delta + (k-1)\xi)$, which is nonzero as a consequence of the assumption on μ in conjunction with the identity $\delta + \epsilon = \pi - \ell\xi$. Again using this identity, we find after some elementary trigonometry that (10) is equivalent to, with $\omega_k = \theta_\mu - 2\delta - k\xi$,

$$c_k \sin(k\xi)\sin(\omega_{\ell+k}) = -c_{k-1}\sin((\ell + 1 - k)\xi)\sin(\omega_{k-1}).$$

To show that this holds for the c_k defined in (9), we observe that

$$\langle \mu, (\mathrm{Ref}_k - \mathrm{Rot}_k)v_1\rangle \sin(\omega_{2(k-1)}) = \langle \mu, (\mathrm{Ref}_{k-1} - \mathrm{Rot}_{k-1})v_1\rangle \sin(\omega_{2k})$$

and that

$$\langle \mu, (\mathrm{Rot}_i - \mathrm{Rot}_j)e_1\rangle = -2\sin((j-i)\xi)\sin(\omega_{i+j}).$$

After some algebra we also find that

$$\begin{aligned} &\sin(k\xi)\sin(\omega_{\ell+k})\sin(\omega_{2(k-1)}) \prod_{0\le i<j\le\ell;\, i,j\ne k} \langle \mu, (\mathrm{Rot}_i - \mathrm{Rot}_j)e_1\rangle \\ =\quad &\sin((\ell - k + 1)\xi)\sin(\omega_{k-1})\sin(\omega_{2k}) \prod_{0\le i<j\le\ell;\, i,j\ne k-1} \langle \mu, (\mathrm{Rot}_i - \mathrm{Rot}_j)e_1\rangle, \end{aligned}$$

and the claim follows.

4.2 Proof of Proposition 1

For simplicity we suppose that $\|\mu\| = 1$. We first investigate the behaviour near zero of $\pi^\mu(x)$, for which we rewrite $e^{\langle\mu,x\rangle}\pi^\mu(x)$ using the determinantal representation (7). A key ingredient is the identity $e^{r\cos(\eta)} = I_0(r) + 2\sum_{n=1}^\infty \cos(n\eta)I_n(r)$, where I_n is the modified Bessel function of the first kind. Using this identity, after absorbing $e^{\langle\mu,x\rangle}$ into the first row, we rewrite the elements on this row as

$$\begin{aligned}
e^{\langle\mu,x\rangle}\pi_j^\mu(x) &= e^{\|x\|\langle\mu,\mathrm{Rot}_j w_\theta\rangle} - \langle\mu,(I-\mathrm{Ref}_j)v_1\rangle \frac{e^{\|x\|\langle\mu,\mathrm{Rot}_j w_\theta\rangle} - e^{\|x\|\langle\mu,\mathrm{Ref}_j w_\theta\rangle}}{\langle\mu,(\mathrm{Rot}_j-\mathrm{Ref}_j)v_1\rangle} \qquad (11)\\
&= I_0(\|x\|) + 2\sum_{n=1}^\infty T_n(\cos(\omega_{2j}-\theta))I_n(\|x\|)\\
&\quad - 2\frac{\langle\mu,(I-\mathrm{Ref}_j)v_1\rangle}{\langle\mu,(\mathrm{Rot}_j-\mathrm{Ref}_j)v_1\rangle}\sum_{n=1}^\infty \Big[T_n(\cos(\omega_{2j}-\theta)) - T_n(\cos(\omega_{2j}+\theta))\Big] I_n(\|x\|)\\
&= I_0(\|x\|) + 2\sum_{n=1}^\infty T_n(\cos(\omega_{2j}-\theta))I_n(\|x\|)\\
&\quad - 2\langle\mu,(I-\mathrm{Ref}_j)v_1\rangle \sum_{n=1}^\infty \sin(n\theta)U_{n-1}(\cos(\omega_{2j}))I_n(\|x\|),
\end{aligned}$$

where we again set $\omega_k = \theta_\mu - 2\delta - k\xi$, and T_n and U_n are the Chebyshev polynomials of the first and second kind, respectively.
In conjunction with some trigonometry, the above reasoning shows that

$$e^{\langle\mu,x\rangle}\pi_j^\mu(x) = I_0(\|x\|) + \frac{2}{\sin(\delta)}\sum_{n=1}^\infty h_{j,n}(\theta)I_n(\|x\|),$$

where $h_{j,n}(\theta)$ is defined as

$$\frac{1}{2}\sin(n\theta+\delta)U_n(\cos(\omega_{2j})) - \langle\mu, v_1/\|v_1\|\rangle\sin(n\theta)U_{n-1}(\cos(\omega_{2j})) + \frac{1}{2}\sin(n\theta-\delta)U_{n-2}(\cos(\omega_{2j})).$$

(We use the convention $U_{-1}(x)=0$.) Therefore, $e^{\langle\mu,x\rangle}\pi^\mu(x)$ can be expanded in terms of modified Bessel functions of the first kind, and for $n\geq 1$ the coefficient in front of $I_n(\|x\|)$ is proportional to

$$\begin{vmatrix}
h_{0,n}(\theta) & h_{1,n}(\theta) & h_{2,n}(\theta) & \cdots & h_{\ell,n}(\theta)\\
U_{\ell-1}(\cos(\omega_0)) & U_{\ell-1}(\cos(\omega_2)) & U_{\ell-1}(\cos(\omega_4)) & \cdots & U_{\ell-1}(\cos(\omega_{2\ell}))\\
\vdots & \vdots & & \vdots & \\
U_0(\cos(\omega_0)) & U_0(\cos(\omega_2)) & U_0(\cos(\omega_4)) & \cdots & U_0(\cos(\omega_{2\ell}))
\end{vmatrix}. \qquad (12)$$

To see how this follows from (7), note that we may apply elementary determinantal operations to replace a row with elements $\cos(\omega_{2j})^m$ by $U_m(\cos(\omega_{2j}))$. The term $I_0(\|x\|)$ is not present in the expansion in view of the last row in (12) with ones.
The condition $\theta_\mu\in\Theta_\ell$ guarantees that none of the $\cos(\omega_{2j})$ are equal, and we conclude that the coefficient of $I_n(\|x\|)$ vanishes for $n<\ell$ and that it is proportional to $\sin(\ell\theta+\delta)$ for $n=\ell$. Since $I_\ell(r)\sim Cr^\ell$ for some constant $C\neq 0$ as $r\to 0$, this yields $\pi^\mu(rw_\theta)\sim C^\mu r^\ell\sin(\ell\theta+\delta)$.

4.3 Proof of Proposition 2

We rely on the following auxiliary lemma, whose proof is inspired by elementary symmetric-function theory. Alternatively, as communicated to us by Sean Meyn, Proposition 2 can be proved using a continuous-space analogue of Theorem 1 of Foster [12]; this can be derived using the general theory of Markov processes.

Lemma 2. *Let $\ell \in \{1,2,\dots\}$ and $\zeta \in \mathbb{R}^{\ell+1}$. For any $y > 0$, the sign of the determinant*

$$\begin{vmatrix} e^{\zeta_0 y} & e^{\zeta_1 y} & \cdots & e^{\zeta_\ell y} \\ \zeta_0^{\ell-1} & \zeta_1^{\ell-1} & \cdots & \zeta_\ell^{\ell-1} \\ \vdots & \vdots & & \vdots \\ \zeta_0 & \zeta_1 & \cdots & \zeta_\ell \\ 1 & 1 & \cdots & 1 \end{vmatrix}$$

equals the sign of $\prod_{0\le i<j\le\ell}[\zeta_i - \zeta_j]$.

Proof. The statement is a continuous analogue of the claim that $s_{(n,0,\dots,0)}(\zeta)$ is nonnegative for $\zeta \ge 0$, where s_λ is a symmetric polynomial known as a Schur polynomial (or, in this special case, a complete homogeneous symmetric polynomial).
By induction on ℓ one can show that the given determinant equals

$$\prod_{0\le i<j\le\ell}[\zeta_i - \zeta_j]\int_{0=z_{-1}\le z_0\le\dots\le z_{\ell-1}\le z_\ell=y} e^{\zeta_0(z_0-z_{-1})}\cdots e^{\zeta_\ell(z_\ell-z_{\ell-1})}dz_0\cdots dz_{\ell-1},$$

and the claim follows. □

We now prove Proposition 2. By the Maximum Principle (see Theorem 2.5 of [21] for a suitable form), neither the minimum nor the maximum of π^μ over S is attained in the open set S^o. We therefore investigate the boundary.
We first prove that $\pi^\mu(x) \to 0$ as $\|x\| \to \infty$ by showing that, for any $x \in S$,

$$\langle \mu, (I - \mathrm{Rot}_k)x\rangle > 0, \qquad k = 0,\dots,\ell,$$
$$\langle \mu, (I - \mathrm{Ref}_k)x\rangle > 0, \qquad k = 1,\dots,\ell.$$

Set $\theta = \arg x$. For the claim involving Rot_k, we observe that

$$\langle \mu, (I - \mathrm{Rot}_k)x\rangle = -2\|x\|\sin(\delta + k\xi)\sin(\theta_\mu - \theta - \delta - k\xi).$$

Since $0 < \delta+k\xi \le \delta+\ell\xi = \pi-\epsilon < \pi$ and we have $\theta < \xi$ and the stability condition $\xi-\epsilon < \theta_\mu < \delta$, we obtain $-\pi < \theta_\mu - \theta - \delta - k\xi < 0$. The same argument works for the claim involving Ref_k, now relying on

$$\langle \mu, (I - \mathrm{Ref}_k)x\rangle = -2\|x\|\sin(\delta + k\xi - \theta)\sin(\theta_\mu - \delta - k\xi)$$

and the assumption $k \ge 1$.
We next prove that the signs of $\pi^\mu(rw_0)$ and $\pi^\mu(rw_\xi)$ are equal and independent of $r > 0$. The equality of the signs follows from Proposition 1 after showing that they do not depend on r. From Lemma 2 with $\zeta_j = \langle \mu, \mathrm{Rot}_j e_1\rangle$ and (11) we conclude this for $\pi^\mu(rw_0)$. Applying the same argument 'clockwise' shows that this also holds for $\pi^\mu(rw_\xi)$.

5 The BAR and a PDE with boundary conditions

This section prepares for the proof of Theorem 1 by relating the stationary density to a partial differential equation (PDE) with boundary conditions involving the pushing directions.

5.1 The BAR

Our proof of Theorem 1 requires the Basic Adjoint Relationship (BAR) as presented in following proposition, which is implied by Propositions 3 and 4 in [6]; see [5] for proofs.

Proposition 3. *Suppose that $\alpha < 1$ and assume the existence and uniqueness of a stationary distribution for the SRBM.*
A nonzero finite measure ν_0 on S is proportional to the stationary distribution if and only if there exist finite measures ν_1 on F_1 and ν_2 on F_2 such that for any $f \in C_b^2(S)$

$$\int_S \left[\frac{1}{2}\Delta f - \langle \mu, \nabla f\rangle\right] d\nu_0 + \int_{F_1} \langle v_1, \nabla f\rangle d\nu_1 + \int_{F_2} \langle v_2, \nabla f\rangle d\nu_2 = 0 \qquad \text{(BAR)}.$$

Let σ and σ_i be the Lebesgue measures on S and F_i, respectively. Write $v_i^* = 2n_i - v_i$.

Proposition 4. *Let $p \in C^2(S)$ be nonnegative and integrable over S.*
If (BAR) is satisfied with $d\nu_0 = p\, d\sigma$ and $d\nu_i = p/2\, d\sigma_i$, then

$$\begin{aligned} \Delta p + 2\langle \mu, \nabla p\rangle &= 0 \quad \text{on } S^o, && (13)\\ \langle v_1^*, \nabla p\rangle + 2\langle \mu, n_1\rangle p &= 0 \quad \text{on } F_1^o, && (14)\\ \langle v_2^*, \nabla p\rangle + 2\langle \mu, n_2\rangle p &= 0 \quad \text{on } F_2^o. && (15) \end{aligned}$$

Conversely, if (13)–(15) hold and moreover $p(0) = 0$, then (BAR) is satisfied with $d\nu_0 = p\, d\sigma$ and $d\nu_i = p/2\, d\sigma_i$.

Proof. We may repeat the arguments in the proof of Lemma 7.1 of Harrison and Williams [17]. The additional assumption $p(0) = 0$ ensures that (7.8) in [17] automatically holds. □

The above proposition motivates investigating sum-of-exponential solutions to (13)–(15).

5.2 Sum-of-exponential solutions to the PDE

In this subsection we study some properties of sum-of-exponential solutions to the PDE (13) plus either boundary condition (14) or (15). We use the following observation, due to Foschini [11, Sec. III.A].

Lemma 3. *Let p be given by $p(x) = \sum_{i=1}^k a_i e^{-\langle c_i, x\rangle}$ for some $k < \infty$, $a_i \neq 0$, and $c_i \neq 0$ such that $c_i \neq c_j$ if $i \neq j$.*
If p satisfies (13) and (14), then for each $i = 1, \ldots, k$ precisely one of the following holds:

1. *$x \mapsto e^{-\langle c_i, x\rangle}$ satisfies (14), or*
2. *there exists a unique $j \neq i$ such that $x \mapsto a_i e^{-\langle c_i, x\rangle} + a_j e^{-\langle c_j, x\rangle}$ satisfies (14) and we have $\langle c_i, w_0\rangle = \langle c_j, w_0\rangle$.*

By symmetry, Lemma 3 also holds when (14) is replaced by (15), provided the condition $\langle c_i, w_0\rangle = \langle c_j, w_0\rangle$ is replaced by $\langle c_i, w_\xi\rangle = \langle c_j, w_\xi\rangle$. The next two lemmas investigate the two scenarios of Lemma 3 in more detail; Lemma 4 may be regarded as a generalisation of Theorem 6.1 of [17] (modulo the discussion of the BAR in Section 6).

Lemma 4. *Let $p(x) = e^{-\langle c, x\rangle}$ for some $c \neq 0$.*

1. *If p satisfies (13) and (14), then either* $p(x) = e^{-\langle \mu,(I-\rho_{2\delta})x\rangle}$ *or* $p(x) = e^{-\langle \mu,(I-R_0)x\rangle}$.

2. *If p satisfies (13) and (15), then either* $p(x) = e^{-\langle \mu,(I-\rho_{-2\epsilon})x\rangle}$ *or* $p(x) = e^{-\langle \mu,(I-R_\xi)x\rangle}$.

Proof. We only prove the first claim, the second being the clockwise analogue. The condition that p satisfies (13) translates to $\|c-\mu\| = \|\mu\|$. We may therefore write $c = \mu - \rho_{-2\gamma}\mu$ for some $\gamma \equiv \gamma(\mu)$. Next we substitute this in (14), yielding $-\langle \mu,(I-\rho_{2\gamma})v_1^*\rangle + 2\langle \mu, n_1\rangle = 0$, which we may rewrite as $\langle \mu, v_1 + \rho_{2\gamma}v_1^*\rangle = 0$ since $v_1^* = 2n_1 - v_1$. Using $v_1^* = -R_\delta\rho_{2\delta}v_1$, we get $\langle \mu,(I-\rho_{2(\gamma-\delta)})v_1\rangle = 0$. This can only be the case if $\gamma = \delta \mod \pi$ or $\gamma = \theta_\mu \mod \pi$. □

The next result investigates the second scenario of Lemma 3.

Lemma 5. *Let p be given by* $p(x) = a_1 e^{-\langle c,x\rangle} + a_2 e^{-\langle d,x\rangle}$ *for some* $a_1, a_2 \neq 0$, $c, d \neq 0$, *and* $c \neq d$.

1. *If* $\langle c, w_0\rangle = \langle d, w_0\rangle$ *and p satisfies (13) and (14), then there exists some* $\gamma \equiv \gamma(\mu) \in (0,\pi)$ *such that p is proportional to* p_γ *defined by*

$$p_\gamma(x) = \langle \mu,(I-\rho_{2\gamma+2\delta})v_1\rangle e^{-\langle \mu,(I-\rho_{2\gamma+2\delta})x\rangle} - \langle \mu,(I-\rho_{2\gamma+2\delta}R_0)v_1\rangle e^{-\langle \mu,(I-\rho_{2\gamma+2\delta}R_0)x\rangle}.$$

2. *If* $\langle c, w_\xi\rangle = \langle d, w_\xi\rangle$ *and p satisfies (13) and (15), then there exists some* $\tilde\gamma \equiv \tilde\gamma(\mu) \in (0,\pi)$ *such that p is proportional to* $\tilde p_{\tilde\gamma}$ *defined by*

$$\tilde p_{\tilde\gamma}(x) = \langle \mu,(I-\rho_{-2\tilde\gamma-2\epsilon})v_2\rangle e^{-\langle \mu,(I-\rho_{-2\tilde\gamma-2\epsilon})x\rangle} - \langle \mu,(I-\rho_{-2\tilde\gamma-2\epsilon}R_\xi)v_2\rangle e^{-\langle \mu,(I-\rho_{-2\tilde\gamma-2\epsilon}R_\xi)x\rangle}.$$

Proof. Again we only prove the first claim. By linear independence both $e^{-\langle c,x\rangle}$ and $e^{-\langle d,x\rangle}$ must satisfy (13) individually. As in the proof of Lemma 4, we may therefore write $c = \mu - \rho_{-2\delta-2\gamma}\mu$ for some $\gamma \equiv \gamma(\mu) \in [0,\pi)$, so that $\langle c,x\rangle = \langle \mu,(I-\rho_{2\gamma+2\delta})x\rangle$. From $\langle c, w_0\rangle = \langle d, w_0\rangle$ and $c \neq d$ we conclude that $\langle d,x\rangle = \langle \mu,(I-\rho_{2\gamma+2\delta}R_0)x\rangle$.
It remains to study a_1 and a_2, for which we use (14). Since $\langle c, w_0\rangle = \langle d, w_0\rangle$ we obtain that on F_1,

$$\langle v_1^*, \nabla p(x)\rangle = -[a_1\langle \mu,(I-\rho_{2\gamma+2\delta})v_1^*\rangle + a_2\langle \mu,(I-\rho_{2\gamma+2\delta}R_0)v_1^*\rangle]e^{-\langle c,x\rangle}.$$

With $v_1^* = 2n_1 - v_1$, we conclude that (14) implies

$$a_1\langle \mu, v_1 + \rho_{2\gamma+2\delta}v_1^*\rangle + a_2\langle \mu, v_1 + \rho_{2\gamma+2\delta}R_0 v_1^*\rangle = 0.$$

The result follows after using $v_1^* = -R_0 v_1$ and noting that $\langle \mu,(I-\rho_{2\gamma+2\delta}R_0)v_1\rangle$ cannot be zero. □

We remark that Lemmas 3–5 show that some structure from the particular example of Section 2 holds in general. Specifically, each exponent in a sum-of-exponential solution equals $-\langle \mu,(I-M)x\rangle$ for some reflection or rotation matrix M.

6 Proof of Theorem 1

To prove our main result, it suffices to show that (i) implies (ii) and that (iii) implies (i). We start with the latter.

6.1 Proof that (iii) implies (i)

By assumption, there is some μ with $\xi - \epsilon < \theta_\mu < \delta$ such that the stationary density has the form

$$p^\mu(x) = \sum_{i=1}^{k} a_i e^{-\langle c_i, x\rangle}. \tag{16}$$

We assume without loss of generality that the a_i are nonzero and that the c_i are distinct. We may also restrict ourselves to the case of more than one summand ($k > 1$), since the $k = 1$ case has already been studied [15, 17].

We next argue that p^μ satisfies (13)–(15). To this end, in view of Proposition 4, we need to show that (BAR) is satisfied with $d\nu_0 = p^\mu\, d\sigma$ and $d\nu_i = p^\mu/2\, d\sigma_i$. We do so using an argument due to Harrison and Williams [16, p. 108]. From Proposition 3 we know that (BAR) holds for some measures ν_1, ν_2. Let $\lambda \in \mathbb{R}^2$ satisfy $\langle \lambda, x\rangle \geq 0$ for all $x \in S$. On substituting $f(x) = e^{-\langle \lambda, x\rangle}$ in (BAR) we find

$$\left[\frac{1}{2}\|\lambda\|^2 + \langle \mu, \lambda\rangle\right] \int_S e^{-\langle \lambda, x\rangle} p^\mu(x) dx - \langle \nu_1, \lambda\rangle \int_{F_1} e^{-\langle \lambda, x\rangle} \nu_1(dx) - \langle \nu_2, \lambda\rangle \int_{F_2} e^{-\langle \lambda, x\rangle} \nu_2(dx) = 0.$$

Let us first focus on ν_1. Write $\lambda = \lambda_1 w_0 + \lambda_2 n_1$, and let $\lambda_2 \to \infty$ after dividing the above equation by λ_2. To evaluate the resulting limit of the first term, we use the initial value theorem to obtain

$$\lim_{\lambda_2 \to \infty} \lambda_2 \int_S e^{-\langle \lambda, x\rangle} p^\mu(x) dx = \int_0^\infty e^{-\lambda_1 s} p^\mu(s w_0) ds.$$

After taking the limits of all other terms as well (recalling that $\langle \nu_1, n_1\rangle = 1$), we conclude that $\int_{F_1} e^{-\lambda_1 x_1} \nu_1(dx) = \int_0^\infty e^{-\lambda_1 z} p^\mu((z, 0)) dz/2$ for $\lambda_1 \geq 0$. The uniqueness theorem for Laplace transforms thus yields $d\nu_1 = p^\mu/2\, d\sigma_1$. A similar argument works to show $d\nu_2 = p^\mu/2\, d\sigma_2$ by studying $\lambda = \lambda_1 w_\xi + \lambda_2 n_2$ for large λ_2.

In the remainder of this subsection, it is our aim to further specify the structure of p^μ defined in (16) when it solves (13)–(15). Lemmas 3–5 play a central role in this analysis.

Graph representation.

It is convenient to represent p^μ by an undirected labelled graph G, with k vertices, as follows. Each vertex represents a summand $a_i e^{-\langle c_i, x\rangle}$ in (16), and we say that two vertices are joined by a *BC1 edge* (respectively *BC2 edge*) if the sum of the terms corresponding to these vertices satisfies (13) and (14) (respectively (13) and (15)). Note that by Lemma 3, at most one BC1 edge and at most one BC2 edge can be incident at any given vertex. Therefore, the degree of the vertices in G cannot exceed two, and BC1 edges and BC2 edges alternate along any path. Here and throughout, we say that a subgraph of G is a *path* if it is connected and acyclic. The *length* of a path equals its number of vertices. The vertex corresponding to the summand $e^{-\langle c, x\rangle}$ is labelled by the matrix M for which $\langle c, x\rangle = \langle \mu, (I - M)x\rangle$, and we refer to this vertex as an M vertex. Note that M is necessarily a reflection or rotation matrix. Although M is uniquely defined for any of the vertices of G, it has several representations—for example, we also refer to an R_ξ vertex as a $\rho_{2\xi} R_0$ vertex. We refer to a path between an M_1 vertex and and M_2 vertex as an '$M_1 - M_2$ path'.

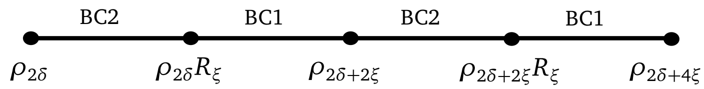

Figure 3: An example labelled graph. By (6), this graph corresponds to the reflection construction in the leftmost diagram of Figure 2. If $\alpha = -2$ then this graph also corresponds to the rightmost diagram of Figure 2.

Mating procedure.

Given the label of any vertex in G, we can specify the labels of all other vertices in the same connected component. For let M_1, M_2 be the labels of two arbitrary vertices joined by a BC1 edge: then by Lemma 5, the sum of the corresponding exponential terms is proportional to p_γ for some $\gamma \in (0, \pi)$. We will say that M_1 is the *BC1 mate* of M_2. By considering separately the cases when M_1 is a reflection and a rotation, it is easy to see that $\{M_1, M_2\} = \{\rho_{2\beta}, \rho_{2\beta-2\xi}R_\xi\}$ for some angle $\beta \in (-\pi, \pi]$. Similarly, the labels of two arbitrary vertices joined by a BC2 edge are $\{\rho_{2\beta}, \rho_{2\beta}R_\xi\}$ for some $\beta \in (-\pi, \pi]$. Any path in G—beginning for example with a BC2 edge—therefore has labels $\rho_{2\beta}, \rho_{2\beta}R_\xi, \rho_{2\beta+2\xi}, \dots$ for some β.

Example.

To illustrate the graph representation and mating procedure, suppose that one summand in (16) has exponent $-\langle \mu, (I - \rho_{2\delta})x \rangle$. In the graph representation, this summand is represented by a vertex with label $\rho_{2\delta}$: suppose there exists a path of length 5 starting at this vertex, and that its first edge is a BC2 edge. By the mating procedure, the vertex labels for this path are $\rho_{2\delta}, \rho_{2\delta}R_\xi, \rho_{2\delta+2\xi}, \rho_{2\delta+2\xi}R_\xi, \rho_{2\delta+4\xi}$, cf. Figure 3. By (6), this path corresponds to the reflection construction in the leftmost diagram of Figure 2.

Proposition 5. *Let the stationary density p^μ be of the form* (16). *If G is the labelled graph corresponding to p^μ then:*

- *G is a $\rho_{2\delta} - \rho_{-2\epsilon}$ path;*
- *The number of vertices in G is odd;*
- *$\alpha = -\ell$ for some integer $\ell \geq 0$.*

Proof. As usual we exclude the case $\alpha = 0$. Since p^μ is a density, each exponent in (16) is nonpositive and $p^\mu(x) \to 0$ as $\|x\| \to \infty$ in S.
A key tool in the proof is the following *range restriction* for the reflection labels. That is, G cannot contain an R_γ vertex if $w_\gamma \in S^o$ or $-w_\gamma \in S^o$. Suppose *a contrario* that G contains an R_γ vertex and that $L_\gamma \cap S^o \neq \emptyset$, where L_γ is the line $\{rw_\gamma : r \in \mathbb{R}\}$. We must then have $\mu \in L_\gamma$ since otherwise the exponent $x \mapsto -\langle \mu, (I - R_\gamma)x \rangle$ changes sign in S^o on either side of L_γ. However, if $\mu \in L_\gamma$ then $-\langle \mu, (I - R_\gamma)x \rangle = 0$ for all x, but such a constant exponent (which must be unique) contradicts $p^\mu(x) \to 0$.
G is acyclic. Suppose that G contains a cycle G_0, of length $2m$ say (note that cycles of odd length are impossible by the mating procedure). Taking an arbitrary rotation label $\rho_{2\beta}$ from G_0, we must have $\rho_{2\beta} = \rho_{2\beta+2m\xi}$ by cyclicity and the mating procedure, so that $\xi = n\pi/m$ for some integer $n \geq 1$. The reflection labels in G_0 are readily seen to be $R_{\beta+k\xi}, k = 1, \dots, m$, so range restriction yields $L_{\beta+k\xi} \cap S^o = \emptyset$ for each $k = 1, \dots, m$. Using this in conjunction with the fact that (by

uniqueness of the reflection labels) $\{L_{\beta+kn\pi/m} : k = 1,\ldots,m\} = \{L_{\beta+k\pi/m} : k = 1,\ldots,m\}$, we deduce that $n = 1$ and that $R_{\beta+k\xi} = R_0$ for some k. By the proof of Lemma 4 R_0 must be a vertex of degree 1, so this is a contradiction.

G does not contain the label R_0, nor does it contain the label R_ξ. We only prove that G does not contain the vertex R_0; similar arguments can be given for R_ξ. Suppose that G contains the vertex R_0, and consider the sum-of-exponentials $p^\mu(x)$ corresponding to G when x lies on the boundary F_1. There is a constant nonzero term since $(I - R_0)w_0 = 0$, so to ensure $p^\mu(rw_0) \to 0$ as $r \to \infty$ there must be another exponent that vanishes on F_1. First observe that R_0 is the only reflection label that can possibly correspond to a constant term on F_1 (as shown already, $\mu \notin L_\gamma$ if G contains the label R_γ so μ cannot be orthogonal to $(I - R_\gamma)w_0$). All rotation vertices in G correspond to exactly the same exponents on F_1 as their possible BC1 mates, which are reflection vertices necessarily different from R_0, whence none of the vertices joined by a BC1 edge can represent the term constant on F_1. Therefore, in view of Lemma 4, the only remaining possibility is that G contain a vertex labelled $\rho_{2\delta}$ which corresponds to the constant on F_1. However, this contradicts our assumption that $\xi - \epsilon < \theta_\mu < \delta$.

We have now proved that G is a $\rho_{2\delta} - \rho_{-2\epsilon}$ path, and by the mating procedure the number of terms is odd, say $2\ell + 1$. The mating procedure also shows that we must have $\rho_{-2\epsilon} = \rho_{2\ell\xi+2\delta}$, so that $\delta + \epsilon + \ell\xi = n\pi$ for some integer $n \geq 1$. The range restriction on the reflection labels entails that $L_{\delta+\xi}, L_{\delta+2\xi}, \ldots, L_{\delta+\ell\xi} \notin S^o$. None of these lines can be equal to L_0, since this would imply that R_0 is a label in G. We deduce that $\delta + \ell\xi < \pi$, and therefore $n = 1$. □

To continue our example, suppose that $\alpha = -2$. Then $\rho_{2\delta+4\xi} = \rho_{-2\epsilon}$ and so the graph in our example also corresponds to the rightmost diagram in Figure 2. Then G has 5 vertices, so that p^μ has $k = 5$ exponential terms.

6.2 Proof that (i) implies (ii)

Suppose that $\alpha = -\ell$ for some $\ell \in \{1, 2, \ldots\}$, and consider a μ with $\theta_\mu \in \Theta_\ell$. We shall use the representation (8) and Lemma 1 to argue that π^μ must equal the stationary density up to a multiplicative constant. We first argue that π^μ satisfies (13)–(15).

The proofs of Lemmas 4 and 5 show that π^μ satisfies (13) and (14). To see that π^μ also satisfies (15) we note that, for any constants $d_0, \ldots, d_\ell$, the function $\tilde\pi^\mu$ defined on S by

$$\tilde\pi^\mu(x) = d_0\langle\mu, (I - \rho_{-2\epsilon})v_2\rangle e^{-\langle\mu,(I-\rho_{-2\epsilon})x\rangle} + d_1\tilde p_\xi(x) + d_2\tilde p_{2\xi}(x) + \ldots + d_\ell \tilde p_{\ell\xi}(x) \tag{17}$$

satisfies (13) and (15) ($\tilde p_\theta$ is defined in Lemma 5). Using $\epsilon + \delta + \ell\xi = \pi$, on investigating the exponents we find that π^μ and $\tilde\pi^\mu$ are linear combinations of the same exponential terms, so it suffices to show that the coefficients are proportional to each other. To do so, we write

$$\begin{aligned}\pi^\mu(x) \quad = \quad & c_\ell\langle\mu, (I - \mathrm{Rot}_\ell)v_1\rangle e^{-\langle\mu,(I-\mathrm{Rot}_\ell)x\rangle} \\ & + \sum_{k=1}^{\ell}\Big[c_{\ell-k}\langle\mu, (I - \mathrm{Rot}_{\ell-k})v_1\rangle e^{-\langle\mu,(I-\mathrm{Rot}_{\ell-k})x\rangle} \\ & \qquad - c_{\ell-k+1}\langle\mu, (I - \mathrm{Ref}_{\ell-k+1})v_1\rangle e^{-\langle\mu,(I-\mathrm{Ref}_{\ell-k+1})x\rangle}\Big].\end{aligned}$$

Equating the coefficients with (17), we find that π^μ satisfies (13)–(15) if for $k = 1, \ldots, \ell$,

$$\begin{aligned}& c_{\ell-k}\langle\mu, (I - \mathrm{Rot}_{\ell-k})v_1\rangle\langle\mu, (I - \mathrm{Ref}_{\ell-k+1})v_2\rangle \\ & \quad = \quad c_{\ell-k+1}\langle\mu, (I - \mathrm{Ref}_{\ell-k+1})v_1\rangle\langle\mu, (I - \mathrm{Rot}_{\ell-k})v_2\rangle,\end{aligned}$$

and this is the recursion given in Lemma 1 (the condition $\theta_\mu \notin \Theta_\ell$ guarantees that none of the four inner products in (10) is zero).
Now that we know that π^μ satisfies (13)–(15), it remains to show that it is a multiple of the stationary density. Proposition 2 and its proof show that π^μ is integrable and single signed. Moreover, Proposition 1 implies that $\pi^\mu(0) = 0$. We therefore conclude from Proposition 4 that (BAR) is satisfied with $d\nu_0 = \pi^\mu \, d\sigma$ and $d\nu_i = \pi^\mu/2 \, d\sigma_i$. The claim thus follows from Proposition 3.

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