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EXCITED RANDOM WALK

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Abstract
A random walk on $\mathbb{Z}^d$ is excited if the first time it visits a vertex there is a bias in one direction, but on subsequent visits to that vertex the walker picks a neighbor uniformly at random. We show that excited random walk on $\mathbb{Z}^d$ is transient iff $d > 1$.

1. Excited Random Walk

A random walk on $\mathbb{Z}^d$ is excited (with bias ε/d) if the first time it visits a vertex it steps right with probability $(1+\varepsilon)/(2d)$ $(\varepsilon > 0)$, left with probability $(1-\varepsilon)/(2d)$, and in other directions with probability $1/(2d)$, while on subsequent visits to that vertex the walker picks a neighbor uniformly at random. This model was studied heavily in the framework of perturbing 1-dimensional Brownian motion, see for instance [5, 14] and reference therein. Excited random walk falls into the notorious wide category of self-interacting random walks, such as reinforced random walk, or self-avoiding walks. These models are difficult to analyze in general. The reader should consult [4, 11, 16, 15, 1], and especially the survey paper [13] for examples. Simple coupling and an additional neat observation allow us to prove that excited random walk is recurrent only in dimension 1. The proof uses and studies a special set of points ("tan points") for the simple random walk.

2. Recurrence in $\mathbb{Z}^1$

It is already known that excited random walk in $\mathbb{Z}^1$ is recurrent, indeed, a great deal more is known about it [6]. But for the reader's convenience we provide a short proof.
On the first visit to a vertex there is probability $p > 1/2$ of going right and $1-p$ of going left, while on subsequent visits the probabilities are $1/2$. Suppose that the walker is at $x > 0$ for the first time, and that all vertices between 0 and x have been visited. The probability that the walker goes to $x+1$ before going to 0 is $p + (1-p)(1-2/(x+1)) = 1-2(1-p)/(x+1)$. Multiplying over the x's, we see that the random walk returns to 0 with probability 1.

3. Transience in $\mathbb{Z}^2$

The simple random walk (SRW) in $\mathbb{Z}^2$ visits about $n/\log n$ points by time n, and if the excited random walk (ERW) gets pushed to the right $n/\log n$ times, it would very quickly depart its start location and never return. But it is not clear what effect that the perturbations have on the number of visited points, and it is not obvious that the excited random walk will visit $n/\log n$ distinct points by time n.

To lower bound the number of points that the excited random walk visits, we couple it with the SRW in the straightforward way, and count the number of "tan points" visited by the SRW. We define the coupling as follows: if the SRW moves up, down, or right, then so does the ERW. If the SRW moves left, then the ERW moves left if it is at a previously visited point, and if the ERW is at a new point, it moves either left or right with suitable probabilities. At all times, the y-coordinates of the SRW and ERW are identical.

To explain the concept of a "tan point", we imagine that the simple random walker leaves behind an opaque trail, and that the sun is shining from infinitely far away in the positive x-direction. If the SRW visits a point (x, y) such that no point (x', y) with $x' > x$ has been visited, then the sun shines upon (x, y), and this point becomes tanned. Formally, we define a *tan point* for the SRW to be a vertex (x, y) that is visited by the SRW before any point of the form (x', y) with $x' > x$. If the sun shines upon the simple random walker the first time it is at (x, y), it is straightforward to check that ERW is at a new point. We will show that with high probability there are many tan points (so the ERW visits many new points), and that this implies that the ERW is transient.

The probability that a point (x, y) will be tan follows from some enumerative work of Bousquet-Mélou and Schaeffer on random walks in the slit plane [3].

Lemma 1. *Let r and θ be the polar coordinates of the point (x, y), i.e. $r \geq 0$, $0 \leq \theta < 2\pi$, $x = r\cos\theta$, and $y = r\sin\theta$. Then*

$$\Pr[(x, y) \textit{ is tan}] = (1 + o(1))\sqrt{\frac{1+\sqrt{2}}{2\pi}\frac{\sin(\theta/2)}{\sqrt{r}}}, \tag{1}$$

where the $o(1)$ term goes to 0 as r tends to ∞.

This equation does not explicitly appear in [3], but all the real work that goes into proving it is in [3]. In the interest of completeness, we explain how this equation follows from explicit results in [3]:

Proof. Let a_n be the number of walks of length n that start from $(0, 0)$, and avoid the non-negative real axis at all subsequent times, and let $p_{x,y,n}$ denote the probability that a random such walk ends at the point (x, y). By reversibility of the random walks,

$$\Pr\left[\begin{array}{r}\text{SRW started from the point } (x,y) \text{ first hits the}\\ \text{nonnegative real axis at the point } (0,0) \text{ and at time } n\end{array}\right] = \frac{a_n}{4^n} \times p_{x,y,n}.$$

Thus $\Pr[(x, y) \text{ is tan}] = \sum_{n=0}^{\infty} a_n/4^n \times p_{x,y,n}$. Theorem 1 of [3] gives

$$\frac{a_n}{4^n} = (1 + o(1))\frac{\sqrt{1+\sqrt{2}}}{2\Gamma(3/4)}n^{-1/4}.$$

Theorem 21 of [3] considers the endpoint (X_n, Y_n) of a random walk started from $(0, 0)$ which avoids the nonnegative real axis, and gives the limiting distribution of $(X_n/\sqrt{n}, Y_n/\sqrt{n})$. This limiting distribution morally determines the asymptotics of $p_{x,y,n}$ — a *local* limit theorem *would* determine the asymptotics — and the authors prove a local limit theorem for $Y_n/\sqrt{n}$ but not $X_n/\sqrt{n}$ let alone the joint distribution $(X_n/\sqrt{n}, Y_n/\sqrt{n})$. However, since the ordinary random walk has a local limit theorem (on vertices such that $x+y \equiv n \mod 2$), one can take the limiting distribution of $X_{(1-\varepsilon)n}, Y_{(1-\varepsilon)n}$ and then run the walk another εn steps; upon sending ε to 0 sufficiently slowly, one can obtain a local limit theorem version of Theorem 21 of [3]:

$$p_{x,y,n} = (1+o(1))\frac{2}{\Gamma(1/4)}\frac{r^{1/2}}{n^{5/4}}e^{-r^2/n}\sin(\theta/2)\times\begin{cases}2 & x+y\equiv n \mod 2\\ 0 & x+y\not\equiv n \mod 2\end{cases}$$

when $r=\Theta(\sqrt{n})$ and θ is bounded away from 0 and 2π. Thus we obtain

$$\begin{aligned}\sum_{n=\Theta(r^2)}\frac{a_n}{4^n}\times p_{x,y,n} &= (1+o(1))\int_0^\infty \frac{\sqrt{1+\sqrt{2}}}{2\Gamma(3/4)}n^{-1/4}\times\frac{2}{\Gamma(1/4)}\frac{r^{1/2}}{n^{5/4}}e^{-r^2/n}\sin(\theta/2)\,dn\\ &= (1+o(1))\sqrt{\frac{1+\sqrt{2}}{2\pi}}\frac{\sin(\theta/2)}{\sqrt{r}}.\end{aligned}$$

Next we check that the terms when $n \ll r^2$ or $n \gg r^2$ contribute negligibly. We may bound $\frac{a_n}{4^n}\times p_{x,y,n}$ by the probability that the walk survives the first $n/2$ steps $(O(1/n^{1/4}))$ times the probability that ordinary SRW for the remaining $n/2$ steps ends at the point (x,y) $(O(1/n))$. Thus

$$\sum_{n\gg r^2}\frac{a_n}{4^n}\times p_{x,y,n}\le\sum_{n\gg r^2}O(1/n^{5/4})\ll 1/r^{1/2}$$

When $n\ll r^2$ we bound $\frac{a_n}{4^n}\times p_{x,y,n}$ by the probability that the walk makes it out to radius $r/2$ without hitting the line (which is $O(1/r^{1/2})$ by [11, Eqn 2.40]) times the probability that the walk ends up at (x,y) (distance $s\ge r/2$ away) at the end of the remaining $m<n$ steps. This latter probability is at most $\frac{1+o(1)}{\pi m}e^{-s^2/(2m)}$, and assuming $n\le r^2/8$, it is upper bounded by $\frac{1+o(1)}{\pi n}e^{-r^2/(8n)}$. Thus

$$\sum_{n\ll r^2}\frac{a_n}{4^n}\times p_{x,y,n}\le O(1/r^{1/2})\times\sum_{n\ll r^2}O(n^{-1}e^{-r^2/(8n)})\ll 1/r^{1/2}$$

Hence when θ is bounded away from 0 and 2π and r is large, the terms when $n\ll r^2$ or $n\gg r^2$ contribute negligibly, so the formula follows in this case. But the formula in the limiting case when θ approaches 0 or 2π follows from the case when θ is bounded away from 0 and 2π, so the formula is valid simply when r is large enough. □

We will use the notation introduced by Knuth where $\Theta(f)$ denotes an expression which is upper-bounded by $C\times f$ and lower-bounded by $c\times f$, where c and C are positive constants. (By contrast, $O(f)$ denotes an expression for which there is an upper bound of $C\times f$, but not necessarily any lower bound.) Using this notation, we may crudely approximate Equation (1) with

$$\Pr[(x,y)\text{ is tan}]=\begin{cases}\Theta(1/r^{1/2}) & \text{when } x\ge 0\\ \Theta(|y|/r^{3/2}) & \text{when } x\le 0.\end{cases}\qquad(2)$$

Equation (2) is useful, but we need two modifications. For our purposes, it would be better to have the probability that a point is tan and that it is reached by time n. Then we would have the expected number of tan points by time n, which would lower-bound the expected number of times that the ERW is pushed to the right. But a lower bound on the expected number is not quite what we need to prove transience of the ERW; what we'd really like to know is that with very high probability, the number of tan points by time n is large.

In order to get the "with very high probability" part of the statement, it would be convenient to be working with independent events. To get this independence, we divide the plane into bands of height $h = h(n)$ to be determined later, and we will focus on every other band, say the even ones. When the SRW first arrives at an even band, let us count the tan points within the band that are encountered before the random walk reaches a different even band. These counts are independent for the different bands. After n steps it is likely that order $\sqrt{n}/h$ even bands have been crossed, so it is likely that the number of tan points dominates a sum of $\Theta(\sqrt{n}/h)$ independent random variables.

Lemma 2. *Consider a band of height h, i.e., $\mathbb{Z} \times [y_0, y_0 + h - 1]$. After the SRW first reaches this band, with probability $\Theta(1)$ the SRW hits $\Theta(h^{3/2})$ tan points within the band before leaving the enclosing band $\mathbb{Z} \times [y_0 - h, y_0 + 2h - 1]$.*

For the $\Theta(1)$ part of this lemma, the following proposition is useful.

Proposition 3. *If X is a real-valued random variable and $\mathbb{E}[X] \geq 0$, then*

$$\Pr\left[X \geq \tfrac{1}{2}\mathbb{E}[X]\right] \geq \frac{\mathbb{E}[X]^2}{4\mathbb{E}[X^2]}.$$

This proposition is essentially exercise 1.3.8 of Durrett [8].

Proof of Lemma 2. Consider a point (x, y) within the band, and the ray $[x, \infty) \times y$ with (x, y) at its tip. Consider the largest circle contained within the enclosing band and centered at the tip, and also a small enough disk centered at the tip. From Equation (2) it follows that we can take the ratios of the two radii to be $\Theta(1)$ and have the property that for any point p in the left half of the small disk and any point q on the outer circle,

$$\Pr[\mathrm{SRW}_p \text{ hits ray at tip}] \geq 2\Pr[\mathrm{SRW}_q \text{ hits ray at tip}],$$

where SRW_p denotes the simple random walk started at point p, and by "hits ray at tip" we mean that the first time the walker hits the ray $(x+, y)$ is at the tip (x, y). Now

$$\Pr[\mathrm{SRW}_p \text{ hits ray at tip}] = \Pr\begin{bmatrix}\mathrm{SRW}_p \text{ hits ray at tip} \\ \text{before leaving circle}\end{bmatrix} + \Pr\begin{bmatrix}\mathrm{SRW}_p \text{ leaves circle and} \\ \text{then hits ray at tip}\end{bmatrix}$$

$$\Pr[\mathrm{SRW}_p \text{ hits ray at tip}] \leq \Pr\begin{bmatrix}\mathrm{SRW}_p \text{ hits ray at tip} \\ \text{before leaving circle}\end{bmatrix} + \max_q \Pr[\mathrm{SRW}_q \text{ hits ray at tip}]$$

$$\Pr[\mathrm{SRW}_p \text{ hits ray at tip}] \leq \Pr\begin{bmatrix}\mathrm{SRW}_p \text{ hits ray at tip} \\ \text{before leaving circle}\end{bmatrix} + \frac{1}{2}\Pr[\mathrm{SRW}_p \text{ hits ray at tip}]$$

$$\Pr\begin{bmatrix}\mathrm{SRW}_p \text{ hits ray at tip} \\ \text{before leaving circle}\end{bmatrix} \geq \frac{1}{2}\Pr[\mathrm{SRW}_p \text{ hits ray at tip}] = \Theta(1/\operatorname{dist}(p, \mathrm{tip})^{1/2})$$

The point where the random walk first enters the band will lie within the left half of the small disk surrounding $\Theta(h^2)$ such points (x, y). In fact, there are $\Theta(h^2)$ such points (x, y) within radius h of where the SRW first hits the band. Thus

$$\mathbb{E}\begin{bmatrix}\#\text{ tan points within band before SRW departs enclosing band} \\ \text{and within radius } h \text{ from where SRW arrives in band}\end{bmatrix} \geq \Theta(h^2)\times\Theta(1/h^{1/2}) = \Theta(h^{3/2}).$$

Next we need a second moment estimate:

$$\begin{aligned}
&\mathbb{E}\left[\begin{pmatrix}\text{\# tan points within band before SRW departs enclosing band}\\ \text{and within radius } h \text{ from where SRW arrives in band}\end{pmatrix}^2\right]\\
&\le \mathbb{E}\left[(\text{\# tan points within radius } h \text{ from where SRW arrives in band})^2\right]\\
&= \sum_{x_1,y_1,x_2,y_2 \text{ within radius } h} \Pr[(x_1,y_1) \text{ and } (x_2,y_2) \text{ tan}]\\
&\le 2 \sum_{x_1,y_1,x_2,y_2 \text{ within radius } h} \Pr[(x_1,y_1) \text{ tan and } (x_2,y_2) \text{ tan after } (x_1,y_1)]
\end{aligned}$$

where by "after" we include the possibility that $(x_2,y_2) = (x_1,y_1)$

$$\begin{aligned}
&= 2\sum \Pr[(x_1,y_1) \text{ tan}] \times \Pr[(x_2,y_2) \text{ after } (x_1,y_1) | (x_1,y_1) \text{ tan}] \times\\
&\qquad \Pr[(x_2,y_2) \text{ tan} | (x_2,y_2) \text{ after } (x_1,y_1) \text{ and } (x_1,y_1) \text{ tan}]\\
&\le 2\sum \Pr[(x_1,y_1) \text{ tan}] \times \Pr\left[\begin{matrix}\text{after } (x_1,y_1),\ (x_2,y_2) \text{ visited before } (x_2+,y_2)|\\ (x_2,y_2) \text{ after } (x_1,y_1) \text{ and } (x_1,y_1) \text{ tan}\end{matrix}\right]\\
&= 2\sum \Pr[(x_1,y_1) \text{ tan}] \Pr[(x_2-x_1, y_2-y_1) \text{ tan in } \mathrm{SRW}_0]\\
&= \Theta(h^3)
\end{aligned}$$

Combining these estimates with Proposition 3, we see that with at least $\Theta(1)$ probability there are at least $\Theta(h^{3/2})$ tan points in the band before the SRW departs the enclosing band. $\square$

Theorem 4. *With probability* 1*, for all but finitely* $n \in \mathbb{N}$*, the excited random walk has drifted right by a distance of at least* $\Theta(n^{3/4}/\log^{5/4} n)$ *at time* n*. In particular, it is transient.*

Proof. Say that the SRW deals with a band if it reaches that band and then reaches a different band of the same parity. Suppose that the random walk starts from an odd band in the middle of a group of $4k+1$ bands of height h. Then the probability that it fails to leave the group of bands after $n = (kh)^2 t$ steps is exponentially small in t. We will optimize k and t later, but we will take t to be large so that with high probability the walk leaves the group of $4k+1$ bands, and in particular deals with at least k even bands.

Rather than run the random walk for exactly n steps, let us run it until it deals with k even bands. Then the number of early tan points in the different even bands are independent of one another, and each one has a $\Theta(1)$ chance of being at least $\Theta(h^{3/2})$. Except with probability $\exp(-\Theta(k))$, the number of tan points in the k even bands will be $\Theta(kh^{3/2})$.

Since there is only a $\exp(-\Theta(t))$ chance that the walk has not dealt with k even bands by time n, we find that, except with probability $\exp(-\Theta(t)) + \exp(-\Theta(k))$, there are $\Theta(kh^{3/2})$ tan points by time n. To optimize our parameters we take $t = k$, and then we have $n = k^3h^2$.

Next we consider the location of the perturbed random walk at time n. Typically the random walk diffuses by $\Theta(\sqrt{n}) = \Theta(k^{3/2}h)$, and drifts right by at least $\Theta(kh^{3/2})$. The probability that it diffuses by more than k^2h is $\exp(-\Theta(k))$, and the probability that it drifts less than $\Theta(kh^{3/2})$ is $< \exp(-\Theta(k))$. We take $k = \Theta(\log n)$, so that the probability of a bad event is $< 1/n^2$, which gives us $h = \Theta(n^{1/2}/\log^{3/2} n)$. Except with probability $< 1/n^2$, the excited random walk has drifted right by at least $\Theta(n^{3/4}/\log^{5/4} n)$. In particular this event fails only finitely often, so the random walk is transient. $\square$

4. Transience in $\mathbb{Z}^d$, $d > 2$

As with ERW in $\mathbb{Z}^2$, in $\mathbb{Z}^d$ we can couple the ERW with SRW, and then (in continuous time) couple the SRW in $\mathbb{Z}^d$ with the SRW in $\mathbb{Z}^2$. For each tan point of the SRW in $\mathbb{Z}^2$ there is a tan point of the SRW in $\mathbb{Z}^d$, so as before the ERW is transient.

5. Speed

We have seen that the excited random walk on $\mathbb{Z}^d$ is transient for $d \geq 2$, but does it have positive speed?

Theorem 5. *Let X_n denote the x-coordinate at time n of the excited random walk on $\mathbb{Z}^d$ with bias ε/d. If $d \geq 4$, then almost surely $\liminf_{n\to\infty} X_n/n \geq 0.659\varepsilon/d$; in particular the speed is positive.*

Proof. Project down the x-coordinate of the ERW and $d-4$ additional coordinates, and consider the resulting SRW on $\mathbb{Z}^3$. Let R_n be the range of the SRW on $\mathbb{Z}^3$ by time n, i.e. the number of points visited by time n. Since the SRW is transient, $\mathbb{E}[R_n]/n \to c$ where c is the escape probability of the SRW. Glasser and Zucker [10] (see also [7]) determined this escape probability c to be

$$c = \frac{32\pi^3}{\sqrt{6}\Gamma(1/24)\Gamma(5/24)\Gamma(7/24)\Gamma(11/24)} = 0.65946\ldots.$$

For our purposes it is not enough to know $\mathbb{E}[R_n]$, what we need is the strong law of large numbers for R_n that was proved by Dvoretzky and Erdős [9]: a.s. $R_n/n \to c$ (see also [2] for even stronger results on R_n). Thus for any $\delta > 0$, a.s. there are only finitely many n for which the ERW has not had $(c-\delta)n$ pushes to the right by time n. The theorem then follows from the ordinary strong law of large numbers. ☐

It seems intuitive that excited random walk in $\mathbb{Z}^3$ also has positive speed, but we do not see a proof. Excited random walk in $\mathbb{Z}^2$ is more delicate, and it is not clear even at an intuitive level whether or not the speed is positive, though we believe that by time n it has traveled distance at least $\Theta(n/\log n)$.

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