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STONE-WEIERSTRASS TYPE THEOREMS FOR LARGE DEVIATIONS

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Abstract
We give a general version of Bryc's theorem valid on any topological space and with any algebra $\mathcal{A}$ of real-valued continuous functions separating the points, or any well-separating class. In absence of exponential tightness, and when the underlying space is locally compact regular and $\mathcal{A}$ constituted by functions vanishing at infinity, we give a sufficient condition on the functional $\Lambda(\cdot)_{|\mathcal{A}}$ to get large deviations with not necessarily tight rate function. We obtain the general variational form of any rate function on a completely regular space; when either exponential tightness holds or the space is locally compact Hausdorff, we get it in terms of any algebra as above. Prohorov-type theorems are generalized to any space, and when it is locally compact regular the exponential tightness can be replaced by a (strictly weaker) condition on $\Lambda(\cdot)_{|\mathcal{A}}$.

1 Introduction

Let (μ_α) be a net of Borel probability measures on a topological space X where compact sets are Borel sets, and let (t_α) be a net in $]0,+\infty[$ converging to 0. For any $[-\infty,+\infty[$-valued Borel measurable function h on X, we write $\mu_\alpha^{t_\alpha}(e^{h/t_\alpha})$ for $(\int_X e^{h(x)/t_\alpha}\mu_\alpha(dx))^{t_\alpha}$, and define $\overline{\Lambda}(h)=\log\limsup\mu_\alpha^{t_\alpha}(e^{h/t_\alpha})$, $\underline{\Lambda}(h)=\log\liminf\mu_\alpha^{t_\alpha}(e^{h/t_\alpha})$, and $\Lambda(h)=\log\lim\mu_\alpha^{t_\alpha}(e^{h/t_\alpha})$ when this limit exists (throughout this paper, "existence" of $\Lambda(h)$ means existence in $[-\infty,+\infty]$). The Bryc's theorem asserts that under exponential tightness hypothesis and when X is completely regular Hausdorff, the existence of $\Lambda(\cdot)$ on the set $\mathcal{C}_b(X)$ of real-valued bounded continuous functions on X, implies a large deviation principle with rate function $J(x)=\sup_{h\in\mathcal{C}_b(X)}\{h(x)-\Lambda(h)\}$ for all $x\in X$ ([3], [6]).
We prove here that this theorem is still true replacing $\mathcal{C}_b(X)$ by the bounded above part $\mathcal{A}_{ba}$ of any algebra $\mathcal{A}$ of real-valued (not necessarily bounded) continuous functions separating the points, or any well-separating class; the rate function is then given by

$$\forall x\in X,\qquad J(x)=\sup_{h\in\mathcal{A}_{ba}}\{h(x)-\Lambda(h)\}=\sup_{h\in\mathcal{A}}\{h(x)-\underline{\Lambda}(h)\}=\sup_{h\in\mathcal{A}}\{h(x)-\overline{\Lambda}(h)\},\qquad(1)$$

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and we can replace in the above expression $\mathcal{A}$ by its negative part $\mathcal{A}_-$ when $\mathcal{A}$ does not vanish identically at any point of X (in particular when $\mathcal{A}$ contains the constants); this is the most tricky part because of the lack of stability by translation of $\mathcal{A}_-$. As a consequence we obtain the variational form of any rate function (under exponential tightness) on such a space in terms of any such a set $\mathcal{A}$; without assuming exponential tightness, we get it only in terms of $\mathcal{C}(X)$ and $\mathcal{C}(X)_-$ (Corollary 1).
In fact, the main result (Theorem 1) gives a general version valid for any topological space X with the restriction that (1) holds only on the "completely regular part" X_0 of X; however, (1) determines completely the rate function when X is regular. Various kinds of hypotheses are proposed, which all reduce to exponential tightness in the completely regular (not necessarily Hausdorff) case; for instance, the simpler one requires that the tightening compact sets be included in X_0. This allows us to extend some known results by relaxing topological assumptions on the space: Corollary 2 is a Prohorov-type result valid on any topological space, when preceding versions assumed complete regularity and Hausdorffness; Corollary 3 consider the well-separating class constituted by continuous affine functionals on any real Hausdorff topological vector space, when a preceding version assumed metrizability.
When exponential tightness fails, but assuming X locally compact regular (not necessarily Hausdorff) and $\mathcal{A}$ an algebra of continuous functions vanishing at infinity which separates the points and does not vanish identically at any point of X, we give a necessary and sufficient condition on $\Lambda(\cdot)_{|\mathcal{A}}$ to get large deviations with a rate function satisfying a property weaker than the tightness, and having still the form (1) (Theorem 2). A similar condition allows to get the large deviation principle for some subnets (Corollary 5). When X is moreover Hausdorff we show that a large deviation principe is always governed by the rate function (1) with any $\mathcal{A}$ as above (Corollary 4).
This is achieved by applying the results of [4] and [5]; we use in particular the notion of approximating class, that is a set of functions on which the existence of $\Lambda(\cdot)$ implies large deviations lower bounds with a function satisfying (2). When X is completely regular, the upper bounds on compact sets hold also with the function (2), which turns to be the rate function when large deviations hold; in the general case some extra conditions are required in order that (2) be a rate function on X_0. Note that in absence of regularity the identification of a rate function is quite difficult by the lack of uniqueness.
More precisely, an approximating class is a set $\mathcal{T}$ of $[-\infty,+\infty[$-valued continuous functions on X such that for each $x \in X$, each open set G containing x, each real $s > 0$, and each real $t > 0$, there exists $h \in \mathcal{T}$ satisfying

$$e^{-t}1_{\{x\}} \leq e^h \leq 1_G \vee e^{-s}.$$

In [4] we proved that if $\Lambda(\cdot)$ exists on $\mathcal{T}$ and under some extra condition (namely, (iii) of Proposition 2), then (μ_α) satisfies a large deviation principle with powers (t_α) and a rate function verifying

$$J(x) = \inf_{t>0} \sup_{\{h\in\mathcal{T}^T : h(x)\geq -t\}} \{-\Lambda(h)\} \qquad \text{for all } x \in X_0, \tag{2}$$

where $\mathcal{T}^T$ denotes the set of elements in $\mathcal{T}$ satisfying the usual tail condition of Varadhan's theorem. We first improve that by showing that the sup in (2) can be taken on $\mathcal{T}$ and $\mathcal{T}_-$ in place of $\mathcal{T}^T$, and $\Lambda(h)$ can be replaced by $\overline{\Lambda}(h)$ and $\underline{\Lambda}(h)$ (Proposition 2). This result is used in a crucial way in the sequel. Indeed, we first show that the existence of $\Lambda(\cdot)$ on $\mathcal{A}_{ba}$ implies the existence of $\Lambda(\cdot)$ on $\mathcal{C}(X)_-$; next, we get large deviations with rate function

satisfying (1) on X_0 by applying Proposition 2 with the approximating class $\mathcal{C}(X)$. The case with well-separating class is proved similarly.
The paper is organized as follows. Section 2 gives the general form of a rate function in terms of an approximating class, strengthening a result of [4]. In Section 3 we establish the general versions of Bryc's theorem; an example of Hausdorff regular space where the usual form does not work only because of one point, but satisfying our general hypotheses is given. In Section 4 we study the case where X is locally compact regular.

2 General form of a rate function

Throughout the paper X denotes a topological space in which compact sets are Borel sets (in particular, no separation axiom is required) and $\mathcal{C}(X)$ denotes the set of all real-valued continuous functions on X. We exclude the trivial case where X is reduced to one point. We recall that a set $\mathcal{C} \subset \mathcal{C}(X)$ "separates the points of X" if for any pair of points $x \neq y$ in X there exists $h \in \mathcal{C}$ such that $h(x) \neq h(y)$. By "$\mathcal{C}$ does not vanish identically at any point of X" we mean that for any $x \in X$ there exists $h \in \mathcal{C}$ such that $h(x) \neq 0$. Note that any well-separating class satisfies the two above properties. Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{K}$ denote respectively the set of closed, open, and compact subsets of X, and let l be a $[0, +\infty]$-valued function on X. We say that (μ_α) satisfies the large deviation upper (resp. lower) bounds with powers (t_α) and function l if

$$\limsup \mu_\alpha^{t_\alpha}(F) \leq \sup_{x \in F} e^{-l(x)} \qquad \text{for all } F \in \mathcal{F}, \tag{3}$$

$$(\text{resp.} \qquad \sup_{x \in G} e^{-l(x)} \leq \liminf \mu_\alpha^{t_\alpha}(G) \qquad \text{for all } G \in \mathcal{G}). \tag{4}$$

When (3) (resp. (3) with $\mathcal{K}$ in place of $\mathcal{F}$) and (4) hold, we say that (μ_α) satisfies a large deviation principle (resp. vague large deviation principle) with powers (t_α); in this case, the lower-regularization of l (i.e., the greatest lower semi-continuous function lesser than l) is called a rate function, which is said to be tight when it has compact level sets. As it is well known ([6], Lemma 4.1.4 and Remark pp. 118), when X is regular a rate function is uniquely determined and coincides with the function l_0 defined by

$$l_0(x) = -\log \inf\{\limsup \mu_\alpha^{t_\alpha}(G) : G \in \mathcal{G}, G \ni x\} \qquad \text{for all } x \in X.$$

The following proposition will be used in the sequel since we will deal with functions which do not necessarily satisfy the usual tail condition

$$\lim_{M \to \infty} \limsup \mu_\alpha^{t_\alpha}(e^{h/t_\alpha} 1_{\{e^h \geq M\}}) = 0. \tag{5}$$

It is easy to see that it generalizes Varadhan's theorem; indeed, since for each $[-\infty, +\infty[$-valued Borel function h on X, for each subnet $(\mu_\beta^{t_\beta})$ of $(\mu_\alpha^{t_\alpha})$ and each real M we have

$$\limsup \mu_\beta^{t_\beta}(e^{h/t_\beta}) = \limsup \mu_\beta^{t_\beta}(e^{h/t_\beta} 1_{\{h < M\}}) \vee \limsup \mu_\beta^{t_\beta}(e^{h/t_\beta} 1_{\{h \geq M\}}),$$

it follows that when $h \in \mathcal{C}(X)$ and satisfies (5), by letting $M \to +\infty$ Proposition 1 implies $\limsup \mu_\beta^{t_\beta}(e^{h/t_\beta}) = \sup_{x \in X} e^{h(x) - l(x)}$ hence the existence of $\Lambda(h)$ (since this expression does not depend on the subnet along which the upper limit is taken).

Proposition 1. *If the large deviation upper (resp. lower) bounds hold with some function l, then for each $h \in \mathcal{C}(X)$ and each real M we have*

$$\limsup \mu_\alpha^{t_\alpha}(e^{h/t_\alpha} 1_{\{h \leq M\}}) \leq \sup_{x \in X, h(x) \leq M} e^{h(x)-l(x)}$$

$$(\textit{resp.} \qquad \liminf \mu_\alpha^{t_\alpha}(e^{h/t_\alpha} 1_{\{h < M\}}) \geq \sup_{x \in X, h(x) < M} e^{h(x)-l(x)}).$$

In particular when the upper and the lower bounds hold with l we have

$$\lim_{M \to +\infty} \liminf \mu_\alpha^{t_\alpha}(e^{h/t_\alpha} 1_{\{h < M\}}) = \lim_{M \to +\infty} \limsup \mu_\alpha^{t_\alpha}(e^{h/t_\alpha} 1_{\{h \leq M\}}) = \sup_{x \in X} e^{h(x)-l(x)}.$$

Proof. For each $[-\infty, +\infty[$-valued Borel measurable function h on X, each $\varepsilon > 0$, and each $x \in X$, we put $G_{e^{h(x)},\varepsilon} = \{y \in X : e^{h(x)} - \varepsilon < e^{h(y)} < e^{h(x)} + \varepsilon\}$ and $F_{e^{h(x)},\varepsilon} = \{y \in X : e^{h(x)} - \varepsilon \leq e^{h(y)} \leq e^{h(x)} + \varepsilon\}$. First assume that the large deviation lower bounds hold with l. For each real M, by Theorem 3.1 of [5] applied to the function $k = h1_{\{h<M\}} - \infty 1_{\{h \geq M\}}$ (with the convention "$\infty \cdot 0 = 0$"), there exists a subnet $(\mu_\beta^{t_\beta})$ of $(\mu_\alpha^{t_\alpha})$ such that

$$\liminf \mu_\alpha^{t_\alpha}(e^{k/t_\alpha} 1_{\{h<M\}}) = \limsup \mu_\beta^{t_\beta}(e^{k/t_\beta} 1_{\{h<M\}})$$

$$\sup_{\{x \in X, \varepsilon > 0 : h(x) < M\}} \{(e^{k(x)} - \varepsilon) \limsup \mu_\beta^{t_\beta}(G_{e^{k(x)},\varepsilon} \cap \{h < M\})\}.$$

Since k coincides with h on $\{h < M\}$ we have $G_{e^{k(x)},\varepsilon} \cap \{h < M\} = G_{e^{h(x)},\varepsilon} \cap \{h < M\}$, hence

$$\liminf \mu_\alpha^{t_\alpha}(e^{h/t_\alpha} 1_{\{h<M\}}) = \limsup \mu_\beta^{t_\beta}(e^{h/t_\beta} 1_{\{h<M\}}) =$$

$$\sup_{\{x \in X, \varepsilon > 0 : h(x) < M\}} \{(e^{h(x)} - \varepsilon) \limsup \mu_\beta^{t_\beta}(G_{e^{h(x)},\varepsilon} \cap \{h < M\})\} \geq \sup_{x \in X, h(x) < M} e^{h(x)} e^{-l(x)}$$

(where the last inequality follows from the large deviations lower bounds), which proves the lower bounds case. Assume now that the large deviation upper bounds hold with l, and suppose that

$$\limsup \mu_\alpha^{t_\alpha}(e^{h/t_\alpha} 1_{\{e^h \leq M\}}) > \sup_{x \in X, h(x) \leq M} e^{h(x)-l(x)}$$

for some real M. Applying Theorem 3.1 of [5] as above with $F_{e^{h(x)},\varepsilon}$ in place of $G_{e^{h(x)},\varepsilon}$ yields

$$\limsup \mu_\alpha^{t_\alpha}(e^{h/t_\alpha} 1_{\{h \leq M\}}) = \sup_{\{x \in X, \varepsilon > 0 : h(x) \leq M\}} \{(e^{h(x)} - \varepsilon) \limsup \mu_\alpha^{t_\alpha}(F_{e^{h(x)},\varepsilon} \cap \{h \leq M\})\},$$

and therefore there exists $x \in X$ and $\varepsilon > 0$ such that

$$(e^{h(x)} - \varepsilon) \limsup \mu_\alpha^{t_\alpha}(F_{e^{h(x)},\varepsilon} \cap \{h \leq M\}) > \sup_{x \in X, h(x) \leq M} e^{h(x)-l(x)}.$$

By the large deviation upper-bounds we have

$$(e^{h(x)} - \varepsilon) \sup_{y \in F_{e^{h(x)},\varepsilon} \cap \{h \leq M\}} e^{-l(y)} > \sup_{x \in X, h(x) \leq M} e^{h(x)-l(x)},$$

and so there exists $x' \in F_{e^{h(x)},\varepsilon} \cap \{h \leq M\}$ such that

$$(e^{h(x)} - \varepsilon) e^{-l(x')} > \sup_{x \in X, h(x) \leq M} e^{h(x)} e^{-l(x)}.$$

Since $e^{h(x')} \geq e^{h(x)} - \varepsilon$ we obtain $e^{h(x')} e^{-l(x')} > \sup_{x \in X, h(x) \leq M} e^{h(x)-l(x)}$ and the contradiction, which proves the upper bounds case. $\square$

We recall here the definition of an approximation class, which involves the set X_0 constituted by the points $x \in X$ which can be suitably separated by a continuous function from any closed set not containing x. Note that $\mathcal{C}(X)_-$ is an approximating class for any space X. It is known that $X_0 = X$ if and only if X is completely regular ([4], Proposition 1). At the other extreme, $X_0 = \emptyset$ when $\mathcal{C}(X)$ is reduced to constants and X is a T_0 space containing more than one point, as it may occur with some regular Hausdorff spaces ([7]). Note also that the negative part $\mathcal{A}_-$ of any approximating class $\mathcal{A}$ is again an approximating class.

Definition 1. Let X_0 be the set of points x of X such that for each $G \in \mathcal{G}$ containing x, each real $s > 0$, and each real $t > 0$, there exists $h \in \mathcal{C}_b(X)$ such that

$$e^{-t}1_{\{x\}} \leq e^h \leq 1_G \vee e^{-s}. \tag{6}$$

A class $\mathcal{T}$ of $[-\infty, +\infty[$-valued continuous functions on X is said to be approximating if for each $x \in X_0$, each $G \in \mathcal{G}$ containing x, each real $s > 0$, and each real $t > 0$, $\mathcal{T}$ contains some function satisfying (6).

We introduce now a strong variant of exponential tightness by requiring that the tightening compact sets be included in X_0. Of course, it coincides with the usual one in the completely regular case.

Definition 2. The net (μ_α) is X_0-exponentially tight with respect to (t_α) if for each $\varepsilon > 0$ there exists a compact set $K \subset X_0$ such that $\limsup \mu_\alpha^{t_\alpha}(X \backslash K) < \varepsilon$.

To any approximating class $\mathcal{T}$ we associate the function $l_{\mathcal{T}}$ defined by

$$l_{\mathcal{T}}(x) = \begin{cases} \inf_{t>0} \sup_{\{h \in \mathcal{T}^T : h(x) \geq -t\}} \{-\overline{\Lambda}(h)\} & if\ x \in X_0 \\ \\ \sup_{G \supset \{x\}, G \in \mathcal{G}} \sup_{0<s<\infty} \inf_{\mathcal{A}_{G,s}} \{-\overline{\Lambda}(h)\} & if\ x \in X \backslash X_0, \end{cases} \tag{7}$$

where $\mathcal{T}^T$ denotes the elements $h \in \mathcal{T}$ satisfying the tail condition (5), and

$$\mathcal{A}_{G,s} = \bigcup_{x \in G \cap X_0, t>0} \{h \in \mathcal{A} : e^h \leq 1_G \vee e^{-s}, h(x) \geq -t\} \quad \text{for all } G \in \mathcal{G} \text{ and } s \in]0, +\infty[.$$

In Theorem 3 of [4] we proved that the existence of $\Lambda(\cdot)$ on $\mathcal{T}$ together with the condition (iii) below imply a large deviation principle with rate function $l_{\mathcal{T}}$; in fact, it is easy to verify that the existence of $\Lambda(\cdot)$ on $\mathcal{T}_-$ together with (iii) are sufficient. The following proposition shows that we can replace $\mathcal{T}^T$ by $\mathcal{T}$ (resp. $\mathcal{T}_-$) and $\overline{\Lambda}(h)$ by $\underline{\Lambda}(h)$ in (7) for the case $x \in X_0$. We can even replace $\overline{\Lambda}(h)$ and $\underline{\Lambda}(h)$ by $\lim_M \underline{\Lambda}(h_M)$, where $h_M = h1_{\{h<M\}} - \infty 1_{\{h \geq M\}}$ for all $h \in \mathcal{T}$ and all reals M. This will be used in the next section in order to obtain the expression of the rate function in terms of the whole algebra (or the well-separating class) since this one may contain unbounded functions.

Proposition 2. *Consider the following statements:*

(i) (μ_α) *is* X_0*-exponentially tight with respect to* (t_α)*;*

(ii) (μ_α) *is exponentially tight with respect to* (t_α) *and* $l_{0|X \backslash X_0} = +\infty$*;*

(iii) For all $F \in \mathcal{F}$*, for all open covers* $\{G_i : i \in I\}$ *of* $F \cap X_0$ *and for all* $\varepsilon > 0$*, there exists a finite set* $\{G_{i_1}, ..., G_{i_N}\} \subset \{G_i : i \in I\}$ *such that*

$$\limsup \mu_\alpha^{t_\alpha}(F) - \limsup \mu_\alpha^{t_\alpha}\Big(\bigcup_{1 \leq j \leq N} \overline{G_{i_j}}\Big) < \varepsilon.$$

The following conclusions hold.

(a) $(i) \Rightarrow (iii)$*,* $(ii) \Rightarrow (iii)$*, and* $(i) \Rightarrow (ii)$ *when* $\mathcal{K} \subset \mathcal{F}$*.*

(b) If (iii) *holds and* $\Lambda(\cdot)$ *exists on the negative part* $\mathcal{T}_-$ *of an approximating class* $\mathcal{T}$*, then* (μ_α) *satisfies a large deviation principle with powers* (t_α) *and rate function* $l_{\mathcal{T}}$*. Moreover,* $l_{\mathcal{T}} = l_{\mathcal{T}_-}$ *and*

$$\forall x \in X_0, \qquad l_{\mathcal{T}}(x) = \inf_{t>0} \sup_{\{h\in\mathcal{T}_-:h(x)\geq -t\}} \{-\Lambda(h)\} = \inf_{t>0} \sup_{\{h\in\mathcal{T}:h(x)\geq -t\}} \{ \lim_{M\to+\infty} -\underline{\Lambda}(h_M)\}. \tag{8}$$

If only one rate function can govern the large deviations, then

$$\forall x \in X, \qquad l_{\mathcal{T}}(x) = \sup_{G\in\mathcal{G},G\ni x} \inf_{y\in G\cap X_0} l_{\mathcal{T}}(y).$$

Proof. $(i) \Rightarrow (ii)$ is clear when $\mathcal{K} \subset \mathcal{F}$, since in this case we have for each compact $K \subset X_0$,

$$\limsup \mu_\alpha^{t_\alpha}(X\backslash K) \geq \sup_{X\backslash K} e^{-l_0} \geq \sup_{X\backslash X_0} e^{-l_0}.$$

Assume that (ii) holds. Let $F \in \mathcal{F}$, let $\{G_i : i \in I\}$ be an open cover of $F \cap X_0$ and let $\varepsilon > 0$. By hypotheses, each $x \in X\backslash X_0$ has an open neighborhood G_x such that $\limsup \mu_\alpha^{t_\alpha}(G_x) < \varepsilon$. Let $K \in \mathcal{K}$ satisfying $\limsup \mu_\alpha^{t_\alpha}(X\backslash K) < \varepsilon$. Then $\{G_i : i \in I\} \cup \{G_x : x \in X\backslash X_0\}$ is an open cover of F and a fortiori of $F \cap K$, hence there exists a finite set $\{G_{i_1}, ..., G_{i_N}, G_{x_1}, ...G_{x_M}\}$ covering $F \cap K$ so that

$$\limsup \mu_\alpha^{t_\alpha}(F) = \limsup \mu_\alpha^{t_\alpha}(F\cap K) \vee \limsup \mu_\alpha^{t_\alpha}(F \cap X\backslash K) \leq \limsup \mu_\alpha^{t_\alpha}(\bigcup_{1\leq j\leq N} G_{i_j}) + \varepsilon,$$

and (iii) holds. If (i) holds, then the finite cover can be obtained from $\{G_i : i \in I\}$ and the above expression is still valid. This proves (a).
Assume that (iii) holds and $\Lambda(\cdot)$ exists on the negative part $\mathcal{T}_-$ of some approximating class $\mathcal{T}$. By Theorem 3 of [4] (and the comment before Proposition 2), (μ_α) satisfies a large deviation principle with powers (t_α) and rate function $l_{\mathcal{T}}$ defined in (7) with moreover

$$-\inf_{x\in G} l_{\mathcal{T}}(x) = \inf_{0<s<\infty} \sup_{h\in\mathcal{T}_{G,s}} \Lambda(h) \qquad \text{for all } G \in \mathcal{G}.$$

By lower semi-continuity the above expression determines $l_{\mathcal{T}}$ and since $\mathcal{T}_{G,s} = (\mathcal{T}_-)_{G,s}$, the same reasoning with the approximating class $\mathcal{T}_-$ yields $l_{\mathcal{T}} = l_{\mathcal{T}_-}$. Let $x \in X_0$ and ν be a real such that $\nu > \sup_{t>0} \inf_{\{h\in\mathcal{T}:h(x)\geq -t\}} \lim_M \underline{\Lambda}(h_M)$. For each $t > 0$, there exists $h_t \in \mathcal{T}$ such that $h_t(x) \geq -t$ and $\nu > \underline{\Lambda}(h_t)$. By Proposition 1 we get

$$\nu \geq \sup_{y\in X} \{h_t(y) - l_{\mathcal{T}}(y)\} \geq h_t(x) - l_{\mathcal{T}}(x) \geq -t - l_{\mathcal{T}}(x),$$

and finally $\nu \geq -l_{\mathcal{T}}(x)$ by letting $t \to 0$. Since ν is arbitrary, it follows that

$$-l_{\mathcal{T}}(x) \leq \sup_{t>0} \inf_{\{h\in\mathcal{T}:h(x)\geq -t\}} \lim_M \underline{\Lambda}(h_M),$$

and since

$$-l_{\mathcal{T}}(x) = -l_{\mathcal{T}_-}(x) = \sup_{t>0} \inf_{\{h\in\mathcal{T}_-:h(x)\geq -t\}} \overline{\Lambda}(h),$$

the two first assertions of (b) are proved. If only one rate function can govern the large deviations, then the lower regularization of the function l defined in Remark 1 coincides with $l_{\mathcal{T}}$; consequently we have for all $x \in X$,

$$l_{\mathcal{T}}(x) = \sup_{G\in\mathcal{G},G\ni x} \inf_{y\in G} l(y) = \sup_{G\in\mathcal{G},G\ni x} \inf_{y\in G\cap X_0} l(y) = \sup_{G\in\mathcal{G},G\ni x} \inf_{y\in G\cap X_0} l_{\mathcal{T}}(y).$$

□

Remark 1. It is easy to see that $l_{\mathcal{T}}(x) = +\infty$ for all $x \in X\backslash\overline{X_0}$ ([4], Remark 1). Condition (iii) in Proposition 2 implies that $\lim \mu_\alpha^{t_\alpha}(F) = 0$ for all closed sets $F \subset X\backslash X_0$. In fact, the proof of Proposition 3 of [4] shows that the large deviation upper (resp. lower) bounds hold also with the function l defined by

$$l(x) = \begin{cases} l_{\mathcal{T}}(x) & \text{if } x \in X_0 \\ \\ +\infty & \text{if } x \in X\backslash X_0; \end{cases}$$

it follows that under exponential tightness (ii) is equivalent to (iii) when l is lower semi-continuous and X regular.

3 Main result

In this section we establish our general version of Bryc's theorem, whose usual algebraic statement in the completely regular Hausdorff case is recovered by taking $\mathcal{A} = \mathcal{C}_b(X)$ in Theorem 1; recall that in this case $X_0 = X$, so that the general hypotheses reduce to exponential tightness, and (9) coincides with (1). The improvement is threefold: first, it allows a general separating algebra (resp. well-separating class) $\mathcal{A}$; secondly, the rate function is obtained in terms of the negative part $\mathcal{A}_-$ of $\mathcal{A}$ when $\mathcal{A}$ does not vanish identically at any point of X; finally, the results hold for any topological space, under the stronger hypothesis of X_0-exponential tightness (or exponential tightness plus some extra conditions), and with the restriction that the usual form of the rate function is obtained only on X_0.
Let us stress that more than the large deviation property itself, the hard part consists in obtaining the rate function in terms of $\mathcal{A}$ and $\mathcal{A}_-$, respectively. To our knowledge, up to now, in the algebraic case such formulas were known only when $\mathcal{A} = \mathcal{C}_b(X)$ and X completely regular Hausdorff. The proof is heavily based on Proposition 2. For instance, it is thanks to the expression (8) of the rate function together with Lemma 1 that we can write (14) and (15) leading to (9).

Lemma 1. *For each set $\mathcal{T} \subset \mathcal{C}(X)$ and each $x \in X$ we have*

$$\sup_{t>0} \inf_{\{h\in\mathcal{T}:h(x)\geq -t\}} \overline{\Lambda}(h) \geq \inf_{h\in\mathcal{T}} \{\overline{\Lambda}(h) - h(x)\}$$

and

$$\sup_{t>0} \inf_{\{h\in\mathcal{T}:h(x)\geq -t\}} \lim_{M\to+\infty} \overline{\Lambda}(h_M) \geq \inf_{h\in\mathcal{T}} \{ \lim_{M\to+\infty} \overline{\Lambda}(h_M) - h(x)\}$$

with equalities when $\mathcal{T}$ is stable by translations. The same holds with $\underline{\Lambda}$ in place of $\overline{\Lambda}$.

Proof. Let δ be a real such that

$$\sup_{t>0} \inf_{\{h\in\mathcal{T}: h(x)\geq -t\}} \overline{\Lambda}(h) < \delta.$$

For each $t > 0$ there exists $h_t \in \mathcal{T}$ such that $\overline{\Lambda}(h_t) < \delta$ and $h_t(x) \geq -t$, hence $\overline{\Lambda}(h_t) - h_t(x) < \delta + t$ and

$$\inf_{h\in\mathcal{T}} \{\overline{\Lambda}(h) - h(x)\} \leq \inf_{t>0} \{\overline{\Lambda}(h_t) - h_t(x)\} \leq \delta,$$

which proves the first inequality; the proof of the second one is similar. The assertion about the equality is clear, as well as the last assertion. □

Theorem 1. *Assume that (μ_α) is exponentially tight with respect to (t_α), and satisfies one of the conditions of Proposition 2 (in particular under X_0-exponential tightness). If $\Lambda(\cdot)$ exists on the bounded-above part $\mathcal{A}_{ba}$ of a set $\mathcal{A}$ of real-valued continuous functions on X, which is either an algebra separating the points or a well-separating class, then (μ_α) satisfies a large deviation principle with powers (t_α) and rate function J verifying $J_{|X\setminus\overline{X_0}} = +\infty$ and*

$$\forall x \in X_0, \qquad J(x) = \sup_{h\in\mathcal{A}_{ba}} \{h(x) - \Lambda(h)\} = \sup_{h\in\mathcal{A}} \{h(x) - \overline{\Lambda}(h)\} = \sup_{h\in\mathcal{A}} \{h(x) - \underline{\Lambda}(h)\} \tag{9}$$

$$= \sup_{h\in\mathcal{A}} \{h(x) - \lim_{M\to+\infty} \underline{\Lambda}(h_M)\},$$

where $h_M = h1_{\{h<M\}} - \infty 1_{\{h\geq M\}}$ for all $h \in \mathcal{A}$ and all reals M. When X is regular (9) determines uniquely the rate function by

$$\forall x \in X, \qquad J(x) = \sup_{G\in\mathcal{G}, G\ni x} \inf_{y\in G\cap X_0} J(y).$$

When $\mathcal{A}$ does not vanish identically at any point of X (in particular when $\mathcal{A}$ contains the constants as in the well-separating case) it is sufficient to assume the existence of $\Lambda(\cdot)$ on the negative part $\mathcal{A}_-$ of $\mathcal{A}$ and we can replace $\mathcal{A}_{ba}$ by $\mathcal{A}_-$ in (9).

Proof. Let $h \in \mathcal{C}(X)_-$ such that $\overline{\Lambda}(h) > -s$ for some $s > 0$, and put $h_s = h \vee -s$. First assume that $\mathcal{A}$ is an algebra separating the points and put $g = \sqrt{-h_s}$. Let $\mathcal{B}$ be the algebra generated by $\mathcal{A} \cup \{c\}$ where c is any nonzero constant function, and note that any element $g \in \mathcal{B}$ has the form $g = k + t$ for some $k \in \mathcal{A}$ and some constant t (i.e., $\mathcal{B} = \mathcal{A} + \mathbb{R}$). By the Stone-Weierstrass theorem, there is a net $(g_i)_{i\in I}$ in $\mathcal{B}$ converging uniformly on compact sets to g. Put $h_i = -g_i^2$, $k_i = h_i \vee -2s$ for all $i \in I$, and note that (h_i) and (k_i) converge uniformly on compact sets to h_s. Let $K \in \mathcal{K}$ such that $\limsup \mu_\alpha^{t_\alpha}(X\setminus K) < e^{-3s}$. Assume that $\Lambda(\cdot)$ exists on $\mathcal{A}_{ba}$, and note that this gives the existence of $\Lambda(\cdot)$ on $\mathcal{B}_-$. Since for each $i \in I$, and each subnet $(\mu_\beta^{t_\beta})$ of $(\mu_\alpha^{t_\alpha})$ we have

$$e^{-2s} \leq e^{\Lambda(k_i)} = \lim \mu_\beta^{t_\beta}(e^{k_i/t_\beta}) = \limsup \mu_\beta^{t_\beta}(e^{k_i/t_\beta} 1_K) \vee \limsup \mu_\beta^{t_\beta}(e^{k_i/t_\beta} 1_{X\setminus K}),$$

and

$$e^{-s} \leq \limsup \mu_\alpha^{t_\alpha}(e^{h_s/t_\alpha}) = \limsup \mu_\alpha^{t_\alpha}(e^{h_s/t_\alpha} 1_K) \vee \limsup \mu_\alpha^{t_\alpha}(e^{h_s/t_\alpha} 1_{X\setminus K}),$$

it follows that $\lim \mu_\alpha^{t_\alpha}(e^{k_i/t_\alpha} 1_K)$ exists with

$$e^{\Lambda(k_i)} = \lim \mu_\alpha^{t_\alpha}(e^{k_i/t_\alpha} 1_K) \qquad \text{for all } i \in I, \tag{10}$$

and

$$\limsup \mu_\alpha^{t_\alpha}(e^{h_s/t_\alpha}) = \limsup \mu_\alpha^{t_\alpha}(e^{h_s/t_\alpha}1_K). \tag{11}$$

The inequalities

$$\log \mu_\alpha^{t_\alpha}(e^{k_i/t_\alpha}1_K) - \sup_{x\in K}|k_i(x)-h(x)| \leq \log \mu_\alpha^{t_\alpha}(e^{h_s/t_\alpha}1_K) \leq$$

$$\log \mu_\alpha^{t_\alpha}(e^{k_i/t_\alpha}1_K) + \sup_{x\in K}|k_i(x)-h_s(x)|$$

combined with (10) and (11) yield

$$\Lambda(k_i) - \sup_{x\in K}|k_i(x)-h_s(x)| \leq \log\liminf \mu_\alpha^{t_\alpha}(e^{h_s/t_\alpha}1_K) \leq \log\liminf \mu_\alpha^{t_\alpha}(e^{h_s/t_\alpha}) \leq$$

$$\log\limsup \mu_\alpha^{t_\alpha}(e^{h_s/t_\alpha}) = \log\limsup \mu_\alpha^{t_\alpha}(e^{h_s/t_\alpha}1_K) \leq \Lambda(k_i) + \sup_{x\in K}|k_i(x)-h_s(x)|,$$

and by taking the limit along i, it follows that $\Lambda(h)$ exists with

$$\Lambda(h) = \Lambda(h_s) = \lim \Lambda(k_i) = \lim \Lambda(h_i). \tag{12}$$

Since h is arbitrary in $\mathcal{C}(X)_-$, $\Lambda(\cdot)$ exists on $\mathcal{C}(X)_-$ and

$$\inf_{h\in\mathcal{C}(X)_-}\{\Lambda(h)-h(x)\} = \inf_{h\in\mathcal{B}_-}\{\Lambda(h)-h(x)\} \qquad \text{for all } x\in X. \tag{13}$$

By Proposition 2 and Lemma 1 with $\mathcal{T} = \mathcal{C}(X)$ it follows that (μ_α) satisfies a large deviation principle with powers (t_α) and rate function $l_{\mathcal{C}(X)}$ taking infinite value outside $\overline{X_0}$ (see Remark 1) and satisfying

$$\forall x\in X_0, \qquad -l_{\mathcal{C}(X)}(x) = \inf_{h\in\mathcal{C}(X)}\{\lim_{M\to+\infty}\underline{\Lambda}(h_M)-h(x)\} = \inf_{h\in\mathcal{C}(X)}\{\underline{\Lambda}(h)-h(x)\} \tag{14}$$

$$= \inf_{h\in\mathcal{C}(X)_-}\{\Lambda(h)-h(x)\} = \inf_{h\in\mathcal{B}_-}\{\Lambda(h)-h(x)\} \geq \inf_{h\in\mathcal{B}_{ba}}\{\Lambda(h)-h(x)\} = \inf_{h\in\mathcal{A}_{ba}}\{\Lambda(h)-h(x)\}$$

$$\geq \inf_{h\in\mathcal{A}}\{\underline{\Lambda}(h)-h(x)\} = \inf_{h\in\mathcal{B}}\{\underline{\Lambda}(h)-h(x)\} \geq \inf_{h\in\mathcal{C}(X)}\{\underline{\Lambda}(h)-h(x)\},$$

where the fourth equality follows from (13), and the last two equalities follow by noting that $\overline{\Lambda}(h) = \overline{\Lambda}(k)+t$ when $h = k+t$ with $k\in\mathcal{A}$ and $t\in\mathbb{R}$. Consequently, the above inequalities are equalities, (9) holds and the first assertion of the algebraic case is proved. The assertion concerning the regular case follows from Proposition 2. The last assertion follows by noting that when $\mathcal{A}$ does not vanish identically at any point of X (in particular when $\mathcal{A}$ contains the constants) we can use $\mathcal{A}$ in place of $\mathcal{B}$, and it is sufficient to assume the existence of $\Lambda(\cdot)$ on $\mathcal{A}_-$.

Assume now that $\mathcal{A}$ is a well-separating class. Let $\mathcal{A}_-^\vee$ be the set constituted by the finite maxima of elements of $\mathcal{A}_-$. By Lemma 4.4.9 of [6] (which remains true for any topological space), for each $K\in\mathcal{K}$ and each $\varepsilon>0$ there exists $h_{K,\varepsilon}\in\mathcal{A}_-^\vee$ such that $\sup_{x\in K}|h_{K,\varepsilon}(x)-h(x)|<\varepsilon$. The nets $(h_i)_{i\in I}$ and $(k_i)_{i\in I}$, where $k_i = h_i\vee -2s$ and $I=\{(K,\varepsilon): K\in\mathcal{K}, \varepsilon>0\}$ (as a product directed set), converge uniformly on compact sets to h. A similar proof as above with $\mathcal{A}_-^\vee$ in place of $\mathcal{B}_-$ gives the existence of $\Lambda(\cdot)$ on $\mathcal{C}(X)_-$, and the large deviation principle with rate function $l_{\mathcal{C}(X)}$ satisfying

$$\forall x\in X_0, \qquad -l_{\mathcal{C}(X)}(x) = \inf_{h\in\mathcal{C}(X)}\{\lim_{M\to+\infty}\underline{\Lambda}(h_M)-h(x)\} = \inf_{h\in\mathcal{C}(X)}\{\underline{\Lambda}(h)-h(x)\} \tag{15}$$

$$= \inf_{h\in\mathcal{C}(X)_-}\{\underline{\Lambda}(h)-h(x)\} = \inf_{h\in\mathcal{A}_-^{\vee}}\{\Lambda(h)-h(x)\} = \inf_{h\in\mathcal{A}_-}\{\Lambda(h)-h(x)\}$$

$$\geq \inf_{h\in\mathcal{C}(X)}\{\underline{\Lambda}(h)-h(x)\},$$

where the last equality follows by noting that $\Lambda(h) = \max_{1\leq j\leq r}\Lambda(h_j)$ when $h = \bigvee_{j=1}^{r} h_j$ with $\{h_j : 1\leq j\leq r\}\subset\mathcal{A}$. Therefore, the above inequality is an equality, which proves the well-separating case. □

The following corollary gives the general variational form of any rate function on a completely regular space in terms of $\mathcal{C}(X)$; this result seems new in absence of exponential tightness. When X is Polish or locally compact Hausdorff, it gives the general form of any tight rate function in terms of any separating algebra or well-separating class, since in this case the exponential tightness holds. In the locally compact Hausdorff case, we may compare it with a similar result obtained in the next section when the exponential tightness fails (Corollary 4).

Corollary 1. *Let X be completely regular. If (μ_α) satisfies a large deviation principle with powers (t_α), then the rate function J is given by*

$$J(x) = \sup_{h\in\mathcal{C}(X)_-}\{h(x)-\Lambda(h)\} = \sup_{h\in\mathcal{C}(X)}\{h(x)-\overline{\Lambda}(h)\} = \sup_{h\in\mathcal{C}(X)}\{h(x)-\underline{\Lambda}(h)\}$$

$$= \sup_{h\in\mathcal{C}(X)}\{h(x)-\lim_{M\to+\infty}\underline{\Lambda}(h_M)\} \qquad \textit{for all } x\in X.$$

If moreover (μ_α) is exponentially tight with respect to (t_α), then we can replace $\mathcal{C}(X)$ by $\mathcal{A}$ in the last two above equalities, and $\mathcal{C}(X)_-$ by $\mathcal{A}_{ba}$ (resp. $\mathcal{A}_-$ when $\mathcal{A}$ does not vanish identically at any point of X) in the first equality (with $\mathcal{A}$ any set of real-valued continuous functions on X, which is either an algebra separating the points or a well-separating class).

Proof. Since X is completely regular, by Theorem 3 of [4] a large deviation principle implies condition (iii) of Proposition 2. The first assertion is then a direct consequence of Proposition 2 and Lemma 1 with $\mathcal{T} = \mathcal{C}(X)$. The existence of $\Lambda(\cdot)$ on $\mathcal{C}(X)_{ba}$ follows from the generalized version of Varadhan's theorem for bounded above continuous functions ([5], Theorem 3.3); the last assertion follows then from Theorem 1. □

The following Prohorov-type result generalizes to any topological space the preceding versions for normal Hausdorff and completely regular spaces given in [5] and [4], respectively. The notations $\Lambda^{(\mu_\beta^{t_\beta})}(h)$ means that the limit is taken along the subnet $(\mu_\beta^{t_\beta})$ (i.e. $\Lambda^{(\mu_\beta^{t_\beta})}(h) = \log\lim\mu_\beta^{t_\beta}(e^{h/t_\beta})$).

Corollary 2. *If (μ_α) is X_0-exponentially tight with respect to (t_α) and $\mathcal{C}(X)$ separates the points, then (μ_α) has a subnet (μ_β) satisfying the conclusions of Theorem 1 with any set $\mathcal{A}$ of real-valued continuous functions on X, which is either an algebra separating the points or a well-separating class.*

Proof. Let $\mathcal{A}\subset\mathcal{C}(X)$ as above and put $\Lambda_\alpha(h) = \log\mu_\alpha^{t_\alpha}(e^h)$ for all $h\in\mathcal{A}$, so that (Λ_α) is a net in the compact set $[-\infty,+\infty]^{\mathcal{A}}$ (provided with the product topology). Therefore, (Λ_α) has a converging subnet (Λ_β), i.e., $\lim\Lambda_\beta(h) = \Lambda^{(\mu_\beta^{t_\beta})}(h)$ exists for all $h\in\mathcal{A}$. The conclusion follows from Theorem 1 applied to $(\nu_\beta^{t_\beta})$. □

Corollary 3 generalizes Theorem 7.1 of [2] where X is required to be metrizable, and also strengthens it by only requiring the existence of $\Lambda(\cdot)$ on $\mathcal{A}_-$ and giving the rate function in terms of $\mathcal{A}_-$ (by continuous affine functionals, we mean those of the form $u+c$, where u belongs to the topological dual and c is a real) .

Corollary 3. *Let X be a locally convex real topological vector space and assume that (μ_α) is exponentially tight with respect to (t_α). If $\Lambda(\cdot)$ exists on the negative part of the set $\mathcal{A}$ of all finite minima of continuous affine functionals, then (μ_α) satisfies a large deviation principle with powers (t_α) and rate function*

$$J(x) = \sup_{h\in\mathcal{A}_-}\{h(x)-\Lambda(h)\} = \sup_{h\in\mathcal{A}}\{h(x)-\overline{\Lambda}(h)\} = \sup_{h\in\mathcal{A}}\{h(x)-\underline{\Lambda}(h)\} =$$

$$\sup_{h\in\mathcal{A}}\{h(x)-\lim_{M\to+\infty}\underline{\Lambda}(h_M)\} \qquad \textit{for all } x\in X.$$

Proof. Since the topological dual of a locally convex real topological vector space separates the points, $\mathcal{A}$ is a well-separating class and the result follows from Theorem 1. □

Example 1. We briefly describe here a regular Hausdorff space X such that $X\backslash X_0$ is a singleton; in other words, X fails to be completely regular because of one point (it is taken from Example 8 of [1], and we refer to this paper for more details). In particular, the usual version of Bryc's theorem does not work, but $\mathcal{C}(X)$ separates the points so that Theorem 1 and Corollary 2 apply. Note that this is not a trivial property when complete regularity fails, since there are regular spaces (with more than one point) where $\mathcal{C}(X)$ is reduced to constants ([7]).
Let H (resp. K) be the set of all irrational numbers belonging (resp. not belonging) to the standard Cantor set, and let $\psi : K \to H$ be a homeomorphism. Let $D =]0,1[^2 \setminus \{(x,x) : x \text{ rational}\}$ and define the following topology on D. Each (x,y) is isolated when $x\neq y$; a basic neighborhood of $(x,x)\in H^2$ (resp. $(x,x)\in K^2$) is of the form $[(x,x),(x+\varepsilon,x)[$ (resp. $[(x,x),(x,x-\varepsilon)[$) for some $\varepsilon>0$. For each integer n let D_n be a copy of D, and let Y be the quotient space of the topological sum of the $D_n' s$, where the equivalence relation is given by identifying each $(x,x)\in K_n^2$ with $(\psi(x),\psi(x))\in H_{n+1}^2$. Put $X = Y\cup\{p\}$ where p is any extra point, and define the basic open neighborhoods of p the sets of the form

$$G_{n,p} = \{p\}\cup(D_n\backslash \text{ diagonal of } D_n)\cup\bigcup_{m>n} D_m$$

for some integer n. It is easy to verify that X is regular Hausdorff with $X\backslash\{p\}\subset X_0$; in fact, every point of $X\backslash\{p\}$ has a clopen neighborhood basis. However, p cannot be separated from $X\backslash G_{n,p}$ (for any $G_{n,p}$) by a continuous function, i.e., $X\backslash X_0 = \{p\}$ and X is not completely regular.

Remark 2. In the algebraic case, when $\mathcal{A}\subset\mathcal{C}_b(X)$ and $\Lambda(\cdot)$ exists on $\mathcal{A}$, then $\Lambda(\cdot)$ exists on the unital algebra $\mathcal{B}=\mathcal{A}+\mathbb{R}$, and on its uniform closure $\mathcal{B}'$, which is stable by finite minima, as it is well known ([12], Lemma 4.3.3). Since $\mathcal{B}'$ is then a well-separating class, in the completely regular Hausdorff case, it follows from Bryc's theorem that under exponential tightness, large deviations hold with rate function

$$\forall x\in X, \qquad J(x) = \sup_{h\in\mathcal{C}_b(X)}\{h(x)-\Lambda(h)\}. \tag{16}$$

If moreover X is metric, then the sup in (16) can be taken on $\mathcal{B}'$ by [9], and then on $\mathcal{B}$ by the continuity of $\Lambda(\cdot)$ with respect to the uniform metric, hence finally on $\mathcal{A}$. This shows that when X is metric and $\mathcal{A} \subset \mathcal{C}_b(X)$ the first conclusion of Theorem 1 follows easily from known results; however, the expression of the rate function in terms of $\mathcal{A}_-$ seems new, except when $\mathcal{A} = \mathcal{C}_b(X)$ and X Polish (see the proof of Bryc's theorem given in [8]). Also in the Polish case, the fact that the existence of $\Lambda(\cdot)$ on $\mathcal{C}_b(X)_-$ implies large deviations was already known (see the proof of Corollary 1.2.5 of [8]).

Remark 3. In the completely regular Hausdorff case, the strengthening of the known versions with well-separating class consists in the form of the rate function in terms of $\overline{\Lambda}(\cdot)$ (resp. $\lim_{M\to+\infty} \underline{\Lambda}(\cdot_M)$) on $\mathcal{A}$, and in terms of $\Lambda(\cdot)$ on $\mathcal{A}_-$; the formula with $\underline{\Lambda}(\cdot)$ was known (see Exercise 4.4.1 of [6]). When $\Lambda(\cdot)$ exists on the whole well-separating class, (9) was proved in [9] for a sequence $(\mu_n^{1/n})$ and X metric (the result is stated there for normal spaces but the author uses a metric in the proof). Unlike [9], Proposition 2 allows to prove (9) without using the stability by finite minima; nevertheless, this property is required to get the existence of $\Lambda(\cdot)$.

4 The case locally compact

In this section we assume that X is locally compact regular, we drop the exponential tightness hypothesis, and we consider an algebra $\mathcal{A}$ of real-valued continuous functions on X vanishing at ∞, which separates the points and does not vanish identically at any point of X.
We first recall some basic topological facts. Here, the notion of local compactness is the one of [11], that is, each point has a compact neighborhood. This definition differs from others ones where it is asked that each point has an open neighborhood with compact closure (see for example [10]). Such a space need not be regular; however, it is known that X is regular if and only if X is completely regular, and this holds in particular when X is Hausdorff. The one-point compactification $\hat{X}$ of X (where the neighborhood of ∞ are given by the complements of closed compact subsets of X) is Hausdorff (resp. regular) if and only if X is Hausdorff (resp. regular) ([11]).
Let us look the following example: Take $X = \mathbb{N}\backslash\{0\}$ and μ_n the uniform probability measure on $\{1, ..., n\}$ for all $n \in \mathbb{N}\backslash\{0\}$; then, large deviations holds with rate function $J = 0$, so that $\Lambda(h) = \sup_{x\in X} h(x) \geq 0$ for all $h \in \mathcal{A}$, which implies

$$\sup_{h\in\mathcal{A}}\{-\Lambda(h)\} \geq \sup_{x\in X, h\in\mathcal{A}}\{h(x) - \Lambda(h)\}.$$

The next theorem shows that the above inequality is in general sufficient (and necessary in the Hausdorff case) to get large deviations with a rate function J satisfying a condition weaker than the tightness (namely (19)), and having still the form (17); indeed, J is tight if and only if the L.H.S. of (19) equals $+\infty$. Theorem 2 supplies a broad class of examples where the classical version of Bryc's theorem does not work, since any net of Borel probability measures on any regular locally compact space satisfying a large deviation principle with no-tight rate function, will satisfy the hypotheses.

Theorem 2. *Let X be locally compact regular, let $\mathcal{A}$ be an algebra of real-valued continuous functions on X vanishing at ∞, which separates the points and does not vanish identically at any point of X, and assume that $\Lambda(\cdot)$ exists on $\mathcal{A}$. The following conclusions hold.*

(a) (μ_α) *satisfies a vague large deviation principle with powers* (t_α) *and rate function*

$$J(x) = \sup_{h\in\mathcal{A}}\{h(x) - \Lambda(h)\} \qquad \text{for all } x \in X. \tag{17}$$

If moreover

$$\sup_{h\in\mathcal{A}}\{-\Lambda(h)\} \geq \sup_{x\in X, h\in\mathcal{A}}\{h(x) - \Lambda(h)\}, \tag{18}$$

then the large deviation principle holds with the same rate function, and

$$\sup_{K\in\mathcal{K}} \inf_{x\in X\setminus K} J(x) \geq \sup_{x\in X} J(x). \tag{19}$$

(b) If X *is Hausdorff and* (μ_α) *satisfies a large deviation principle with powers* (t_α)*, then the rate function is given by (17), and (19) implies (18).*

(c) (μ_α) *is exponentially tight with respect to* (t_α) *if and only if*

$$\sup_{h\in\mathcal{A}}\{-\Lambda(h)\} = +\infty.$$

Proof. Let $\phi : X \to \hat{X}$ be the canonical imbedding of X in its one-point compactification $\hat{X} = X \cup \{\infty\}$, let $\mathcal{C}_0(X)$ be the algebra of continuous functions on X vanishing at ∞, and identify $\mathcal{C}_0(X)$ with the set of continuous functions h on $\hat{X}$ such that $h(\infty) = 0$. The hypotheses imply that the algebra $\mathcal{A}+\mathbb{R}$ separates the points of $\hat{X}$. By the Stone-Weierstrass theorem in $\hat{X}$, for each $g \in \mathcal{C}_0(X)$ and each $\varepsilon > 0$, there exists $h \in \mathcal{A}$ and $c \in \mathbb{R}$ such that $\sup_{x\in\hat{X}} |g(x) - (h(x) + c)| \leq \varepsilon$, hence $\sup_{x\in X} |g(x) - h(x)| \leq 2\varepsilon$ since $g(\infty) = h(\infty) = 0$. It follows that $\mathcal{A}$ is uniformly dense in $\mathcal{C}_0(X)$, hence $\Lambda(\cdot)$ exists on $\mathcal{C}_0(X)$. For each $x \in X$ and each $G \in \mathcal{G}$ containing x, there exists $h_{G,x} \in \mathcal{C}_0(X)$ satisfying $1_{\{x\}} \leq h_{G,x} \leq 1_G$, so that $1_{\{x\}} \leq e^{sh_{G,x}-s} \leq 1_G \vee e^{-s}$ for all $s > 0$; in particular, when $G = X\setminus K$ for some $K \in \mathcal{K}$, we have $1_{\{x\}} \leq e^{sh_{G,x}-s} \leq 1_{\hat{X}\setminus K} \vee e^{-s}$. For each $K \in \mathcal{K}$, there is a continuous function $h_{K,\infty}$ on $\hat{X}$ such that $1_{\{\infty\}} \leq h_{K,\infty} \leq 1_{\hat{X}\setminus K}$, so that $1_{\{\infty\}} \leq e^{s(h_{K,\infty}-1)} \leq 1_{\hat{X}\setminus K} \vee e^{-s}$ for all $s > 0$, with $h_{K,\infty} - 1 \in \mathcal{C}_0(X)$. Thus, the algebra $\mathcal{B} = \mathcal{C}_0(X) + \mathbb{R}$ is an approximating class for $\hat{X}$. Since $\lim \phi[\mu_\alpha]^{t_\alpha}(e^{h/t_\alpha})$ exists for all $h \in \mathcal{B}$, it follows from Corollary 3 of [4] that $(\phi[\mu_\alpha])$ satisfies a large deviation principle with powers (t_α). Since $\mathcal{B}$ satisfies the hypotheses of Corollary 2 of [4], the rate function $\hat{J}$ is given by

$$\hat{J}(x) = \sup_{\{h\in\mathcal{B}:h(x)=0\}}\{-\Lambda(h)\} = \sup_{h\in\mathcal{B}}\{h(x) - \Lambda(h)\} = \sup_{h\in\mathcal{C}_0(X)}\{h(x) - \Lambda(h)\} \tag{20}$$

$$= \sup_{h\in\mathcal{A}}\{h(x) - \Lambda(h)\} \qquad \text{for all } x \in \hat{X},$$

where the last equality follows by noting that for each $h \in \mathcal{C}_0(X)$, $\Lambda(h) = \lim \Lambda(h_i)$ for all nets (h_i) in $\mathcal{A}$ converging uniformly to h. In particular

$$\hat{J}(\infty) = \sup_{h\in\mathcal{A}}\{-\Lambda(h)\}. \tag{21}$$

The complete regularity of $\hat{X}$ yields

$$e^{-\hat{J}(\infty)} = \inf_{K\in\mathcal{K}} \limsup \phi[\mu_\alpha]^{t_\alpha}(\hat{X}\setminus K) = \inf_{K\in\mathcal{K}} \limsup \mu_\alpha^{t_\alpha}(X\setminus K),$$

which proves (c). Then, we have

$$\liminf \mu_\alpha^{t_\alpha}(G) = \liminf \phi[\mu_\alpha]^{t_\alpha}(G) \geq \sup_{x \in G} e^{-\hat{J}(x)} \qquad \text{for all } G \in \mathcal{G}, \tag{22}$$

and

$$\limsup \mu_\alpha^{t_\alpha}(K) = \limsup \phi[\mu_\alpha]^{t_\alpha}(K) \leq \sup_{x \in K} e^{-\hat{J}(x)} \qquad \text{for all } K \in \mathcal{K}, \tag{23}$$

and so, (μ_α) satisfies a vague large deviation principle with power (t_α) and rate function (17), since $J = \hat{J}_{|X}$. This proves the first assertion of (a).
Assume moreover that (18) holds. By (20) and (21), (18) is equivalent to

$$e^{-\hat{J}(\infty)} \leq \inf_{x \in X} e^{-\hat{J}(x)}, \tag{24}$$

hence for each $F \in \mathcal{F}$,

$$\limsup \mu_\alpha^{t_\alpha}(F) = \limsup \phi[\mu_\alpha]^{t_\alpha}(F \cup \{\infty\}) \leq \sup_{x \in F \cup \{\infty\}} e^{-\hat{J}(x)} \leq \sup_{x \in F} e^{-\hat{J}(x)},$$

which proves the large deviation upper bounds with J. The lower semi-continuity of $\hat{J}$ at ∞, and (24) imply

$$\sup_{x \in X} J(x) \leq \hat{J}(\infty) = \sup_{K \in \mathcal{K}} \inf_{x \in \hat{X} \setminus K} \hat{J}(x) = \sup_{K \in \mathcal{K}} \inf_{x \in X \setminus K} J(x),$$

and (19) holds. The second assertion of (a) is proved.
Assume that X is Hausdorff and (μ_α) satisfies a large deviation principle with powers (t_α). Then, $\Lambda(\cdot)$ exists on $\mathcal{C}_0(X)$, and (as proved before) $(\phi[\mu_\alpha])$ satisfies a large deviation principle with powers (t_α) and rate function $\hat{J}$ defined in (20). Therefore, (μ_α) satisfies a vague large deviation principle with power (t_α) and rate function $\hat{J}_{|X}$, and by uniqueness of a vague rate function on a locally compact Hausdorff space, the rate function coincides with $\hat{J}_{|X}$. This proves the first conclusion of (b). Assume that (19) holds, and define the function $\hat{l}$ on $\hat{X}$ by

$$\hat{l}(x) = \begin{cases} J(x) & \text{if } x \neq \infty \\ \sup_{y \in X} J(y) & \text{if } x = \infty. \end{cases}$$

If $\hat{l}(\infty) > \sup_{K \in \mathcal{K}} \inf_{x \in \hat{X} \setminus K} \hat{l}(x)$, then

$$\hat{l}(\infty) > \sup_{K \in \mathcal{K}} \inf_{x \in X \setminus K} \hat{l}(x) = \sup_{K \in \mathcal{K}} \inf_{x \in X \setminus K} J(x),$$

which contradicts (19). Therefore, $\hat{l}$ is lower semi-continuous at ∞, and so $\hat{l}$ is lower semi-continuous on $\hat{X}$. The large deviations for $(\mu_\alpha^{t_\alpha})$ with rate function J implies for each $F \in \mathcal{F}$ and each $K \in \mathcal{K}$ with $F \subset X \backslash K$,

$$\limsup \mu_\alpha^{t_\alpha}(F) = \limsup \phi[\mu_\alpha]^{t_\alpha}(F \cup \{\infty\}) \leq \sup_{x \in F} e^{-J(x)} = \sup_{x \in F \cup \{\infty\}} e^{-\hat{l}(x)}$$

$$\leq \sup_{x \in X \setminus K} e^{-J(x)} \leq \sup_{x \in \hat{X} \setminus K} e^{-\hat{l}(x)} \leq \liminf \mu_\alpha^{t_\alpha}(X \backslash K) = \liminf \phi[\mu_\alpha]^{t_\alpha}(\hat{X} \backslash K).$$

Together with (22) and (23), this shows that $(\phi[\mu_\alpha])$ satisfies a large deviation principle with powers (t_α) and rate function $\hat{l}$, hence $\hat{l} = \hat{J}$ and so $\hat{J}(\infty) = \sup_{x \in X} J(x)$, which implies (18) by (20) and (21). $\square$

The two following corollaries are the analogues of Corollaries 1 and 2 respectively, for locally compact spaces when the tightness fails (exponential or of the rate function).

Corollary 4. *Let X be locally compact Hausdorff. If (μ_α) satisfies a large deviation principle with powers (t_α), then the rate function J is given by*

$$J(x) = \sup_{h\in\mathcal{A}}\{h(x) - \Lambda(h)\} \qquad \textit{for all } x \in X,$$

where $\mathcal{A}$ is any algebra of real-valued continuous functions on X vanishing at ∞, which separates the points and does not vanish identically at any point of X.

Proof. The large deviation principle implying the existence of $\Lambda(\cdot)$ on $\mathcal{A}$ by the tightness-free version of Varadhan's theorem ([5]), the conclusion follows from Theorem 2 (b). $\square$

Corollary 5. *Let X be locally compact regular. Let $\mathcal{A}$ be an algebra of real-valued continuous functions on X vanishing at ∞, which separates the points and does not vanish identically at any point of X. If*

$$\sup_{h\in\mathcal{A}}\{-\overline{\Lambda}(h)\} \geq \sup_{x\in X, h\in\mathcal{A}}\{h(x) - \underline{\Lambda}(h)\},$$

then (μ_α) has a subnet (μ_β) satisfying a large deviation principle with powers (t_β) and rate function

$$J(x) = \sup_{h\in\mathcal{A}}\{h(x) - \Lambda^{(\nu_\beta^{t_\beta})}(h)\} \qquad \text{for all } x \in X;$$

moreover we have

$$\sup_{K\in\mathcal{K}} \inf_{x\in X\setminus K} J(x) \geq \sup_{x\in X} J(x).$$

Proof. The same argument as for Corollary 2 gives the existence of $\Lambda^{(\nu_\beta^{t_\beta})}(\cdot)$ on $\mathcal{A}$ for some subnet $(\nu_\beta^{t_\beta})$. The conclusion follows from Theorem 2 (a) since the hypothesis implies (18) with $\Lambda^{(\nu_\beta^{t_\beta})}(h)$ in place of $\Lambda(h)$. $\square$

Remark 4. The assumption of regularity in Theorem 2 is necessary to have the complete regularity of the one-point compactification; if one drop it, then we fall in the general case of the preceding section. The Hausdorff assumption in (b) is only required to ensure the uniqueness of the rate function for vague large deviations; consequently, the same conclusions hold for any regular locally compact space satisfying this property. The assumption that $\mathcal{A}$ does not vanish identically at any point of X is crucial in the proof: it ensures that $\mathcal{A} + \mathbb{R}$ separates the points of $\hat{X}$, which allows to prove the first assertion of (a), part on which is built the rest of the proof.

Remark 5. Under the hypotheses of Theorem 2 and when X is not compact, (18) is equivalent to

$$\sup_{h\in\mathcal{A}}\{-\Lambda(h)\} = \sup_{x\in X, h\in\mathcal{A}}\{h(x) - \Lambda(h)\}. \tag{25}$$

In particular the exponential tightness holds if and only if

$$\sup_{x\in X} J(x) = +\infty.$$

Indeed, assume that (18) holds. The set $\{\hat{J} > \sup_{x\in X}\hat{J}(x)\}$ is open in $\hat{X}$ and since $\{\infty\}$ is not open when X is not compact, we have $\hat{J}(\infty) \leq \sup_{x\in X}\hat{J}(x)$; since the converse inequality holds by (18), we have $\hat{J}(\infty) = \sup_{x\in X}\hat{J}(x)$, which is equivalent to (25).

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