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EQUIDISTANT SAMPLING FOR THE MAXIMUM OF A BROWNIAN MOTION WITH DRIFT ON A FINITE HORIZON

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Abstract

A Brownian motion observed at equidistant sampling points renders a random walk with normally distributed increments. For the difference between the expected maximum of the Brownian motion and its sampled version, an expansion is derived with coefficients in terms of the drift, the Riemann zeta function and the normal distribution function.

1 Introduction

Let $\{B(t)\}_{t\geq 0}$ denote a Brownian motion with drift coefficient μ and variance parameter σ^2, so that

$$B(t) = \mu t + \sigma W(t), \tag{1}$$

with $\{W(t)\}_{t\geq 0}$ a Wiener process (standard Brownian motion). Without loss of generality, we set $B(0) = 0$, $\sigma = 1$ and consider the Brownian motion on the interval $[0, 1]$. When we sample the Brownian motion at time points $\frac{n}{N}$, $n = 0, 1, \ldots N$, the resulting process is a random walk with normally distributed increments (Gaussian random walk). The fact that Brownian motion evolves in continuous space and time leads to great simplifications in determining its properties. In contrast, the Gaussian random walk, moving only at equidistant points in time, is an object much harder to study. Although it is obvious that, for $N \to \infty$, the behavior of the Gaussian random walk can be characterized by the continuous time diffusion equation, there are many effects to take into account for finite N. This paper deals with the expected maximum of the Gaussian random

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walk and, in particular, its deviation from the expected maximum of the underlying Brownian motion. This relatively simple characteristic already turns out to have an intriguing description. In Section 2 we derive an expansion with coefficients in terms of the Riemann zeta function and (the derivatives of) the normal distribution function. Some historical remarks follow, and the proof is presented in Section 3.

2 Main result and discussion

By Spitzer's identity (see [19, 14]) we have

$$\mathbb{E} \max_{n=0,\ldots,N} B(n/N) = \sum_{n=1}^{N} \frac{1}{n} \mathbb{E}B^{+}(n/N), \tag{2}$$

where $B^{+}(t) = \max\{0, B(t)\}$. The monotone convergence theorem, in combination with a Riemann sum approximation of the right-hand side of (2), gives (see [1])

$$\mathbb{E} \max_{0\le t\le 1} B(t) = \int_0^1 \frac{1}{t} \mathbb{E}B^{+}(t)dt. \tag{3}$$

The mean sampling error, as a function of the number of sampling points is then given by

$$\mathbb{E}\Delta_N(\mu) = \int_0^1 \frac{1}{t} \mathbb{E}B^{+}(t)dt - \sum_{n=1}^{N} \frac{1}{n} \mathbb{E}B^{+}(n/N). \tag{4}$$

Since $B(t)$ is normally distributed with mean μt and variance t one can compute

$$\mathbb{E}B^{+}(t) = \mu t \Phi(\mu\sqrt{t}) + \left(\frac{t}{2\pi}\right)^{1/2} e^{-\frac{1}{2}\mu^2 t}, \tag{5}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}u^2} du$. Substituting (5) into (4) yields

$$\mathbb{E}\Delta_N(\mu) = \int_0^1 g(t)dt - \frac{1}{N} \sum_{n=1}^{N} g(n/N), \tag{6}$$

where

$$g(t) = \mu \Phi(\mu\sqrt{t}) + \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}\mu^2 t}. \tag{7}$$

We are then in the position to present our main result.

Theorem 1. *The difference in expected maximum between $\{B(t)\}_{0\le t\le 1}$ and its associated Gaussian random walk obtained by sampling $\{B(t)\}_{0\le t\le 1}$ at N equidistant points, for $|\mu/\sqrt{N}| < 2\sqrt{\pi}$, is given by*

$$\begin{aligned}\mathbb{E}\Delta_N(\mu) = &-\frac{\zeta(1/2)}{\sqrt{2\pi N}} - \frac{2g(1)-\mu}{4N} - \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!} \frac{g^{(2k-1)}(1)}{N^{2k}} \\ &- \frac{1}{\sqrt{2\pi N}} \sum_{r=0}^{\infty} \frac{\zeta(-1/2-r)(-1/2)^r}{r!(2r+1)(2r+2)} \left(\frac{\mu}{\sqrt{N}}\right)^{2r+2} + O(1/N^{2p+2}),\end{aligned} \tag{8}$$

with O uniform in μ, ζ the Riemann zeta function, p some positive integer, B_n the Bernoulli numbers, and $g^{(k)}$ defined as the kth derivative of g in (7).

$\mathbb{E}\Delta_N(\mu)$ shows up in a range of applications. Examples are sequentially testing for the drift of a Brownian motion [7], corrected diffusion approximations [17], simulation of Brownian motion [1, 5], option pricing [3], queueing systems in heavy traffic [12, 13, 15], and the thermodynamics of a polymer chain [8].
The expression in (8) for $\mathbb{E}\Delta_N(\mu)$ involves terms $c_j N^{-j/2}$ with

$$c_1 = -\frac{\zeta(1/2)}{\sqrt{2\pi}},\ c_2 = -\frac{\mu - 2\mu\Phi(-\mu) + 2\phi(\mu)}{4},\ c_3 = -\frac{\zeta(-1/2)\mu^2}{2\sqrt{2\pi}},\ c_4 = \frac{\phi(\mu)}{24}, \tag{9}$$

$\phi(x) = e^{-x^2/2}/\sqrt{2\pi}$ and $c_j = 0$ for $j = 6, 10, 14, \ldots$. The first term c_1 has been identified by Asmussen, Glynn & Pitman [1], Thm. 2 on p. 884, and Calvin [5], Thm. 1 on p. 611, although Calvin does not express c_1 in terms of the Riemann zeta function. The second term c_2 was derived by Broadie, Glasserman & Kou [3], Lemma 3 on p. 77, using extended versions of the Euler-Maclaurin summation formula presented in [1]. To the best of the authors' knowledge, all higher terms appear in the present paper for the first time.
The distribution of the maximum of Brownian motion with drift on a finite interval is known to be (see Shreve [18], p. 297)

$$\mathbb{P}(\max_{0\le t\le T} B(t) \le x) = \Phi\Big(\frac{x-\mu T}{\sqrt{T}}\Big) - e^{2\mu x}\Phi\Big(\frac{-x-\mu T}{\sqrt{T}}\Big), \quad x \ge 0, \tag{10}$$

and integration thus yields

$$\mathbb{E}(\max_{0\le t\le T} B(t)) = \frac{1}{2\mu}(2\Phi(\mu\sqrt{T}) - 1) + \Phi(\mu\sqrt{T})\mu T + \phi(\mu\sqrt{T})\sqrt{T}. \tag{11}$$

A combination of (11) and (8) leads to a full characterization of the expected maximum of the Gaussian random walk. Note that the mean sampling error for the Brownian motion defined in (1) on $[0, T]$, sampled at N equidistant points, is given by $\sigma\sqrt{T}\cdot\mathbb{E}\Delta_N(\mu\sqrt{T}/\sigma)$.
When the drift μ is negative, results can be obtained for the expected all-time maximum. That is, for the special case $\mu < 0$, $\sigma = 1$, $T = N$ and $N \to \infty$, one finds that $\lim_{N\to\infty}\sqrt{N}\cdot\mathbb{E}\Delta_N(\mu\sqrt{N})$ is equal to

$$-\frac{\zeta(1/2)}{\sqrt{2\pi}} + \frac{1}{4}\mu - \frac{\mu^2}{\sqrt{2\pi}}\sum_{r=0}^{\infty}\frac{\zeta(-1/2-r)}{r!(2r+1)(2r+2)}\left(\frac{-\mu^2}{2}\right)^r, \tag{12}$$

for $-2\sqrt{\pi} < \mu < 0$. Note that (12) follows from Theorem 1. The result, however, was first derived by Pollaczek [16] in 1931 (see also [11]). Apparently unaware of this fact, Chernoff [7] obtained the first term $-\zeta(1/2)/\sqrt{2\pi}$, Siegmund [17], Problem 10.2 on p. 227, obtained the second term 1/4 and Chang & Peres [6], p. 801, obtained the third term $-\zeta(-1/2)/2\sqrt{2\pi}$. The complete result was rediscovered by the authors in [9], and more results for the Gaussian random walk were presented in [9, 10], including series representations for all cumulants of the all-time maximum.

3 Proof of Theorem 1

We shall treat separately the cases $\mu < 0$, $\mu > 0$ and $\mu = 0$. The proof for $\mu < 0$ in Subsection 3.1 largely builds upon Euler-Maclaurin summation and the result in Section 4 of [9] on the expected value of the all-time maximum of the Gaussian random walk. The result for $\mu > 0$ in Subsection

3.2 then follows almost immediately due to convenient symmetry properties of Φ. Finally, in Subsection 3.3, the issue of uniformity in μ is addressed and the result for $\mu = 0$ is established in two ways: First by taking the limit $\mu \uparrow 0$ and subsequently by a direct derivation that uses Spitzer's identity (4) for $\mu = 0$ and an expression for the Hurwitz zeta function.

3.1 The negative-drift case

Set $\mu = -\gamma$ with $\gamma > 0$. We have from (6)

$$\mathbb{E}\Delta_N(\mu) = \left\{ \int_0^\infty g(t)dt - \frac{1}{N}\sum_{n=1}^\infty g(n/N) \right\} - \left\{ \int_1^\infty g(t)dt - \frac{1}{N}\sum_{n=N+1}^\infty g(n/N) \right\}. \tag{13}$$

We compute by partial integration

$$\begin{aligned}\int_0^\infty g(t)dt &= -\int_0^\infty \gamma\Phi(-\gamma\sqrt{t})dt + \frac{1}{\sqrt{2\pi}}\int_0^\infty t^{-1/2}e^{-\frac{1}{2}\gamma^2 t}dt \\ &= \frac{-1}{2\gamma} + \frac{1}{\gamma} = \frac{1}{2\gamma}.\end{aligned} \tag{14}$$

Furthermore, with $\beta = \gamma/\sqrt{N}$,

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^\infty g(n/N) &= \frac{1}{N}\sum_{n=1}^\infty \Big(-\gamma\Phi(-\gamma\sqrt{n}/\sqrt{N}) + \frac{1}{\sqrt{2\pi n/N}}e^{-\frac{1}{2}\gamma^2 n/N} \Big) \\ &= \frac{1}{\sqrt{N}}\sum_{n=1}^\infty \Big(\frac{e^{-\frac{1}{2}\beta^2 n}}{\sqrt{2\pi n}} - \beta\Phi(-\beta\sqrt{n}) \Big) = \frac{\mathbb{E}M}{\sqrt{N}},\end{aligned} \tag{15}$$

with $\mathbb{E}M$ as in (4.1) of [9]. From (14), (15) and [9], (4.25), it follows that

$$\begin{aligned}\int_0^\infty g(t)dt - \frac{1}{N}\sum_{n=1}^\infty g(n/N) &= \frac{-\zeta(1/2)}{\sqrt{2\pi N}} - \frac{\gamma}{4N} \\ &\quad - \frac{1}{\sqrt{2\pi N}}\sum_{r=0}^\infty \frac{\zeta(-1/2-r)(-1/2)^r}{r!(2r+1)(2r+2)}\left(\frac{\gamma}{\sqrt{N}}\right)^{2r+2}.\end{aligned} \tag{16}$$

This handles the first term on the right-hand side of (13).

For the second term, we use Euler-Maclaurin summation (see De Bruijn [4], Sec. 3.6, pp. 40-42) for the series $\frac{1}{N}\sum_{n=N+1}^\infty g(n/N)$. With

$$f(x) = \frac{1}{N}g(x/N), \quad x \geq N, \tag{17}$$

we have for $p = 1, 2, \dots$

$$\begin{aligned}\sum_{n=N+1}^{\infty} f(n) &= -f(N) + \sum_{n=N}^{\infty} f(n) \\ &= -f(N) + \lim_{M\to\infty} \left[\int_N^M f(x)dx + \tfrac{1}{2} f(N) + \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(M) - f^{(2k-1)}(N) \right) \right. \\ &\quad \left. - \int_N^M f^{(2p)}(x) \frac{B_{2p}(x - \lfloor x \rfloor)}{(2p)!} dx \right] \\ &= -\tfrac{1}{2} f(N) + \int_N^{\infty} f(x)dx - \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(N) + R_{p,N},\end{aligned} \tag{18}$$

where $B_n(t)$ denotes the nth Bernoulli polynomial, $B_n = B_n(0)$ denotes the nth Bernoulli number, and

$$R_{p,N} = -\int_N^{\infty} f^{(2p)}(x) \frac{B_{2p}(x - \lfloor x \rfloor)}{(2p)!} dx. \tag{19}$$

Since $f^{(l)}(x) = g^{(l)}(x/N)/N^{l+1}$, we thus obtain

$$\frac{1}{N} \sum_{n=N+1}^{\infty} g(n/N) = \frac{-1}{2N} g(1) + \int_1^{\infty} g(x)dx - \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!} \frac{1}{N^{2k}} g^{(2k-1)}(1) + R_{p,N}, \tag{20}$$

where

$$R_{p,N} = -\frac{1}{N^{2p}} \int_1^{\infty} g^{(2p)}(x) \frac{B_{2p}(Nx - \lfloor Nx \rfloor)}{(2p)!} dx. \tag{21}$$

From the definition of g in (7) it is seen that $g^{(2p)}$ is smooth and rapidly decaying, hence $R_{p,N} = O(1/N^{2p})$. Since

$$R_{p,N} = -\frac{B_{2p+2}}{(2p+2)!} \frac{1}{N^{2p+2}} g^{(2p+1)}(1) + R_{p+1,N}, \tag{22}$$

we even have $R_{p,N} = O(1/N^{2p+2})$. Therefore, from (20),

$$\int_1^{\infty} g(t)dt - \frac{1}{N} \sum_{n=N+1}^{N} g(n/N) = \frac{1}{2N} g(1) + \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!} \frac{1}{N^{2k}} g^{(2k-1)}(1) + O(1/N^{2p+2}). \tag{23}$$

Combining (16) and (23) completes the proof, aside from the uniformity issue, for the case that $\mu = -\gamma < 0$.

3.2 The positive-drift case

The analysis so far was for the case with negative drift $\mu = -\gamma$ with $\gamma > 0$. The results can be transferred to the case that $\mu > 0$ as follows. Note first from $\Phi(-x) = 1 - \Phi(x)$ that $g(t) = \mu - \Phi(-\mu\sqrt{t}) + (2\pi t)^{-1/2} \exp(-\frac{1}{2}(-\mu)^2 t)$. Therefore, by (6)

$$\mathbb{E}\Delta_N(\mu) = \mathbb{E}\Delta_N(-\mu), \tag{24}$$

since the term μ vanishes from the right-hand side of (6). Then use the result already proved with $-\mu < 0$ instead of μ. This requires replacing $g(t)$ from (7) by

$$-\mu\Phi(-\mu\sqrt{t}) + (2\pi t)^{-1/2} e^{-\frac{1}{2}(-\mu)^2 t} \tag{25}$$

and μ by $-\mu$ everywhere in (8). The term $2g(t) - \mu$ then becomes

$$\begin{aligned} &2\left(-\mu\Phi(-\mu\sqrt{t}) + (2\pi t)^{-1/2}\exp(-\tfrac{1}{2}(-\mu)^2 t)\right) - (-\mu) = \\ &2\left(\mu\Phi(\mu\sqrt{t}) + (2\pi t)^{-1/2}\exp(-\tfrac{1}{2}\mu^2 t)\right) - \mu, \end{aligned} \tag{26}$$

which is in the form $2g(t) - \mu$ with g from (7). Next we compute

$$\begin{aligned} g'(t) &= \frac{-1}{2\sqrt{2\pi}} t^{-3/2} e^{-\frac{1}{2}\mu^2 t} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left[-\mu\Phi(-\mu\sqrt{t}) + (2\pi t)^{-1/2} e^{-\frac{1}{2}(-\mu)^2 t}\right]. \end{aligned} \tag{27}$$

Finally, the infinite series with the ζ-function involves μ quadratically. Thus writing down (8) with $-\mu < 0$ instead of μ turns the right-hand side into the same form with g given by (7). This completes the proof of Theorem 1 for $\mu \neq 0$.

3.3 The zero-drift case

We shall first establish the uniformity in $\mu < 0$ of the error term O in (8), for which we need that

$$R_{p,N} = \frac{-1}{N^{2p}} \int_1^\infty g^{(2p)}(x) \frac{B_{2p}(Nx - \lfloor Nx \rfloor)}{(2p)!} dx \tag{28}$$

can be bounded uniformly in $\mu < 0$ as $O(N^{-2p})$. Write $\nu = \frac{1}{2}\mu^2$, and observe from (27) and Newton's formula that for $k = 1, 2, \ldots$

$$\begin{aligned} g^{(k)}(t) &= \frac{-1}{2\sqrt{2\pi}} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{k-1} \left[t^{-3/2} e^{-\nu t}\right] \\ &= \frac{(-1)^k}{2\sqrt{2\pi}} e^{-\nu t} \sum_{n=0}^{k-1} \binom{k-1}{n} \tfrac{3}{2} \cdot \tfrac{5}{2} \cdot \cdots \cdot (\tfrac{3}{2} + n - 1) \nu^{k-1-n} t^{-3/2-n}. \end{aligned} \tag{29}$$

Hence, $g^{(2p)}(t) > 0$ and $g^{(2p-1)}(1) < 0$ for $p = 1, 2, \ldots$. Therefore, with C an upper bound for

$$\left|B_{2p}(Nx - \lfloor Nx \rfloor)/(2p)!\right|, \tag{30}$$

we have

$$\begin{aligned} |R_{p,N}| &\leq \frac{C}{N^{2p}} \int_1^\infty g^{(2p)}(t) dt \;=\; -\frac{C}{N^{2p}} g^{(2p-1)}(1) \\ &= \frac{C}{N^{2p}} \frac{e^{-\nu}}{2\sqrt{2\pi}} \sum_{n=0}^{2p-2} \binom{2p-2}{n} \tfrac{3}{2} \cdot \tfrac{5}{2} \cdot \cdots \cdot (\tfrac{3}{2} + n - 1) \nu^{2p-2-n}, \end{aligned} \tag{31}$$

which is bounded in $\nu > 0$ when $p = 1, 2\ldots$ is fixed. This settles the uniformity issue and thus the case $\mu = 0$ by letting $\mu \uparrow 0$.

A direct derivation of the result (8) for the case $\mu = 0$ is also possible. When $\zeta(s,x)$ is the analytic continuation to $\mathbb{C} \setminus \{-1\}$ of the function

$$\zeta(s,x) = \sum_{n>-x} (n+x)^{-s}, \quad \mathrm{Re}(s) > 1, \ x \in \mathbb{R}, \tag{32}$$

then for $s \in \mathbb{C} \setminus \{-1\}$ and $p = 1,2,\ldots$ with $2p+1 > -\mathrm{Re}(s)$, there holds (see Borwein, Bradley & Crandall [2], Section 3, for similar expressions)

$$\begin{aligned}\zeta(s,x) = \sum_{-x<n\le N} (n+x)^{-s} - \frac{(x+N)^{1-s}}{1-s} - \tfrac{1}{2}(x+N)^{-s} \\ - \sum_{k=1}^{p} \binom{1-s}{2k} \frac{B_{2k}}{1-s} (x+N)^{-s-2k+1} + O(N^{-s-2p-1}).\end{aligned} \tag{33}$$

Combination of

$$\mathbb{E}\Delta_N(0) = \frac{1}{\sqrt{2\pi}} \left(2 - \frac{1}{N^{1/2}} \sum_{n=1}^{N} n^{-1/2} \right) \tag{34}$$

and (33) with $s = 1/2$, $x = 1$ and N replaced by $N-1$, leads to

$$\mathbb{E}\Delta_N(0) = -\frac{\zeta(1/2)}{\sqrt{2\pi N}} - \frac{1}{2N\sqrt{2\pi}} - \frac{2}{\sqrt{2\pi}} \sum_{k=1}^{p} \binom{1/2}{2k} B_{2k} N^{-2k} + O(N^{-2p-2}). \tag{35}$$

Note that

$$\frac{2}{\sqrt{\pi}} \binom{1/2}{2k} B_{2k} = \frac{B_{2k}}{(2k)!} h^{(2k-1)}(t)\Big|_{t=1} \quad ; \quad h(t) = \frac{1}{\sqrt{2\pi t}}, \tag{36}$$

and so (35) corresponds to (8) with $\mu = 0$, indeed.

References

[1] Asmussen, S., Glynn, P. and Pitman, J. (1995). Discretization error in simulation of one-dimensional reflecting Brownian motion. *Annals of Applied Probability* **5**: 875-896. MR1384357

[2] Borwein, J.M., Bradley, D.M., Crandall, R.E. (2000). Computational strategies for the Riemann zeta function. *Journal of Computational and Applied Mathematics* **121**: 247-296. MR1780051

[3] Broadie, M., Glasserman, P. and Kou, S.G. (1999). Connecting discrete and continuous path-dependent options. *Finance and Stochastics* **3**: 55-82. MR1805321

[4] De Bruijn, N.G. (1981). *Asymptotic Methods in Analysis*. Dover Publications, New York. MR0671583

[5] Calvin, J. (1995). Average performance of nonadaptive algorithms for global optimization. *Journal of Mathematical Analysis and Applications* **191**: 608-617. MR1325821

[6] Chang, J.T. and Peres, Y. (1997). Ladder heights, Gaussian random walks and the Riemann zeta function. *Annals of Probability* **25**: 787-802. MR1434126

[7] Chernoff, H. (1965). Sequential test for the mean of a normal distribution IV (discrete case). *Annals of Mathematical Statistics* **36**: 55-68. MR0170443

[8] Comtet, A. and Majumdar, S.N. (2005). Precise asymptotics for a random walker's maximum. *Journal of Statistical Mechanics: Theory and Experiment*, P06013.

[9] Janssen, A.J.E.M. and van Leeuwaarden, J.S.H. (2007). On Lerch's transcendent and the Gaussian random walk. *Annals of Applied Probability* **17**: 421-439. MR2308331

[10] Janssen, A.J.E.M. and van Leeuwaarden, J.S.H. (2007). Cumulants of the maximum of the Gaussian random walk. *Stochastic Processes and Their Applications* 117: 1928-1959. MR2437736

[11] Janssen, A.J.E.M. and van Leeuwaarden, J.S.H. (2008). Back to the roots of queueing theory and the works of Erlang, Crommelin and Pollaczek. *Statistica Neerlandica* 62: 299-313. MR2441855

[12] Janssen, A.J.E.M., van Leeuwaarden, J.S.H. and Zwart, B. (2008). Corrected diffusion approximations for a multi-server queue in the Halfin-Whitt regime. *Queueing Systems* 58: 261-301. MR2415323

[13] Jelenković, P., Mandelbaum, A. and Momčilović, P. (2004). Heavy traffic limits for queues with many deterministic servers. *Queueing Systems* **47**: 54-69. MR2074672

[14] Kingman, J.F.C. (1962). Spitzer's identity and its use in probability theory. *Journal of the London Mathematical Society* **37**: 309-316. MR0143262

[15] Kingman, J.F.C. (1965). The heavy traffic approximation in the theory of queues. In: *Proc. Symp. on Congestion Theory*, eds. W.L. Smith and W. Wilkinson (University of North Carolina Press, Chapel Hill) pp. 137-169. MR0198566

[16] Pollaczek, F. (1931). Über zwei Formeln aus der Theorie des Wartens vor Schaltergruppen. *Elektrische Nachrichtentechnik* **8**: 256-268.

[17] Siegmund, D. (1985). *Corrected Diffusion Approximations in Certain Random Walk Problems.* Springer-Verlag, New York.

[18] Shreve, S.E. (2004). *Stochastic Calculus for Finance II. Continuous-Time Models.* Springer-Verlag, New York. MR2057928

[19] Spitzer, F.L. (1956). A combinatorial lemma and its application to probability theory. *Transactions of the American Mathematical Society* **82**: 323-339. MR0079851