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BROWNIAN EXCURSION CONDITIONED ON ITS LOCAL TIME

DAVID J. ALDOUS[1]
Department of Statistics
University of California
Berkeley CA 94720
e-mail: aldous@stat.berkeley.edu
http://www.stat.berkeley.edu/users/aldous

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1 Introduction

Let $(B_u, 0 \le u \le 1)$ be standard Brownian excursion and $(L_s, 0 \le s < \infty)$ its local time, more precisely its local time at time 1:

$$\int_0^h L_s \ ds = \int_0^1 1_{(B_u \le h)} \ du, \quad h \ge 0.$$

Biane - Yor [4] give an extensive treatment, including an elegant description of the law of L as a random time-change of the Brownian excursion:

$$(\tfrac{1}{2} L_{s/2}, s \ge 0) \ \stackrel{d}{=} \ (B_{\tau^{-1}(s)}, s \ge 0) \text{ for } \tau(t) = \int_0^t 1/B_s \ ds$$

where $\stackrel{d}{=}$ indicates equality in law. Takács [14] gives a combinatorial approach to formulas for the marginal law of L_s. Bertoin - Pitman [3] discuss transformations between Brownian excursion and other Brownian-type processes. References to further papers on standard Brownian excursion can be found in those references.
Consider the question

> Given a function $\ell = (\ell(s), 0 \le s < \infty)$, can we define a process $B^\ell = (B^\ell_u, 0 \le u \le 1)$ whose law $\psi(\ell)$ is, in some sense, the conditional law of B given $L = \ell$?

As discussed in section 1.1, Warren and Yor [16] have recently given a quite different analysis of a similar question, and related ideas appeared earlier in the superprocesses literature. Of course,

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the joint law of (B, L) implicitly gives us conditional laws, i.e. specifies B^ℓ for almost all ℓ with respect to the law of L. One consequence of our results is that in fact B^ℓ exists much more generally.
We first define two (not quite usual) function spaces. First, let $C_{\mathrm{exc}}[0,1]$ be the set of continuous functions $f : [0,1] \to [0,\infty)$ which are "excursions" in the sense

$$f(0) = f(1) = 0, \;\; f(u) > 0 \text{ for } 0 < u < 1. \tag{1}$$

Give $C_{\mathrm{exc}}[0,1]$ the topology of convergence in measure:

$$f_n \to f \text{ iff } \int_0^1 \max(1, |f_n(x) - f(x)|) \; dx \to 0.$$

Let $\mathcal{P}_{\mathrm{exc}}$ be the space of probability laws on $C_{\mathrm{exc}}[0,1]$, with the topology of weak convergence.
Second, let $\mathcal{L}$ be the set of Borel measurable functions $\ell : [0,\infty) \to [0,\infty)$ such that
(i) $s^* = s^*(\ell) := \sup\{s : \ell(s) > 0\} < \infty$
(ii) $\int_0^{s^*} \ell(s) \; ds = 1$
(iii) $\int_a^b 1/\ell(s) \; ds < \infty$ for all $0 < a < b < s^*$
(iv) $\int_0^a 1/\ell(s) \; ds = \infty$ for all $a > 0$.
Give $\mathcal{L}$ the topology: $\ell_m \to \ell$ iff

$$\int_0^\infty \max(1, |\ell_m(s) - \ell(s)|) \; ds \to 0$$

and

$$\int_a^b \left| \tfrac{1}{\ell_m(s)} - \tfrac{1}{\ell(s)} \right| \; ds \to 0 \text{ for all } 0 < a < b < s^*(\ell) \; .$$

The purpose of this paper is to present a construction, which can be outlined as follows.

Construction 1 *Let $\ell \in \mathcal{L}$. There is a certain consistent family $(\mathcal{R}_k^\ell, k \geq 1)$ of k-leaf random trees, defined in section 2.1. Applying the general correspondence [2] between consistent families of trees and excursion functions, we obtain (section 2.2) a $C_{\mathrm{exc}}[0,1]$-valued process B^ℓ. The local time for B^ℓ is ℓ; that is,*

$$\int_0^h \ell(s) \; ds = \int_0^1 1_{(B_u^\ell \leq h)} \; du, \quad h \geq 0.$$

The map $\ell \to \mathrm{law}(B^\ell)$ is continuous from $\mathcal{L}$ into $\mathcal{P}_{\mathrm{exc}}$.

The construction does not directly involve any "Brownian" ingredients, but the next theorem (proved in section 3.2) shows that B^ℓ can be interpreted as Brownian excursion conditioned to have local time ℓ. An intuitive explanation of why everything works out is in section 3.3.

Theorem 2 *For $\ell \in \mathcal{L}$ write $\psi(\ell) = \mathrm{law}(B^\ell)$. If B is standard Brownian excursion and L its local time, then $\psi(\ell)$ is a version of the conditional law of B given $L = \ell$.*

The Biane-Yor description easily implies that L takes values in $\mathcal{L}$ and that the support of the law of L is the whole space $\mathcal{L}$. Thus by the continuity assertion of the construction, $\psi(\ell)$ is specified *uniquely* "by continuity" for all $\ell \in \mathcal{L}$. We emphasize this uniqueness because our definition of

$\psi(\ell)$ will be somewhat indirect, and without knowing continuity one might suspect there could be different extensions of ψ from the set of "typical paths of L" to larger spaces such as $\mathcal{L}$.

We chose to present results in the setting of excursions so that we could appeal directly to the results of [2] giving a correspondence between trees and excursion functions. Straightforward modifications give parallel results (outlined in section 4) for reflecting Brownian bridge conditioned on its local time.

Let us mention two open problems suggested by Theorem 2.

(a) Find explicit formulas, in terms of ℓ, for the law $\psi(\ell)$ or the marginal laws of B^ℓ_u for $0 < u < 1$. Our definition of B^ℓ via (2) and (8) isn't very helpful.

(b) It is clear that conditions (i) and (ii) on ℓ are necessary. It turns out that condition (iv) is necessary to ensure that B^ℓ is strictly positive on $(0, 1)$: see (4). However, condition (iii) is not quite necessary: one can make examples where $\int 1/\ell(s)$ diverges at some point $s_0 \in (0, s^*)$, and where the process B^ℓ has only one upcrossing and downcrossing over height s_0. Perhaps the most general setting is where $ds/\ell(s)$ is a sigma-finite measure on $(0, s^*)$.

1.1 Related work

Warren and Yor [16] study the analogous question with standard Brownian excursion replaced by reflecting Brownian motion B_{ref} killed upon first hitting $+1$, when it has local time L_{ref}. They introduce a *Brownian burglar* process $\hat{B}$ and give a representation of B_{ref} in terms of the independent pair $(\hat{B}, L_{\mathrm{ref}})$. This leads to a description of the conditional laws B^ℓ_{ref} which is more explicit than ours. Warren (personal communication) observes that the processes B^ℓ_{ref} and B^ℓ cannot be expected to be semimartingales.

There are some conceptually related results in the more sophisticated setting of superprocesses. As Le Gall [9] and others have observed,

(a) the Dawson-Watanabe superprocess can be constructed by running conditionally independent copies of the underlying Markov process along the branches of a "genealogical tree"

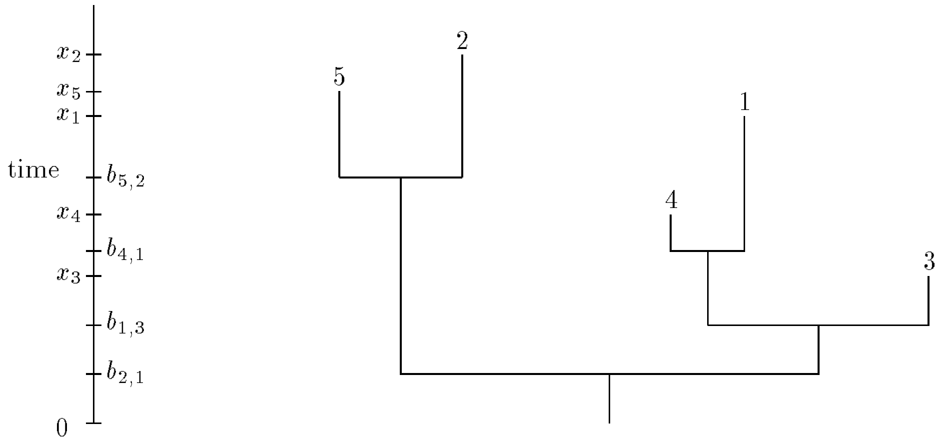
(b) the genealogical tree can be constructed from the excursions of a Brownian-type process, with the "total mass process" being the local time of the Brownian process.

And Perkins [13] showed that for a superprocess one can condition the total mass process to be a specified continuous function ℓ, in other words can condition the genealogical tree on the local time process. See Donnelly and Kurtz [7] for a recent "coalescing particle" derivation. Thus implicit in this circle of ideas is the idea of conditioning an excursion on its local time. To make this explicit one needs a careful treatment of the correspondence between an excursion function (i.e. element of $C_{\mathrm{exc}}[0, 1]$) and trees (which we call *continuum trees*). This general correspondence was treated in Aldous [2], and the present paper is an illustration of the uses of this general theory.

2 The construction

2.1 A non-homogeneous analog of Kingman's coalescent

Fix $\ell \in \mathcal{L}$. For each integer $k \geq 1$ we will define a process of k coalescing particles (later rephrased as a random tree). Here is a verbal description of the process. Take "time" t decreasing from ∞ to 0. Let each of k particles be born at independent random times with probability density ℓ (here we use condition (ii) of the definition of $\mathcal{L}$). Particles coalesce into clusters according to the rule: in time $[t, t - dt]$, each pair of clusters has chance $\frac{4}{\ell(t)} dt$ to merge into a single cluster.



The figure shows a realization of the process for $k = 5$. It is clear that we can regard the process as a random tree $\mathcal{R}_k^\ell$. The range space of $\mathcal{R}_k^\ell$ is the set of *ordered real k-trees* $\mathbf{t}$, defined as follows. The tree $\mathbf{t}$ has k leaves labeled $\{1, 2, \ldots, k\}$ at real-valued positive heights $x_1, \ldots, x_k$ (x_i being the birth time of particle i), where the root at height 0 has degree 1 (see remark below (4)). The internal vertices (branchpoints) have degree 3, and we distinguish the two branches at a branchpoint as "left" and "right". Such a tree has a "shape" σ: in the figure, the shape records the information that particle 1 merges with particle 4 at some unspecified time $b_{4,1}$ with particle 4 on the left of particle 1; then at some time $b_{1,3}$ the cluster $\{1, 4\}$ merges with particle 3 which is on its right; and so on. The tree $\mathbf{t}$ is completely specified by the triple $(\sigma, \mathbf{x}, \mathbf{b})$, where σ is the shape, $\mathbf{x} = (x_i, \ldots, x_k)$ is the vector of leaf-heights, and $\mathbf{b} = (b_j, j \in J_\sigma)$ is the set of heights of branchpoints (the exact convention for the index set J_σ of branchpoints is unimportant). In the random tree $\mathcal{R}_k^\ell$, write X_i^ℓ and B_j^ℓ for the heights of the labeled leaves and the branchpoints, and assign branches to left/right at random. The law of $\mathcal{R}_k^\ell$ may be described by a density $f_k^\ell(\sigma, \mathbf{x}, \mathbf{b})$, whose interpretation is that for each shape σ

$$P(\text{shape}(\mathcal{R}_k^\ell) = \sigma, X_i^\ell \in [x_i, x_i + dx_i] \forall i, B_j^\ell \in [b_j, b_j + db_j] \forall j) = f_k^\ell(\sigma, \mathbf{x}, \mathbf{b})\; d\mathbf{x} d\mathbf{b}.$$

It is easy to see that the verbal description above is equivalent to the density formula

$$f_k^\ell(\sigma, \mathbf{x}, \mathbf{b}) = 2^{-(k-1)} \frac{\prod_{i=1}^k \ell(x_i)}{\prod_{j \in J_\sigma} \frac{1}{4}\ell(b_j)} \quad \exp\left(- \int_0^\infty \binom{n(s)}{2} \frac{4}{\ell(s)}\; ds \right) \tag{2}$$

where $n(s) = |\{i : x_i > s\}| - |\{j : b_j > s\}|$ is the number of edges at height s and $\binom{n}{2} = 0$ for $n = 0, 1$. In (2), the term $2^{-(k-1)}$ is the chance of a particular set of left/right assignments, $\prod_i \ell(x_i)$ is the density function of the k leaves, and the remaining terms are the density of the $k - 1$ branchpoints.

Note that an ordered k-tree is equipped with a distance d: for points v_1 and v_2 with branchpoint w,

$$d(v_1, v_2) = (\text{height}(v_1) - \text{height}(w)) + (\text{height}(v_2) - \text{height}(w)). \tag{3}$$

Note also that the specialization of (2) to $k = 2$ is

$$f_k^\ell(\sigma, x_1, x_2, b) = \frac{2\ell(x_1)\ell(x_2)}{\ell(b)} \exp\left(-\int_b^{\min(x_1,x_2)} \tfrac{4}{\ell(s)}\, ds\right). \tag{4}$$

So condition (iv) in the definition of $\mathcal{L}$ ensures that the height B of the branchpoint in $\mathcal{R}_2^\ell$ is strictly positive.

Remarks. (a) Kingman's coalescent [12] is the analogous process with the k particles born at time 0, with time running from 0 to ∞ and with each pair of clusters merging at rate 1. We later need the easy fact that, in Kingman's coalescent, the number $N_k(t)$ of clusters at time $t > 0$ satisfies

$$N_k(t) \uparrow N_\infty(t) < \infty \text{ a.s.} \tag{5}$$

As noted by Kingman [11] the non-homogeneous case is just a deterministic time-change of the homogeneous case. While many variations have been considered in population genetics [15], our "random birth times" setting has no visible biological interpretation and so has apparently not been studied explicitly.

(b) In our context it would be more natural (cf. Theorem 3 later) to use $2B$ as our "standard" version of Brownian excursion; with this standardization, the factor 4 in the coalescence rate $4/\ell(s)$ would become 1.

2.2 Representing continuum trees by excursion functions

It is clear from the verbal description of the coalescing particle process that the family $(\mathcal{R}_k^\ell, k \geq 1)$ is *consistent*, in the sense

$$\text{the subtree of } \mathcal{R}_k \text{ spanned by the root and vertices } \{1, 2, \ldots, k-1\}$$

$$\text{is distributed as } \mathcal{R}_{k-1}, \text{ for each } k \geq 2. \tag{6}$$

This ties in with the following general theory from [2]. Given $f \in C_{\text{exc}}[0,1]$ satisfying minor extra conditions, and given $u_1, \ldots, u_k \in (0,1)$, we can specify an ordered real k-tree $\mathbf{t}(f, u_1, \ldots, u_k)$ by:

(a) the root is at height 0

(b) there are leaves $1, \ldots, k$, with leaf i at height $f(u_i)$

(c) for the paths from the root to leaves i and j, the branchpoint is at height $\inf_{\min(u_i,u_j) \leq u \leq \max(u_i,u_j)} f(u)$.

If we allow f to be random and take $U_1, \ldots, U_k$ to be independent $U(0,1)$ independent of f, then

$$\mathcal{R}_k = \mathbf{t}(f, U_1, \ldots, U_k) \tag{7}$$

defines a family $(\mathcal{R}_k, k \geq 1)$ which is clearly consistent in the sense (6). Theorem 15 of [2] gives a converse: if a family $(\mathcal{R}_k, k \geq 1)$ is consistent then, under two technical conditions, there exists a random $C_{\text{exc}}[0,1]$-valued function f such that (7) holds, and f is unique in law. In the next section we state the technical conditions (10, 11) and verify them for the family $(\mathcal{R}_k^\ell, k \geq 1)$, where $\ell \in \mathcal{L}$. Then [2] Theorem 15 yields a random function, which we now call B^ℓ, such that

$$\mathcal{R}_k^\ell \stackrel{d}{=} \mathbf{t}(B^\ell, U_1, \ldots, U_k). \tag{8}$$

This representation shows in particular that the heights $(B^\ell(U_i), 1 \le i < \infty)$ (the birth-times of particles in the coalescent process) are independent with density $\ell(\cdot)$, implying by the Glivenko-Cantelli theorem that the local time process for B^ℓ is indeed a.s. equal to the deterministic function $\ell(\cdot)$.
To obtain the continuity assertion of Construction 1, consider $\ell_n \to \ell$ in $\mathcal{L}$. From the density formula (2) and the definition of the topology on $\mathcal{L}$,

$$\sum_\sigma \int\int \left(f^{\ell_n}(\sigma,\mathbf{x},\mathbf{b}) - f^\ell(\sigma,\mathbf{x},\mathbf{b})\right)^- \ d\mathbf{x}d\mathbf{b} \quad \to 0.$$

This implies convergence in total variation of law$(\mathcal{R}_k^{\ell_n})$ to law$(\mathcal{R}_k^\ell)$. In particular, writing X_i for the height of vertex i,

$$(X_1^{\ell_n}, \ldots, X_k^{\ell_n}) \stackrel{d}{\to} (X_1^\ell, \ldots, X_k^\ell).$$

By the representation (8), this is equivalent to

$$(B^{\ell_n}(U_1), \ldots, B^{\ell_n}(U_k)) \stackrel{d}{\to} (B^\ell(U_1), \ldots, B^\ell(U_k)). \tag{9}$$

But it is not hard to show (cf. [5]) that (9) is equivalent to weak convergence $B^{\ell_n} \stackrel{d}{\to} B^\ell$ when $C_{\text{exc}}[0,1]$ is given the topology of convergence in measure.
In the next section we check the technical conditions (10, 11), and thereby complete Construction 1.

2.3 Checking the technical conditions

The consistent family $(\mathcal{R}_k^\ell, k \ge 1)$ specifies, by Kolmogorov extension, a tree $\mathcal{R}_\infty^\ell$ with an infinite number of leaves $V_1, V_2, \ldots$. The first technical condition ([2] equation (7)) is that the set of leaves be precompact with respect to the natural distance d at (3). One formulation of precompactness is: for each $\varepsilon > 0$ there exists an a.s. finite set of points $(Z_j, 1 \le j \le M_\varepsilon)$ such that

$$\sup_{1 \le i < \infty} \min_{1 \le j \le M_\varepsilon} d(V_i, Z_j) \le 2\varepsilon. \tag{10}$$

To establish this, for $h > 0$ let S_h be the set of points of $\mathcal{R}_\infty^\ell$ at height h which are on the path from the root to some V_i at height $\ge h + \varepsilon$. Clearly (10) holds for $\{Z_j\} = \cup_{0 \le i \le s^*/\varepsilon} S_{\varepsilon i}$ (note we are using condition (i) of the definition of $\mathcal{L}$), so it is enough to show that the cardinality $|S_h|$ is a.s. finite. But ignoring births, the coalescing particle process of section 2.1 evolves as a deterministic time-change of Kingman's coalescent, so that in the notation of (5)

$$|S_h| \stackrel{d}{=} N_{Q_h}\left(\int_h^{h+\varepsilon} \tfrac{4}{\ell(s)}\ ds\right)$$

where $Q_h \le \infty$ is the number of branches of $\mathcal{R}_\infty^\ell$ at height $h+\varepsilon$. By the Cauchy-Schwarz inequality

$$\varepsilon^2 = \left(\int_h^{h+\varepsilon} 1\ ds\right)^2 \le \left(\int_h^{h+\varepsilon} \ell(s)\ ds\right)\left(\int_h^{h+\varepsilon} 1/\ell(s)\ ds\right) \le \int_h^{h+\varepsilon} 1/\ell(s)\ ds.$$

Since $N_k(t) \le N_\infty(t)$ and $t \to N_\infty(t)$ is decreasing, we deduce

$$|S_h| \text{ is stochastically smaller than } N_\infty(\varepsilon^2/4).$$

So $|S_h|$ is a.s. finite, establishing (10).
The second technical condition ([2] Theorem 15 condition (a)) is as follows. In $\mathcal{R}^{\ell}_{\infty}$, condition on height$(V_1) = x_1$. Then for each interval $[y, y+\delta] \subset [0, x_1]$ it is required that some vertex V_i $(2 \le i < \infty)$ satisfy

$$\text{height}(V_i) \le y + \delta, \quad \text{height}(B_{1,i}) \ge y \tag{11}$$

where $B_{1,i}$ is the branchpoint of V_1 and V_i. To verify this requirement, consider the coalescing particle process of section 2.1, and let $(N^*_k(t), y+\delta \ge t \ge y)$ be the number of points of the tree at height t which are on the path from the root to some V_j $(2 \le j \le k)$ with height$(V_j) \le y + \delta$. Then

$$P(\text{ (11) holds for some } 2 \le i \le k) = 1 - E\exp\left(-\int_y^{y+\delta} N^*_k(s)\ ds\right) \tag{12}$$

because the conditional probability of a cluster coalescing with the cluster containing particle 1 during $[s, s-ds]$ equals $N^*(s)\ ds$. Now $N^*_k(s) \uparrow N^*_\infty(s)$, say, in probability, and it suffices to show that $\int_y^{y+\delta} N^*_\infty(s)\ ds = \infty$. But as s decreases, $N^*_k(s)$ is the non-homogeneous Markov process with transition rates

$$\begin{array}{ll} n \to n+1 & \text{rate } k\ell(s) \\ n \to n-1 & \text{rate } \binom{n}{2}/\ell(s). \end{array}$$

Clearly N^*_∞ cannot be bounded throughout any interval of time, implying $N^*_\infty(s) = \infty$ on $y+\delta > s \ge y$. Letting $k \to \infty$ in (12) establishes (11).

3 Proof of Theorem 2

3.1 Discrete trees and Brownian excursion

Here we recall a background result, Theorem 3, needed in the next section. Consider a tree in the usual combinatorial sense, with each edge having length 1. There is a classical one-to-one correspondence between rooted ordered trees on m vertices and walk-excursions $\mathbf{w} = (0 = w(0), w(1), \ldots, w(2m) = 0)$ with $w(i) > 0, 1 \le i \le 2m-1$ and $|w(i+1) - w(i)| = 1$. See e.g. [1] section 2.2 for details: briefly, each step $(i, i+1)$ of the walk corresponds to traversing an edge of the tree from height $w(i)$ to height $w(i+1)$, and each edge is traversed once in each direction. Call $\mathbf{w}$ the *depth-first walk* associated with the tree. Such a walk $\mathbf{w}$ may be rescaled to define $\tilde{w} \in C_{\text{exc}}[0,1]$ by setting

$$\tilde{w}(\tfrac{i}{2m}) = w(i),\ 0 \le i \le 2m, \text{ with linear interpolation over } (\tfrac{i}{2m}, \tfrac{i+1}{2m}). \tag{13}$$

Cayley's formula says there are m^{m-1} rooted trees on m labeled vertices. Let $\mathcal{T}_m$ be a uniform random rooted tree on m labeled vertices. Make $\mathcal{T}_m$ into an ordered tree by assigning uniform random order to the children of each vertex. Write $\mathbf{W}_m$ for the depth-first walk associated with $\mathcal{T}_m$, and $\widetilde{W}_m$ for its rescaling (13). Write $\mathbf{Q}_m = (1 = Q_m(0), Q_m(1), \ldots)$ where $Q_m(h)$ is the number of vertices of $\mathcal{T}_m$ at height h. Call $\mathbf{Q}_m$ the *height profile* of $\mathcal{T}_m$. Rescale $\mathbf{Q}_m$ to obtain a $D[0,\infty)$-valued process

$$\widetilde{Q}_m(s) = Q_m(\lfloor 2m^{1/2}s \rfloor),\ 0 \le s < \infty. \tag{14}$$

Theorem 3 $(\frac{1}{2}m^{-1/2}\widetilde{W}_m, 2m^{-1/2}\widetilde{Q}_m) \stackrel{d}{\to} (B, L)$, *where L is local time for standard Brownian excursion B.*

Proof. The result $\frac{1}{2}m^{-1/2}\widetilde{W}_m \stackrel{d}{\to} B$ is a special case of [2] Theorem 23. This implies an integrated form of joint convergence, as follows:

$$(\tfrac{1}{2}m^{-1/2}\widetilde{W}_m, 2m^{-1/2}I_m) \stackrel{d}{\to} (B, I) \tag{15}$$

where $I_m(s) = \int_0^s \widetilde{Q}_m(y)\ dy$ and $I(s) = \int_0^s L(y)\ dy$. The stronger assertion $2m^{-1/2}\widetilde{Q}_m \stackrel{d}{\to} L$ was proved by Drmota and Gittenberger [8]. Convergence of the marginal processes in Theorem 3 implies tightness of the joint processes, and then (15) identifies the limit and hence establishes joint convergence. ■

In Theorem 3 the convergence in distribution was for random elements of $C[0,1] \times D[0,\infty)$. Then because L is continuous and is non-zero on the interior on its support $[0, \sup_u B_u]$, we also have

$$2m^{-1/2}\widetilde{Q}_m \stackrel{d}{\to} L \text{ as random elements of } \mathcal{L}. \tag{16}$$

3.2 Compatibility with standard Brownian excursion

Write B and L for standard Brownian excursion and its local time. The assertion of Theorem 2 which remains to be proved is the "conditional law" assertion, which is equivalent to the assertion

$$(B, L) \stackrel{d}{=} (B^L, L)$$

where B^ℓ has the law specified by the representation (8). By that representation, it is enough to show that for each k

$$(\mathbf{t}(B, U_1, \ldots, U_k), L) \stackrel{d}{=} (\mathcal{R}_k^L, L) \tag{17}$$

where $\mathcal{R}_k^\ell$ is the random ordered real k-tree from section 2.1. We shall derive (17) from a simple discrete analog (19) using the weak convergence result of Theorem 3.

Write $\mathbf{q} = (q(0), \ldots, q(H))$, for integers $1 = q(0), q(1), \ldots, q(H)$ with each $q(i) \geq 1$, and let $\sum_i q(i) = m$. Define a random rooted unlabeled m-vertex tree $\mathcal{T}^{\mathbf{q}}$ as follows. For each $1 \leq i \leq H$, there are $q(i)$ vertices at height i, and each is linked to a uniform random vertex at height $i-1$, independently for each vertex. (This is just a non-homogeneous variation of the classical Wright-Fisher process, cf. [6]). Associated with the tree $\mathcal{T}^{\mathbf{q}}$ is its depth-first walk $(\mathbf{W}^{\mathbf{q}}(i), 0 \leq i \leq 2m)$ from section 3.1. Recall that $\mathcal{T}_m$ is the uniform random rooted tree on m labeled vertices, that $\mathbf{Q}_m$ is its height profile, and that $\mathbf{W}_m$ is the associated depth-first walk.

Lemma 4 *The conditional law of $\mathcal{T}_m$ given $\mathbf{Q}_m = \mathbf{q}$ is the law of $\mathcal{T}^{\mathbf{q}}$. So in particular $(\mathbf{W}^{\mathbf{Q}_m}, \mathbf{Q}_m) \stackrel{d}{=} (\mathbf{W}_m, \mathbf{Q}_m)$.*

Proof. Condition on the sets of height-i vertices of $\mathcal{T}_m$ being the sets $(A_i, i \geq 0)$. The conditional law of $\mathcal{T}_m$ is now uniform on the subset of allowable trees, i.e. trees such that each vertex $v \in A_{i+1}$ has a parent in A_i. Removing labels, it is clear this uniform law is the same as the law of $\mathcal{T}^{\mathbf{q}}$ for $\mathbf{q} = (|A_i|, i \geq 0)$. ■

Now choose uniformly at random k vertices of $\mathcal{T}^{\mathbf{q}}$, label them $\{1, \ldots, k\}$, and consider the subtree $\mathcal{S}_k^{\mathbf{q}}$ spanned by these vertices and the root. After randomly ordering the children of each vertex, we may regard $\mathcal{S}_k^{\mathbf{q}}$ as an ordered k-tree, with leaves and branchpoints at integer heights (the set

of possible trees is actually larger than in the section 2.1 definition, e.g. because branchpoints may have degree > 3, but this makes no essential difference).
Recall the one-to-one correspondence between rooted ordered trees $\mathbf{t}$ and walks $\mathbf{w}$, and consider a corresponding pair $\mathbf{t}$ and $\mathbf{w}$. For this pair there is a $2-1$ map $\phi : \{0, 1, \ldots, 2m-1\} \to \{\text{vertices of } \mathbf{t}\}$ such that each step $(i, i+1)$ of the depth-first walk corresponds to traversing the edge from $\phi(i)$ to its parent or from its parent to $\phi(i)$. And the heights of vertex $\phi(i)$ and its parent are the decreasing arrangement of $(w(i), w(i+1))$. Thus we can construct a uniform random vertex of $\mathbf{t}$ as $\phi(U_m)$, where

$$\begin{aligned} U & \quad \text{has } U[0,1] \text{ distribution} \\ U_m & = j \text{ or } j+1, \text{ where } \tfrac{j}{2m} \le U \le \tfrac{j+1}{2m} \end{aligned}$$

and the height of this random vertex is $\tilde{w}(U_m)$. Here and below the interpretation of "or" is that a certain choice makes the assertion correct. Now the subtree $\mathcal{S}_1^{\mathbf{q}}$ of $\mathcal{T}^{\mathbf{q}}$ consisting of an edge from the root to a random vertex of $\mathcal{T}^{\mathbf{q}}$ can be represented as

$$\mathbf{t}(\mathbf{W}, U_m) \stackrel{d}{=} \mathcal{S}_1^{\mathbf{q}}$$

where the definition of $\mathbf{t}(\cdot)$ from section 2.2 extends naturally from the continuous to the discrete setting. Similarly, the subtree spanned by the root and k random vertices can be represented as

$$\mathbf{t}(\mathbf{W}^{\mathbf{q}}, U_{m,1}, \ldots, U_{m,k}) \stackrel{d}{=} \mathcal{S}_k^{\mathbf{q}} \tag{18}$$

where $(U_i, 1 \le i \le k)$ are independent $U(0,1)$ and

$$U_{m,i} = j \text{ or } j+1 \text{ for } j/2m \le U_i \le (j+1)/2m.$$

In deriving (18) we use the fact that the height of the branchpoint of vertices v_1 and v_2 is the minimum of the depth-first walk between v_1 and v_2. So in particular

$$(\mathbf{t}(\mathbf{W}^{\mathbf{Q}_m}, U_{m,1}, \ldots, U_{m,k}), \mathbf{Q}_m) \stackrel{d}{=} (\mathcal{S}_k^{\mathbf{Q}_m}, \mathbf{Q}_m). \tag{19}$$

For deterministic $\mathbf{q}_m$ define $\tilde{q}_m$ as at (14). Write $\frac{1}{2}m^{-1/2}\mathbf{t}$ for the tree $\mathbf{t}$ with edge-lengths rescaled by $\frac{1}{2}m^{-1/2}$.

Lemma 5 *If $2m^{-1/2}\tilde{q}_m \to \ell$ in $\mathcal{L}$ then $\frac{1}{2}m^{-1/2}\mathcal{S}_k^{\mathbf{q}_m} \xrightarrow{d} \mathcal{R}_k^{\ell}$.*

The proof is deferred. Combining Lemma 5 and (16) we obtain convergence of the rescaled right side of (19) to the right side of (17):

$$\left(\tfrac{1}{2}m^{-1/2}\mathcal{S}_k^{\mathbf{Q}_m}, 2m^{-1/2}\widetilde{Q}_m\right) \xrightarrow{d} (\mathcal{R}_k^{L}, L).$$

By Lemma 4 $(\mathbf{W}^{\mathbf{Q}_m}, \mathbf{Q}_m) \stackrel{d}{=} (\mathbf{W}_m, \mathbf{Q}_m)$, so by Theorem 3 we obtain convergence of the rescaled left side of (19) to the left side of (17):

$$(\tfrac{1}{2}m^{-1/2}\mathbf{t}(\mathbf{W}^{\mathbf{Q}_m}, U_{m,1}, \ldots, U_{m,k}), 2m^{-1/2}\widetilde{Q}_m) \xrightarrow{d} (\mathbf{t}(B, U_1, \ldots, U_k), L).$$

Thus (17) holds as the limit of the equality (19).
Remark. Lemma 5 says that genealogies in a non-homogeneous Wright-Fisher process are converging to a non-homogeneous coalescent. In the usual population genetics setting (all particles

born at the same time) this fact provided the original motivation for studying the coalescent, and non-homogeneous versions have been studied (see [10] for references). So we will only outline the proof in our "random birth-times" setting.

Outline proof of Lemma 5. Let $\mathbf{t}$ be an ordered k-tree with shape σ for which the heights (x_i^*) of labeled leaves and the heights (b_j^*) of branchpoints are all distinct integers. Let $n(h)$ be the number of edges of $\mathbf{t}$ from height $h+1$ to height h. Write $(m)_n := m(m-1)\dots(m-n+1)$. It is easy to see

$$\begin{aligned}
P(\mathcal{S}_k^{\mathbf{q}} = \mathbf{t}) &= 2^{-(k-1)} && \text{left/right assignments}\\
\times \prod_{1\le h\le H,\ h\neq \text{ any } x_i^* \text{ or } b_j^*} & \frac{(q(h))_{n(h)}}{(q(h))^{n(h)}} && \text{distinct parents at height } h\\
&\times \frac{1}{m^k}\prod_{i=1}^{k} q(x_i^*) && \text{heights of leaves}\\
&\times \prod_{i=1}^{k} \frac{(q(x_i^*)-1)_{n(x_i^*)}}{(q(x_i^*))^{n(x_i^*)}} && \text{no branchpoint at height } x_i^*\\
&\times \prod_{j\in J_\sigma} \frac{(q(b_j^*)-1)_{n(b_j^*)-2}}{(q(b_j^*))^{n(b_j^*)-1}} && \text{one branchpoint at height } b_j^*.
\end{aligned}$$

To prove Lemma 5, let $2m^{-1/2}\tilde{q}_m \to \ell$ in $\mathcal{L}$. It is enough to show that

$$\sum_{\mathbf{t}} \left| P(\mathcal{S}_k^{\mathbf{q}_m} = \mathbf{t}) - \ (\tfrac{1}{2}m^{-1/2})^{2k-1} f_k^\ell\left(\sigma, \tfrac{\mathbf{x}^*}{2m^{1/2}}, \tfrac{\mathbf{b}^*}{2m^{1/2}}\right) \right| \to 0 \tag{20}$$

for f_k^ℓ defined at (2). Looking at terms in the formula above for $P(\mathcal{S}_k^{\mathbf{q}} = \mathbf{t})$,

$$\begin{aligned}
\text{third term} &\sim m^{-k}(\tfrac{1}{2}m^{1/2})^k \prod_i \ell\left(\tfrac{x_i^*}{2m^{1/2}}\right)\\
\text{fourth term} &\to 1\\
\text{fifth term} &\sim \prod_j \frac{2m^{-1/2}}{\ell\left(\frac{b_j^*}{2m^{1/2}}\right)}.
\end{aligned}$$

It is not hard to see that proving (20) reduces to showing that if $\frac{1}{2}m^{-1/2}\mathbf{t}_m \to \mathbf{t}$ then

$$\prod_{2m^{1/2}a\le h\le 2m^{1/2}b} \frac{(q_m(h))_{n_m(h)}}{(q_m(h))^{n_m(h)}} \to \exp\left(-\int_a^b \binom{n(s)}{2}\frac{4}{\ell(s)}\,ds\right),\ 0<a<b<s^*.$$

This in turn reduces to showing

$$\int_a^b \left| \frac{2m^{1/2}}{q_m(\lfloor 2m^{1/2}s\rfloor)} - \frac{4}{\ell(s)} \right| ds \to 0$$

which is a consequence of $2m^{-1/2}\tilde{q}_m \to \ell$ in $\mathcal{L}$.

3.3 Why did the construction work?

The central mathematical idea is Lemma 4. Consider the random m-tree and its height profile process. One can associate this with the depth-first walk and its local time process, and also one can associate this with the non-homogeneous Wright-Fisher process. Taking weak limits enables us to associate Brownian excursion and its local time with the non-homogeneous coalescent. This idea was the motivation for the construction of the non-homogeneous coalescent. It is remarkable that, while Lemma 4 is almost obvious in the discrete setting, there seems no way to state a continuous space analog directly.

4 The bridge setting

Define $C_{\mathrm{bridge}}[0,1]$ by relaxing requirement (1) of $C_{\mathrm{exc}}[0,1]$ to

$$f(0) = f(1) = 0, \ f(u) \geq 0 \text{ for } 0 < u < 1.$$

Define $\mathcal{L}^*$ by removing from the definition of $\mathcal{L}$ the requirement (iv). Construction 1 can be extended to $\ell \in \mathcal{L}^*$, provided we allow the root of $\mathcal{R}_k^\ell$ to have arbitrary degree, and we obtain a $C_{\mathrm{bridge}}[0,1]$ -valued process B^ℓ. And Theorem 2 remains true, with standard Brownian excursion replaced by standard reflecting Brownian bridge.
These assertions can be proved by minor modifications of the proofs in this paper and [2] Theorem 15 – we omit details.

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