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ON THE QUADRATIC WIENER FUNCTIONAL ASSOCIATED WITH THE MALLIAVIN DERIVATIVE OF THE SQUARE NORM OF BROWNIAN SAMPLE PATH ON INTERVAL

SETSUO TANIGUCHI[1]
Faculty of Mathematics, Kyushu University, Fukuoka 812-8581, Japan
email: taniguch@math.kyushu-u.ac.jp

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Abstract
Exact expressions of the stochastic oscillatory integrals with phase function $\int_0^T(\int_t^T w(s)ds)^2dt$, $\{w(t)\}_{t\geq 0}$ being the 1-dimensional Brownian motion, are given. As an application, the density function of the distribution of the half of the Wiener functional is given.

1 Introduction and statement of result

The study of quadratic Wiener functionals, i.e., elements in the space of Wiener chaos of order 2, goes back to Cameron-Martin [1, 2] and Lévy [8]. While a stochastic oscillatory integral with quadratic Wiener functional as phase function has a general representation via Carleman-Fredholm determinant ([3, 6, 10]), in our knowledge, a few examples, where the integrals are represented with more concrete functions like the ones used by Cameron-Martin and Lévy, are available. See [1, 2, 8, 6, 10] and references therein. In this paper, we study a new quadratic Wiener functional which admits a concrete expression of stochastic oscillatory integral, and apply the expression to compute the density function of the Wiener functional.
Let $T > 0$, $\mathcal{W}$ be the space of all $\mathbf{R}$-valued continuous functions w on $[0,T]$ with $w(0) = 0$, and P be the Wiener measure on $\mathcal{W}$. The Wiener functional investigated in this paper is

$$q(w) = \int_0^T \left(\int_t^T w(s)ds\right)^2 dt, \quad w \in \mathcal{W}.$$

The functional q interests us because it is a key ingredient in the study of asymptotic theory on $\mathcal{W}$. Namely, recall the Wiener functional

$$q_0(w) = \int_0^T w(t)^2 dt, \quad w \in \mathcal{W},$$

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which was studied first by Cameron-Martin [1, 2, 8]. As is well-known ([15]), the stochastic oscillatory integral

$$\int_{\mathcal{W}} \exp(\zeta q_0/2)\delta_y(w(T))dP,$$

where $\delta_y(w(T))$ is Watanabe's pull back of the Dirac measure δ_y concentrated at $y \in \mathbf{R}$ via $w(T)$, relates to the fundamental solution to the heat equation associated with the Schrödinger operator $(1/2)\{(d/dx)^2 + \zeta x^2\}$, which describes the quantum mechanics of harmonic oscillator. If we denote by $\mathcal{H}$ the Cameron-Martin subspace of $\mathcal{W}$ ($\equiv$ the subspace of all absolutely continuous $h \in \mathcal{W}$ with square integrable derivative $\dot{h}$) and set $\langle h, g\rangle_{\mathcal{H}} = \int_0^T \dot{h}(t)\dot{g}(t)dt$ and $\|h\|_{\mathcal{H}}^2 = \langle h, h\rangle_{\mathcal{H}}$ for $h, g \in \mathcal{H}$, then it is straightforward to see that

$$q = \frac{1}{4}\|\nabla q_0\|_{\mathcal{H}}^2,$$

where ∇ denotes the Malliavin gradient. Thus q determines the stationary points of q_0. It should be noted that, in the context of the Malliavin calculus, the set of stationary points of q_0, i.e. the set $\{\nabla q_0 = 0\} = \{q = 0\}$ is determined uniquely up to equivalence of quasi-surely exceptional sets. On account of the stationary phase method on finite dimensional spaces (cf.[4]), q would play an important role in the study of asymptotic behavior of the stochastic oscillatory integral $\int_{\mathcal{W}} \exp(\zeta q_0)\psi dP$ with amplitude function ψ (cf. [9, 11, 12], in particular [13, 14]).

The aim of this paper is to show

Theorem 1. *(i) For sufficiently small $\lambda > 0$, the following identities hold.*

$$\int_{\mathcal{W}} \exp(\lambda q/2)\, dP = \left\{\frac{1}{\cosh(\lambda^{1/4}T)\,\cos(\lambda^{1/4}T)}\right\}^{1/2}, \tag{1}$$

$$\int_{\mathcal{W}} \exp(\lambda q/2)\delta_0(w(T))\, dP = \frac{\lambda^{1/8}}{\sqrt{\pi}\left\{\sin(\lambda^{1/4}T)\cosh(\lambda^{1/4}T) + \sinh(\lambda^{1/4}T)\cos(\lambda^{1/4}T)\right\}^{1/2}}. \tag{2}$$

(ii) Define $\theta(u;x)$ and $p_T(x)$ for $u \in [0, \pi/2]$ and $x \geq 0$ by

$$\theta(u;x) = \sum_{k=-\infty}^{\infty} (-1)^k \frac{\{u + (2k+1)\pi\}^3 e^{-x\{u+(2k+1)\pi\}^4/T^4}}{\sqrt{\cosh(u + (2k+1)\pi)}},$$

$$p_T(x) = \frac{4}{\pi T^4}\int_0^{\pi/2} \frac{\theta(u;x)}{\sqrt{\cos u}}du.$$

Then p_T is the density function of the distribution of $q/2$ on $\mathbf{R}$;

$$P(q/2 \in dx) = p_T(x)\chi_{[0,\infty)}(x)dx, \tag{3}$$

where $\chi_{[0,\infty)}$ denotes the indicator function of $[0, \infty)$.

The assertion (i) of Theorem 1 will be shown in Section 2 and (ii) will be proved in Section 3.

2 Proof of Theorem 1 (i)

In this section, we shall show the identities (1) and (2). The proof is broken into several steps, each being a lemma. We first show

Lemma 1. *Define the Hilbert-Schmidt operator $A:\mathcal{H}\to\mathcal{H}$ by*

$$Ah(t)=\int_0^t ds\int_s^T du\int_0^u dv\int_v^T da\, h(a),\quad h\in\mathcal{H},\, t\in[0,T].$$

Then it holds that

$$q=Q_A+\frac{T^4}{6}, \tag{4}$$

where $Q_A=(\nabla^*)^2A$, ∇^* being the adjoint operator of the Malliavin gradient ∇. Moreover, A is of trace class and $\operatorname{tr}A=T^4/6$. *In particular,* $q=Q_A+\operatorname{tr}A$.

Proof. Due to the integration by parts on $[0,T]$, it is easily seen that

$$\langle\nabla^2 q, h\otimes k\rangle_{\mathcal{H}^{\otimes 2}}=2\int_0^T\left(\int_t^T h(s)ds\right)\left(\int_t^T k(s)ds\right)dt=2\langle Ah,k\rangle_{\mathcal{H}} \tag{5}$$

for $h,k\in\mathcal{H}$, where $\mathcal{H}^{\otimes 2}$ denotes the Hilbert space of all Hilbert-Schmidt operators on $\mathcal{H}$, and $\langle\cdot,\cdot\rangle_{\mathcal{H}^{\otimes 2}}$ does its inner product. Hence

$$\nabla^2 q=2A. \tag{6}$$

Let $\mathfrak{C}_2$ be the space of Wiener chaos of order 2. Since

$$w(s)w(u)-s=w(s)^2-s+w(s)\{w(u)-w(s)\}\in\mathfrak{C}_2\quad\text{for } u\ge s,$$

we have that

$$q-\frac{T^4}{6}=2\int_0^T\int_t^T\int_s^T(w(s)w(u)-s)dudsdt\in\mathfrak{C}_2.$$

From this and (6), we can conclude the identity (4).
Let $\{h_n\}_{n=1}^\infty$ be an orthonormal basis of $\mathcal{H}$, and define $k_t\in\mathcal{H}$, $t\in[0,T]$, by

$$k_t(s)=\int_0^s(T-\max\{t,u\})du,\quad s\in[0,T].$$

Since $\int_t^T h_n(s)ds=\langle k_t,h_n\rangle_{\mathcal{H}}$, due to (5), we obtain that

$$\sum_{n=1}^\infty\langle Ah_n,h_n\rangle_{\mathcal{H}}=\int_0^T\sum_{n=1}^\infty\langle k_t,h_n\rangle_{\mathcal{H}}^2dt=\int_0^T\|k_t\|_{\mathcal{H}}^2dt=\frac{T^4}{6}.$$

Thus A is of trace class and $\operatorname{tr}A=T^4/6$. ∎

We next recall the following assertion achieved in [5, 7].

Lemma 2. *Let $U : \mathcal{H} \to \mathcal{H}$ be a Hilbert-Schmidt operator admitting a decomposition $U = U_V + U_F$ with a Volterra operator $U_V : \mathcal{H} \to \mathcal{H}$ and a bounded operator $U_F : \mathcal{H} \to \mathcal{H}$ possessing the finite-dimensional range $R(U_F)$.*
(i) For sufficiently small $\lambda \in \mathbf{R}$, it holds that

$$\int_{\mathcal{W}} \exp(\lambda Q_U/2)dP = \left\{\det\left(I - \lambda U_F(I - \lambda U_V)^{-1}\right)\right\}^{-1/2} e^{-(\lambda/2)\operatorname{tr} U_F}. \tag{7}$$

(ii) Let E be a subspace of $R(U_F)$ and $\{\eta_1, \ldots, \eta_d\}$ be a basis of E. Define the Wiener functional $\eta : \mathcal{W} \to \mathbf{R}^d$ by $\eta = (\nabla^*\eta_1, \ldots, \nabla^*\eta_d)$. Then, for sufficiently small $\lambda \in \mathbf{R}$, it holds that

$$\begin{aligned}\int_{\mathcal{W}} &\exp(\lambda Q_U/2)\delta_0(\eta)\,dP \\ &= \frac{1}{\sqrt{(2\pi)^d \det C(\eta)}}\{\det(I - \lambda U_1^\natural(I - \lambda U_V)^{-1})\}^{-1/2} e^{-(\lambda/2)\operatorname{tr} U_F},\end{aligned} \tag{8}$$

where $U_1^\natural = -\pi_E U_V + (I - \pi_E)U_F$, $\pi_E : \mathcal{H} \to \mathcal{H}$ being the orthogonal projection onto E, and $C(\eta) = \left(\langle \eta_i, \eta_j \rangle_{\mathcal{H}}\right)_{1 \le i,j \le d}$.

Proof. The essential part of the proof can be found in [5, 7]. For the completeness, we give the proof.
Due to the splitting property of the Wiener measure, it holds that

$$\int_{\mathcal{W}} \exp(\lambda Q_U/2)dP = \left\{\det_2(I - \lambda U)\right\}^{-1/2},$$

where $\det_2$ denotes the Carleman-Fredholm determinant. For example, see [3, 7]. Observe that, for Hilbert-Schmidt operators $C, D : \mathcal{H} \to \mathcal{H}$ such that C is of trace class, it holds that

$$\det_2(I + C)(I + D) = \det(I + C)\det_2(I + D)e^{-\operatorname{tr} C(I+D)}. \tag{9}$$

Since $\det_2(I - \lambda U_V) = 1$, substituting $C = -\lambda U_F(I - \lambda U_V)^{-1}$ and $D = -\lambda U_V$ into (9), we obtain that

$$\det_2(I - \lambda U) = \det(I - \lambda U_F(I - \lambda U_V)^{-1})e^{\lambda \operatorname{tr} U_F}.$$

Thus (7) has been shown.
Put $U_0 = (I - \pi_E)U(I - \pi_E)$ and $U_1 = \pi_E U \pi_E$. Then it holds ([7, 12]) that

$$\int_{\mathcal{W}} \exp(\lambda Q_U/2)\delta_0(\eta)\,dP = \frac{1}{\sqrt{(2\pi)^d \det C(\eta)}}\{\det_2(I - \lambda U_0)\}^{-1/2} e^{-(\lambda/2)\operatorname{tr} U_1}.$$

Setting $U^\natural = (I - \pi_E)U$, and substituting $C = -\lambda U_1^\natural(I - \lambda U_V)^{-1}$ and $D = -\lambda U_V$ into (9), we see that

$$\det_2(I - \lambda U_0) = \det_2(I - \lambda U^\natural) = \det(I - \lambda U_1^\natural(I - \lambda U_V)^{-1})e^{\lambda \operatorname{tr} U_1^\natural}.$$

Since $\operatorname{tr} U_1^\natural + \operatorname{tr} U_1 = \operatorname{tr} U_F$, we obtain (8). □

It is not known if, by just watching specific shape of quadratic Wiener functional, one can tell that the associated Hilbert-Schmidt operator admits a decomposition as a sum of a Volterra operator and a bounded operator with finite dimensional range. However, in our situation, we know a priori that the operator A admits such a decomposition. Namely, the Hilbert-Schmidt operator B associated with q_0 admits such a decomposition ([7]). Being equal to the square of B (see Remark 1 below), so does A. The following lemma gives the concrete expression of the decomposition of A.

Lemma 3. *Define $\mathcal{I}, A_V, A_F : \mathcal{H} \to \mathcal{H}$ by*

$$\mathcal{I}h(t) = \int_0^t h(s)ds, \quad t \in [0,T],$$
$$A_V h = \mathcal{I}^4 h, \quad A_F h = \Big\{\frac{T^2}{2}\mathcal{I}h(T) - \mathcal{I}^3 h(T)\Big\}\eta_1 - \frac{1}{6}\mathcal{I}h(T)\eta_2, \quad h \in \mathcal{H},$$

where $\eta_j(t) = t^{2j-1}$, $t \in [0,T]$, $j = 1,2$. Then (i) $A = A_V + A_F$, (ii) A_V is a Volterra operator, (iii) $R(A_F) = \{a\eta_1 + b\eta_2 \,|\, a,b \in \mathbf{R}\}$, (iv) $\operatorname{tr} A_F = \operatorname{tr} A$, and (v) for $\lambda > 0$, it holds that

$$(I - \lambda A_V)^{-1}h(t) = \frac{1}{2}\int_0^t \dot{h}(s)\big\{\cosh(\lambda^{1/4}(t-s)) + \cos(\lambda^{1/4}(t-s))\big\}ds,$$
$$h \in \mathcal{H},\ t \in [0,T]. \quad (10)$$

Proof. The assertions (i) and (ii) follow from the very definitions of A and A_V. The assertion (iv) is an immediate consequence of these and Lemma 1. By the definition of A_F, the inclusion $R(A_F) \subset \{a\eta_1 + b\eta_2 \,|\, a,b \in \mathbf{R}\}$ is obvious. To see the converse inclusion, it suffices to notice that $A_F\eta_1 = (5T^4/24)\eta_1 - (T^2/12)\eta_2$ and $A_F\eta_2 = (7T^6/60)\eta_1 - (T^4/24)\eta_2$. Thus (iii) has been verified.

To see (v), let $(I - \lambda A_V)g = h$ and $f = \mathcal{I}^4 g$. It then holds that $f^{(4)} - \lambda f = h$, where $f^{(n)} = (d/dt)^n f$. This leads us to the ordinary differential equation;

$$\frac{d}{dt}\begin{pmatrix} f \\ f^{(1)} \\ f^{(2)} \\ f^{(3)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} f \\ f^{(1)} \\ f^{(2)} \\ f^{(3)} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ h \end{pmatrix}, \quad \begin{pmatrix} f(0) \\ f^{(1)}(0) \\ f^{(2)}(0) \\ f^{(3)}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

It is then easily seen that

$$f^{(3)}(t) = \frac{1}{2}\int_0^t h(s)\{\cosh(\lambda^{1/4}(t-s)) + \cos(\lambda^{1/4}(t-s))\}ds.$$

Since $g = f^{(4)}$, this implies the identity (10). $\square$

Lemma 4. *The identity* (1) *holds.*

Proof. Let $\eta_1, \eta_2 \in \mathcal{H}$ be as described in Lemma 3, and put $f_j = (I - \lambda A_V)^{-1}\eta_j$, $j = 1,2$. By virtue of Lemma 3, we have that

$$\mathcal{I}f_1(t) = \frac{\lambda^{-1/2}}{2}\{\cosh(\lambda^{1/4}t) - \cos(\lambda^{1/4}t)\},$$
$$\mathcal{I}^3 f_1(t) = \frac{\lambda^{-1}}{2}\{\cosh(\lambda^{1/4}t) + \cos(\lambda^{1/4}t) - 2\},$$
$$\mathcal{I}f_2(t) = 3\lambda^{-1}\{\cosh(\lambda^{1/4}t) + \cos(\lambda^{1/4}t) - 2\},$$
$$\mathcal{I}^3 f_2(t) = 3\lambda^{-3/2}\{\cosh(\lambda^{1/4}t) - \cos(\lambda^{1/4}t)\} - 3\lambda^{-1}t^2.$$

Hence, if we set $\alpha_\lambda = \cosh(\lambda^{1/4}T)$ and $\beta_\lambda = \cos(\lambda^{1/4}T)$, then

$$\begin{aligned}
&(I - \lambda A_F(I-\lambda A_V)^{-1})\eta_1 \\
&\quad = \Big\{ -\frac{T^2\lambda^{1/2}}{4}(\alpha_\lambda - \beta_\lambda) + \frac{1}{2}(\alpha_\lambda + \beta_\lambda)\Big\}\eta_1 + \frac{\lambda^{1/2}}{12}(\alpha_\lambda - \beta_\lambda)\eta_2, \\
&(I - \lambda A_F(I-\lambda A_V)^{-1})\eta_2 \\
&\quad = \Big\{ -\frac{3T^2}{2}(\alpha_\lambda + \beta_\lambda) + 3\lambda^{-1/2}(\alpha_\lambda - \beta_\lambda)\Big\}\eta_1 + \frac{1}{2}(\alpha_\lambda + \beta_\lambda)\eta_2,
\end{aligned}$$

Thus, by virtue of (iii), it holds that

$$\begin{aligned}
&\det\big(I - \lambda A_F(I-\lambda A_V)^{-1}\big) \\
&\quad = \det\begin{pmatrix} -\frac{T^2\lambda^{1/2}}{4}(\alpha_\lambda - \beta_\lambda) + \frac{1}{2}(\alpha_\lambda + \beta_\lambda) & \frac{\lambda^{1/2}}{12}(\alpha_\lambda - \beta_\lambda) \\ -\frac{3T^2}{2}(\alpha_\lambda + \beta_\lambda) + 3\lambda^{-1/2}(\alpha_\lambda - \beta_\lambda) & \frac{1}{2}(\alpha_\lambda + \beta_\lambda) \end{pmatrix} = \alpha_\lambda\beta_\lambda.
\end{aligned}$$

This implies the identity (1), because Lemmas 1, 2, and 3 yield that

$$\int_{\mathcal{W}} \exp(\lambda q/2)dP = \{\det(I - \lambda A_F(I - \lambda A_V)^{-1})\}^{-1/2}.$$

$\square$

Lemma 5. *The identity* (2) *holds.*

Proof. Let η_j, $j = 1, 2$, be as in Lemma 3 (iii), and $E = \{c\eta_1 \mid c \in \mathbf{R}\}$. Define $A_1^\natural$ as described in Lemma 2 with $U = A$, $U_V = A_V$, and $U_F = A_F$. Since $\pi_E h = (h(T)/T)\eta_1$ for any $h \in \mathcal{H}$, we have that

$$A_1^\natural h = \Big\{ -\frac{1}{T}\mathcal{I}^4 h(T) + \frac{T^2}{6}\mathcal{I}h(T)\Big\}\,\eta_1 - \frac{1}{6}\mathcal{I}h(T)\,\eta_2.$$

Let f_1, f_2 be as in the proof of Lemma 4. Then we see that

$$\begin{aligned}
\mathcal{I}^4 f_1(t) &= \frac{\lambda^{-5/4}}{2}\{\sinh(\lambda^{1/4}t) + \sin(\lambda^{1/4}t)\} - \lambda^{-1}t, \\
\mathcal{I}^4 f_2(t) &= 3\lambda^{-7/4}\{\sinh(\lambda^{1/4}t) - \sin(\lambda^{1/4}t)\} - \lambda^{-1}t^3.
\end{aligned}$$

Hence, if we put $\sigma_\lambda = \sinh(\lambda^{1/4}T)$ and $\tau_\lambda = \sin(\lambda^{1/4}T)$, then

$$\begin{aligned}
&(I - \lambda A_1^\natural(I-\lambda A_V)^{-1})\eta_1 \\
&\quad = \Big\{ \frac{\lambda^{-1/4}}{2T}(\sigma_\lambda + \tau_\lambda) - \frac{T^2\lambda^{1/2}}{12}(\alpha_\lambda - \beta_\lambda)\Big\}\eta_1 + \frac{\lambda^{1/2}}{12}(\alpha_\lambda - \beta_\lambda)\eta_2, \\
&(I - \lambda A_1^\natural(I-\lambda A_V)^{-1})\eta_2 \\
&\quad = \Big\{ \frac{\lambda^{-3/4}}{T}(\sigma_\lambda - \tau_\lambda) - \frac{T^2}{2}(\alpha_\lambda + \beta_\lambda)\Big\}\eta_1 + \frac{1}{2}(\alpha_\lambda + \beta_\lambda)\eta_2.
\end{aligned}$$

Since $R(A_1^\natural) \subset R(A_F)$, by Lemma 3 (ii), this yields that

$$\begin{aligned}
&\det(I - \lambda A_1^\natural(I-\lambda A_V)^{-1}) \\
&\quad = \det\begin{pmatrix} \frac{\lambda^{-1/4}}{2T}(\sigma_\lambda + \tau_\lambda) - \frac{T^2\lambda^{1/2}}{12}(\alpha_\lambda - \beta_\lambda) & \frac{\lambda^{1/2}}{12}(\alpha_\lambda - \beta_\lambda) \\ \frac{\lambda^{-3/4}}{T}(\sigma_\lambda - \tau_\lambda) - \frac{T^2}{2}(\alpha_\lambda + \beta_\lambda) & \frac{1}{2}(\alpha_\lambda + \beta_\lambda) \end{pmatrix} \\
&\quad = \frac{\lambda^{-1/4}}{2T}\{\sigma_\lambda\beta_\lambda + \tau_\lambda\alpha_\lambda\}.
\end{aligned}$$

The identity (2) follows from this, because Lemmas 1, 2, and 3 imply that

$$\int_{\mathcal{W}} \exp(\lambda q/2)\delta_0(w(T))dP = \int_{\mathcal{W}} \exp(\lambda Q_A/2)\delta_0(\nabla^*\eta_1)dP\, e^{(\lambda/2)\operatorname{tr} A}$$
$$= \frac{1}{\sqrt{2\pi T}}\{\det(I-\lambda A_1^\natural(I-\lambda A_V)^{-1})\}^{-1/2}.$$

□

Remark 1. It may be interesting to see that (1) is also shown by using the infinite product expression. Namely, define $B:\mathcal{H}\to\mathcal{H}$ by

$$Bh(t)=\int_0^t\int_s^T h(u)du\,ds, \quad h\in\mathcal{H},\, t\in[0,T].$$

Then there exists an orthonormal basis $\{h_n\}_{n=0}^\infty$ of $\mathcal{H}$ so that

$$B=\sum_{n=0}^\infty\left(\frac{T}{(n+\frac{1}{2})\pi}\right)^2 h_n\otimes h_n.$$

See [10]. Since $A=B^2$, it holds that

$$A=\sum_{n=0}^\infty\left(\frac{T}{(n+\frac{1}{2})\pi}\right)^4 h_n\otimes h_n. \tag{11}$$

In conjunction with Lemma 1, this implies that

$$q=Q_A+\operatorname{tr} A=\sum_{n=0}^\infty\left(\frac{T}{(n+\frac{1}{2})\pi}\right)^4 (\nabla^* h_n)^2.$$

Due to the splitting property of the Wiener measure, we then obtain that

$$\int_{\mathcal{W}}\exp(\lambda q/2)dP=\left(\prod_{n=0}^\infty\left\{1-\lambda\left(\frac{T}{(n+\frac{1}{2})\pi}\right)^4\right\}\right)^{-1/2}$$
$$=\left(\prod_{n=0}^\infty\left\{1+\lambda^{1/2}\left(\frac{T}{(n+\frac{1}{2})\pi}\right)^2\right\}\prod_{n=0}^\infty\left\{1-\lambda^{1/2}\left(\frac{T}{(n+\frac{1}{2})\pi}\right)^2\right\}\right)^{-1/2}.$$

Due to the infinite product expressions of $\cosh x$ and $\cos x$, this implies (1).

3 Proof of Theorem 1 (ii)

In this section, we shall show Theorem 1 (ii).
We first describe how we realize $\{\cosh z\cos z\}^{1/2}$ for complex number z. Represent $z\in\mathbf{C}$ as $z=re^{i\theta}$ with $r\geq 0$ and $-\frac{1}{2}\pi\leq\theta<\frac{3}{2}\pi$ to define $\sqrt{z}=r^{1/2}e^{i\theta/2}$, where $i^2=-1$. The

Riemann surface of the 2-valued function $z^{1/2}$ is realized by switching $\sqrt{z}$ and $-\sqrt{z}$ on the half line consisting of $i\xi$, $\xi<0$. Set

$$G(z)=\begin{cases}\sqrt{\cos z}, & \text{if } a)\,|\mathrm{Re}\,z|<\frac{\pi}{2},\ \text{or}\\ & \quad b)\,\mathrm{Im}\,z>0,\ -\frac{3\pi}{2}+4k\pi\le \mathrm{Re}\,z<\frac{\pi}{2}+4k\pi\ (k\in\mathbf{Z}),\ \text{or}\\ & \quad c)\,\mathrm{Im}\,z<0,\ -\frac{\pi}{2}+4k\pi\le \mathrm{Re}\,z<\frac{3\pi}{2}+4k\pi\ (k\in\mathbf{Z}),\\ -\sqrt{\cos z}, & \text{if } a)\,\mathrm{Im}\,z>0,\ \frac{\pi}{2}+4k\pi\le \mathrm{Re}\,z<\frac{5\pi}{2}+4k\pi\ (k\in\mathbf{Z}),\ \text{or}\\ & \quad b)\,\mathrm{Im}\,z<0,\ \frac{3\pi}{2}+4k\pi\le \mathrm{Re}\,z<\frac{7\pi}{2}+4k\pi\ (k\in\mathbf{Z}).\end{cases}$$

Then G is holomorphic on $\mathbf{C}\setminus\{\xi\,|\,\xi\in\mathbf{R}, |\xi|\ge\pi/2\}$, and realizes $\{\cos z\}^{1/2}$. Hence $G(z)G(iz)$ is holomorphic on $D_0\equiv\mathbf{C}\setminus\{\xi, i\xi\,|\,\xi\in\mathbf{R},\ |\xi|\ge\pi/2\}$ and does not vanish in D_0. Recalling that $\cosh z=\cos(iz)$, we write $\{\cosh z\cos z\}^{1/2}$ for $G(z)G(iz)$.
We next extend the identity (1) holomorphically. Since there exists $\delta>0$ such that $\exp(\delta q/2)$ is integrable with respect to P and $q\ge 0$, the mapping

$$\{z\in\mathbf{C}\,|\,\mathrm{Re}\,z<\delta\}\ni z\mapsto\int_{\mathcal{W}}\exp(zq/2)dP$$

is holomorphic. $\{\cosh(zT)\cos(zT)\}^{-1/2}$ being holomorphic in D_0, we can find a domain $D\subset\mathbf{C}$ such that

$$D\supset\left\{re^{i\theta}\,\middle|\,r\ge 0,\,\theta\in\bigcup_{k=0}^{3}\Big[\frac{\pi}{8}+\frac{k\pi}{2},\frac{3\pi}{8}+\frac{k\pi}{2}\Big]\right\},\quad\text{and}$$
$$\int_{\mathcal{W}}\exp(z^4q/2)dP=\frac{1}{\{\cosh(zT)\cos(zT)\}^{1/2}}\quad\text{for every } z\in D. \tag{12}$$

By (11) and Lemma 1, as an easy application of the Malliavin calculus, we see that the distribution of $q/2$ on $\mathbf{R}$ admits a smooth density function $p_T(x)$ ([14, Lemma 3.1]). Since $q\ge 0$, $p_T(x)=0$ for $x\le 0$. Hence, in what follows, we always assume that $x>0$. By the inverse Fourier transformation, we have that

$$p_T(x)=\frac{1}{2\pi}\int_{\mathbf{R}}e^{-ixt}I(t)dt,\quad\text{where } I(t)=\int_{\mathcal{W}}\exp(itq/2)dP. \tag{13}$$

For $R>0$, let $\Gamma_+(R)$ (resp. $\Gamma_-(R)$) be the directed line segment in $\mathbf{C}$ starting at the origin and ending at $Re^{i\pi/8}$ (resp. $Re^{-i\pi/8}$). Then, parameterizing $\Gamma_\pm(R)$ by $t^{1/4}e^{\pm i\pi/8}$, $t\in[0,R^4]$, we have that

$$\int_{\Gamma_\pm(R)}f(z^4)z^3dz=\pm\frac{i}{4}\int_0^{R^4}f(\pm it)dt$$

for any piecewise continuous function f on $i\mathbf{R}$, where and in the sequel, the symbol $\pm$ takes $+$ or $-$ simultaneously. Plugging this into (13), and then substituting (12), we obtain that

$$2\pi p_T(x)=\lim_{R\to\infty}\left\{4i\int_{\Gamma_-(R)}\frac{z^3e^{-xz^4}}{\{\cosh(zT)\cos(zT)\}^{1/2}}dz\right.$$
$$\left.-4i\int_{\Gamma_+(R)}\frac{z^3e^{-xz^4}}{\{\cosh(zT)\cos(zT)\}^{1/2}}dz\right\}. \tag{14}$$

Thanks to the estimation that

$$|\cosh(u+iv)\cos(u+iv)|^2 \geq \sinh^2 u \max\{\cos^2 u, \sinh^2 v\},$$

it is a routine exercise of complex analysis to show that

$$\begin{aligned}\lim_{R\to\infty}\int_{\Gamma_\pm(R)} \frac{z^3 e^{-xz^4}}{\{\cosh(zT)\cos(zT)\}^{1/2}}dz \\ = \int_0^\infty \frac{u^3 e^{-xu^4}}{\lim_{h\downarrow 0}\{\cosh(uT\pm ih)\cos(uT\pm ih)\}^{1/2}}du. \quad (15)\end{aligned}$$

Moreover, by the definition of $\{\cosh z\cos z\}^{1/2}$, we have that

$$\begin{aligned}&\lim_{h\downarrow 0}\{\cosh(uT\pm ih)\cos(uT\pm ih)\}^{1/2} \\ &= \begin{cases} \sqrt{\cosh(uT)\cos(uT)}, & \text{if } -\pi-(\pm\frac{\pi}{2})+4k\pi \leq uT < \pi-(\pm\frac{\pi}{2})+4k\pi, \\ -\sqrt{\cosh(uT)\cos(uT)}, & \text{if } \pi-(\pm\frac{\pi}{2})+4k\pi \leq uT < 3\pi-(\pm\frac{\pi}{2})+4k\pi, \end{cases}\end{aligned}$$

Substitute this and (15) into (14) to see that

$$2\pi p_T(x) = 8i\sum_{k=0}^\infty \int_{\{(\pi/2)+2k\pi\}/T}^{\{(3\pi/2)+2k\pi\}/T} \frac{(-1)^k u^3 e^{-xu^4}}{\sqrt{\cosh(uT)\cos(uT)}}du.$$

This implies Theorem 1 (ii), because

$$\begin{aligned}&\int_{\{(\pi/2)+2k\pi\}/T}^{\{(3\pi/2)+2k\pi\}/T} \frac{u^3 e^{-xu^4}}{\sqrt{\cosh(uT)\cos(uT)}}du \\ &= \frac{1}{iT^4}\int_0^{\pi/2} \frac{\{v+(2k+1)\pi\}^3 e^{-x\{v+(2k+1)\pi\}^4/T^4}}{\sqrt{\cosh\{v+(2k+1)\pi\}\cos v}}dv \\ &\quad - \frac{1}{iT^4}\int_0^{\pi/2} \frac{\{v-(2k+1)\pi\}^3 e^{-x\{v-(2k+1)\pi\}^4/T^4}}{\sqrt{\cosh\{v-(2k+1)\pi\}\cos v}}dv.\end{aligned}$$

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