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ON THE DUALITY BETWEEN COALESCING BROWNIAN PARTICLES AND THE HEAT EQUATION DRIVEN BY FISHER-WRIGHT NOISE

TIM HOBSON
Mathematics Institute, University of Warwick, Coventry, England, CV4 7AL
email: thobson@maths.warwick.ac.uk

ROGER TRIBE
Mathematics Institute, University of Warwick, Coventry, England, CV4 7AL
email: tribe@maths.warwick.ac.uk

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Abstract

This paper concerns the Markov process duality between the one-dimensional heat equation driven by Fisher-Wright white noise and slowly coalescing Brownian particles. A representation is found for the law of the solution $x \to U_t(x)$ to the stochastic PDE, at a fixed time, in terms of a labelled system of such particles.

1 Statement of the main result

We consider solutions $U_t(x) \geq 0$ to the stochastic PDE

$$\frac{\partial U}{\partial t} = \Delta U + \sqrt{\lambda U(1-U)}\, dW \tag{1}$$

on the circle $\mathcal{T} = [0,1)$ mod(1). Here W is a space-time white noise on $[0,\infty) \times \mathcal{T}$ and $\lambda > 0$. In [7], Shiga introduced equation (1) as a model for population genetics and established a formula for product moments of U using a Markov duality function and a system of slowly coalescing Brownian particles. Let $((X_t^i)_{i \leq N_t} : t \geq 0)$ be the positions of a finite random number N_t of particles evolving as follows. Each particle follows an independent Brownian motion on the circle $\mathcal{T}$ up to its time of extinction. For each pair of particles we let $L_t^{i,j}$ denote the intersection local time, that is the local time of $X_t^i - X_t^j$ at zero. Independently for each pair, at the rate $\lambda\, dL_t^{i,j}$, the two particles i and j coalesce, that is one of the two particles disappears. Shiga [7] states the formula

$$\mathbf{E}_f \left[U_t(x^1) \dots U_t(x^m) \right] = \mathbf{E}_x \left[\prod_{i=1}^{N_t} f(X_t^i) \right] \tag{2}$$

where the left hand expectation is for a solution to (1) started at continuous $f : \mathcal{T} \to [0,1]$, and the right hand expectation is for the coalescing particle system started from m single particles with initial positions coded by $x = (x^1, \ldots, x^m)$.

We attach labels $W_t^i(dx)$, taking values in $\mathcal{M}(\mathcal{T})$ the space of finite measures on $\mathcal{T}$, to each particle. These labels remain unchanged during the evolution except at coalescent events, when the measure attached to the coalesced particle is the sum of the measures attached to the two coalescing particles. The evolution of the measures W_t^i is therefore determined by the initial values and the particle system. Suppose the initial measure W_0^i is a point mass $\delta_{X_0^i}$ at the position of particle. Then the measure for particle X_t^j at a time $t > 0$ is a sum of point masses, from which one can read off all the initial positions of particles that have coalesced to form X_t^j.

In Section 2 we establish the following duality for the system $((X_t^i, W_t^i)_{i \le N_t} : t \ge 0)$. Write $W_t^i(\phi)$ for the integral of ϕ against the measure $W_t^i(dx)$. Then for measurable $\phi : \mathcal{T} \to [0,\infty)$ and $f : \mathcal{T} \to [0,1]$

$$\mathbf{E}_f \left[\prod_{i=1}^m \left(1 - (1 - e^{-w^i(\phi)}) U_t(x^i) \right) \right] = \mathbf{E}_{(x,w)} \left[\prod_i \left(1 - (1 - e^{-W_t^i(\phi)}) f(X_t^i) \right) \right] \tag{3}$$

where the right hand expectation is for the coalescing particle system started from m single particles with initial positions and labels coded by $x = (x^1, \ldots, x^m)$ and $w = (w^1, \ldots, w^m)$. Note that (2) follows from (3) by letting $\phi \uparrow \infty$, changing f to $1 - f$, and noting that $1 - U_t$ remains a solution to (1).

In Section 3 we construct a continuum limit of our coalescing system. Take, as an initial condition, particles X_0^i with labels $W_0^i = (1/n)\delta_{X_0^i}$, where (X_0^i) form a Poisson point process on $\mathcal{T}$ with intensity $n\,dx$. Then the duality (3) becomes, via Campbell's formula for Poisson processes,

$$\mathbf{E}_f \left[\exp\left(-n \int_{\mathcal{T}} \left(1 - e^{-\frac{1}{n}\phi(x)} \right) U_t(x) dx \right) \right] = \mathbf{E}_{\mathrm{Poisson}(n)} \left[\prod_i \left(1 - (1 - e^{-W_t^i(\phi)}) f(X_t^i) \right) \right] \tag{4}$$

where the expectation on the right hand side is for this Poissonized initial condition (and an empty product takes the value 1). Let $\rho_t^{(n)}$ be the law of $((X_t^i, W_t^i) : i \le N_t)$ under this Poissonized initial condition. We will show that $\rho_t^{(n)} \to \rho_t$ for each $t > 0$ as $n \to \infty$. The convergence here is in distribution and the exact state space is detailed in Section 3. Moreover the family $\rho = (\rho_t : t > 0)$ forms an entrance law for the finite coalescing particle system. In particular under each law ρ_t the total number of particles is finite.

We write $U_t(\phi)$ for $\int_{\mathcal{T}} U_t(x)\phi(x)\,dx$. Passing to the limit in (4) as $n \to \infty$ we obtain (see Section 3.2) for continuous f, ϕ

$$\mathbf{E}_f \left[\exp\left(-U_t(\phi) \right) \right] = \mathbf{E}_\rho \left[\prod_i \left(1 - (1 - e^{-W_t^i(\phi)}) f(X_t^i) \right) \right]$$

where the right hand side denotes an expectation under the entrance law ρ. We may rewrite this as

$$\mathbf{E}_f \left[\exp\left(-U_t(\phi) \right) \right] = \mathbf{E}_\rho \left[\exp\left(-\sum_i W_t^i(\phi) \mathbf{I}(U^i \le f(X_t^i)) \right) \right]$$

where we have introduced an independent I.I.D. family (U^i) of uniform variables on $[0,1]$. This equality between Laplace transforms of random measures on $\mathcal{T}$ shows our main result.

Theorem 1 *Fix $t > 0$. Then $U_t(x)dx$ under $\mathbf{P}_f$ has the same distribution as*

$$\sum_i W_t^i(dx)\mathbf{I}(U^i \leq f(X_t^i)) \tag{5}$$

under the entrance law $\mathbf{P}_\rho$.

Remarks

1. The result should be thought of as the analogue of the duality for the Voter model and coalescing random walks. The long range voter model, under suitable rescaling (see [5]), converges to the stochastic PDE (1). As in the discrete setting, there should be a coupling of the continuum coalescing Brownian particle system together with a solution to the stochastic PDE so that a duality relation holds simultaneously at all times. For details on this, and the extensions below, see the Warwick thesis [4].

2. The same calculations apply in dimension $d = 0$ (that is where we suppress spatial motion) and lead to a representation for the solution X_t to the SDE $dX = \sqrt{\lambda X(1-X)}dB$.

3. The representation can be used to shed light on the compact support property of solutions to (1). The idea is that the indicator variables in the representation (5) allow the solution to be identically zero. For example the fact that there are only finitely many particles at any time $t > 0$ implies, when $f \not\equiv 1$, that $\mathbf{P}_f[U_t \equiv 0] > 0$ for all $t > 0$. The analogous representation holds for solutions on the real line (but one must allow an infinite system of particles). When f is compactly supported, the solution $U_t(x)$ remains compactly supported on $\mathbf{R}$. This can also be deduced from the representation, the key being that only finitely many Brownian particles are used to represent the values of $(U_t(x) : x \in A)$ when A is a bounded interval.

2 The extended duality function

We consider a family of duality functions, one for each measurable $\phi : \mathcal{T} \to [0,\infty)$. For $u \in C(\mathcal{T},[0,1])$ and $(x,w) = ((x^1,w^1),\ldots,(x^n,w^n)) \in (\mathcal{T}\times\mathcal{M}(\mathcal{T}))^n$ define

$$H_\phi(u,(x,w)) := \prod_i H_\phi(u,x^i,w^i) := \prod_i \left(1-(1-e^{-w^i(\phi)})u(x^i)\right).$$

Write L_U and $L_{(X,W)}$ for the generators of the stochastic PDE and the dual weighted particle system. Then formally applying L_U to the function $u \to H_\phi(u,(x,w))$ we have

$$\begin{aligned} & L_U H_\phi(u,(x,w)) \\ = \;& \sum_i \left(\prod_{k\neq i} H_\phi(u,x^k,w^k)\right) \Delta u(x^i)\left(e^{-w^i(\phi)}-1\right) \qquad (6)\\ & \quad + \sum_{i<j}\left(\prod_{k\neq i,j} H_\phi(u,x^k,w^k)\right)\left(1-e^{-w^i(\phi)}\right)\left(1-e^{-w^j(\phi)}\right)\left(u(x^i)-u^2(x^i)\right)\lambda\,\delta_{x^i=x^j} \end{aligned}$$

where $\delta_{x=y}$ is used to represent the spatial covariance of the space-time noise W.
When applying $L_{(X,W)}$ to the function $(x,w) \to H_\phi(u,(x,w))$ we obtain the first term on the right hand side of (6) from the action of the Brownian motions. The coalescence of a pair of particles i and j leads to a jump term of the form

$$\left(\prod_{k\neq i,j} H_\phi(u,x^k,w^k)\right)\left[H_\phi(u,x^i,w^i+w^j) - H_\phi(u,x^i,w^i)H_\phi(u,x^j,w^j)\right]\,\lambda\,\delta_{x^i=x^j}$$

where here we have used $\delta_{x^i=x^j}\,dt$ as a formal expression for the rate of the local time interaction $dL_t^{i,j}$. A small miracle occurs and this simplifies to

$$\left(\prod_{k\neq i,j} H_\phi(u,x^k,w^k)\right)\left(1-e^{-w^i(\phi)}\right)\left(1-e^{-w^j(\phi)}\right)\left(u(x^i)-u^2(x^j)\right)\,\lambda\,\delta_{x^i=x^j}.$$

Summing over all pairs we can match $L_{(X,W)}H_\phi(u,(x,w))$ with the expression in (6). This suggests, via standard duality ideas summarized in Ethier and Kurtz [3] Section 4.4, that the duality relation (3) should hold. We omit the details of a formal proof of this relation since it follows closely the lines of Theorem 1 in Athreya and Tribe [1] (although our case is easier since $0 \le H_\phi(f,(x,w)) \le 1$ and there are no integrability problems). The main step of the proof is to smooth the duality functions to allow them to be in the true domain of the generator.
Remarks

1. For the proof of the duality formula it is convenient to work with continuous ϕ and twice differentiable f. The formula holds however for any measurable ϕ and f. Note the usual vector space monotone class theorems do not apply here. However the set of ϕ for which the duality holds is closed under bounded pointwise limits and contains the continuous functions. Set valued monotone class arguments allow one to show the duality holds for finite sums $\phi(x) = \sum_i c_i \mathbf{I}(x \in \Lambda_i)$, where $\Lambda_i \subseteq \mathcal{T}$ are measurable. Standard approximations then extend this to measurable ϕ. To allow measurable f we need solutions to (1) started from measurable initial conditions (which are not considered in [1], [6], [7]). Take continuous (or smooth) $f^{(n)}$ converging almost everywhere to a measurable f. Using the Green's function representation for solutions, the corresponding solutions $(t,x) \to U_t^{(n)}(x)$ can be checked to satisfy Kolmogorov's tightness criterion as elements of $\mathcal{C}([\delta,T]\times\mathcal{T})$ for any $0<\delta<T<\infty$. Also $t \to U_t^{(n)}(\phi)$ is tight in $\mathcal{C}([0,T])$ for any continuous ϕ. Any limit point satisfies (2) and hence has unique one dimensional marginals. These facts can be used to show that the $U^{(n)}$ converge in distribution to a continuous limit $(U_t(x) : t>0,\, x\in\mathcal{T})$ which uniquely solves (1), and where $U_t(\phi) \to f(\phi)$ as $t \to 0$ for any continuous ϕ. Another monotone class argument shows that the duality formula (3) holds also for measurable f.

2. The duality function will work in a slightly more general setting. The labels can be elements of a vector space V, which add upon coalescence. The duality function $H_\phi(u,x,w)$ can then be replaced by $H_{v^*}(u,x,w) = 1-(1-\exp(\langle v^*,w\rangle))u(x)$, where v^* is an element in the (algebraic) dual vector space V^*, and $\langle v^*,w\rangle$ is the dual pairing.

As a special case of the duality we obtain the law of the sum of the measures labelling particles located in any measurable $\Lambda \subseteq \mathcal{T}$, which we denote by

$$W_t^\Lambda(dx) := \sum_i W_t^i(dx)\,\mathbf{I}(X_t^i \in \Lambda).$$

Taking $f = \mathbf{I}(\Lambda)$ in (3) we obtain the Laplace functional of W_t^Λ

$$\mathbf{E}_{\mathbf{I}_\Lambda}\left[\prod_i \left(1-(1-e^{-w^i(\phi)})U_t(x^i)\right)\right] = \mathbf{E}_{(x,w)}\left[\exp\left(-W_t^\Lambda(\phi)\right)\right].$$

We can deduce formulae for the first and second moments.

$$\mathbf{E}_{(x,w)}\left[W_t^\Lambda(\phi)\right] = \sum_i w^i(\phi)\mathbf{E}_{\mathbf{I}_\Lambda}\left[U_t(x^i)\right],$$

and

$$\begin{aligned}\mathbf{E}_{(x,w)}\left[\left(W_t^\Lambda(\phi)\right)^2\right] &= \sum_{i,j} w^i(\phi)w^j(\phi)\mathbf{E}_{\mathbf{I}_\Lambda}\left[U_t(x^i)U_t(x^j)\right] \\ &\quad + \sum_i \left(w^i(\phi)\right)^2 \mathbf{E}_{\mathbf{I}_\Lambda}\left[U_t(x^i)(1-U_t(x^i))\right]. \qquad (7)\end{aligned}$$

3 The entrance law

3.1 Tightness

A suitable state space for the finite particle system is

$$E = \bigcup_{n=0}^{\infty} \left(\mathcal{T}\times\mathcal{M}(\mathcal{T})\right)^n / \sim .$$

This is the disjoint union of n-fold product spaces, each taken under the quotient of allowing permutation of the coordinates. If we give $\mathcal{M}(\mathcal{T})$ the topology of weak convergence of measures then E is a locally compact metrizable space. Indeed, write $|(x,w)| = n$ when $(x,w) \in (\mathcal{T}\times\mathcal{M}(\mathcal{T}))^n$, and call this the length of (x,w). Then the sets of elements of length at most n are compact subsets.

Recall from Section 1 that $\rho_t^{(n)}$ is the law on E of $((X_t^i, W_t^i) : i \le N_t)$ when started according to $\mathbf{P}_{\text{Poisson}(n)}$. To check tightness we need only consider the number $N_t = |(X_t, W_t)|$ of particles at time t. The Poissonized version of the duality (2) is

$$\mathbf{E}_f\left[\exp\left(-n\int_{\mathcal{T}}(1-U_t(x))\,dx\right)\right] = \mathbf{E}_{\text{Poisson}(n)}\left[\prod_i f(X_t^i)\right]. \qquad (8)$$

Taking $f \equiv \theta \in (0,1)$ gives

$$\mathbf{E}_{\text{Poisson}(n)}\left[\theta^{N_t}\right] = \mathbf{E}_{1-\theta}\left[\exp\left(-n\int U_t(x)\,dx\right)\right] \ge \mathbf{P}_{1-\theta}\left[U_t \equiv 0\right].$$

The right hand side is independent of n. Moreover we claim that it converges to 1 as $\theta \uparrow 1$. This implies, by a Tchebychev argument ($\mathbf{P}[N \ge K] = \mathbf{P}[1-\theta^N \ge 1-\theta^K] \le \mathbf{E}[1-\theta^N]/(1-\theta^K)$), that

$$\sup_n \mathbf{P}_{\text{Poisson}(n)}\left[N_t \ge K\right] \to 0 \quad \text{as } K \to \infty$$

which establishes tightness.

To show the claim we establish death estimates for (1). These show that for small initial conditions the death probabilities are close to those of a super Brownian motion.

Lemma 2 *For a solution U_t to (1) and $\delta \in [0,1]$ let $\tau_\delta := \inf\{t : \sup_x U_t(x) \geq \delta\}$. Recall that $f(1) = \int_{\mathcal{T}} f(x)dx$. Then*

$$\exp\left(-\frac{2\,f(1)}{\lambda(1-\delta)t}\right) - \mathbf{P}_f[\tau_\delta \leq t] \leq \mathbf{P}_f\left[U_t \equiv 0\right] \leq \exp\left(-\frac{2\,f(1)}{\lambda t}\right).$$

Proof. Define $\eta_s = 2/\lambda s$, a solution to $\dot{\eta} = -(\lambda/2)\eta^2$ on $(0,\infty)$. Fix $t > 0$ and define $M_s := \exp\left(-\eta_{t-s} U_s(1)\right)$ for $s < t$ and $M_t := \mathbf{I}(U_t \equiv 0)$. Then Itô's formula shows that M_s is a bounded supermartingale and taking expectations at times $s = t$ and $s = 0$ yields the required upper bound. For the lower bound we use $\eta_s = 2/\lambda(1-\delta)s$. Then M_s is a bounded submartingale up to time τ_δ. Taking expectations at times $s = \tau_\delta \wedge t$ and $s = 0$ we obtain

$$\mathbf{P}_f\left[U_t \equiv 0\,;\, \tau_\delta \geq t\right] + \mathbf{E}_f\left[\exp\left(-\eta_{t-\tau_\delta} U_{\tau_\delta}(1)\right)\,;\, \tau_\delta < t\right] \geq \exp\left(-\frac{2\,f(1)}{\lambda(1-\delta)t}\right)$$

which after rearrangement implies the required lower bound. ∎

Now we can finish the tightness claim. Note that $\mathbf{P}_{1-\theta}\left[U_t \equiv 0\right] \geq \mathbf{P}_{1-\theta}\left[U_{t(\theta)} \equiv 0\right]$ provided that $t(\theta) \leq t$. Choose $t(\theta) = (1-\theta)^{1/2}$ and apply the lower bound from Lemma 2 with $t = t(\theta)$. The term $\exp(-2\,(1-\theta)/\lambda(1-\delta)t(\theta))$ converges to 1 as $\theta \to 1$. We can bound the second term

$$\mathbf{P}_{1-\theta}\left[\tau_\delta \leq t(\theta)\right] \leq \mathbf{P}_{\delta/2}\left[\tau_\delta \leq t(\theta)\right] \tag{9}$$

once $1-\theta \leq \delta/2$. But by the continuity of sample paths this converges to 0 as $t(\theta) \to 0$. The final inequality (9) follows since one can couple two solutions of (1), with initial conditions $U_0 \equiv 1-\theta$ and $V_0 \equiv \delta/2$, so that $U_t(x) \leq V_t(x)$ for all t, x. Such coupling results are standard for equations with Lipschitz coefficients, and follow in our case by an approximation argument (see Pardoux [6] Theorem I.3.1 for such a construction).

3.2 Construction of an entrance law

Write $T_t f((x,w))$ for the expectation of $f((X_t^i, W_t^i))$ when the particle system is started at (x,w). It is straightforward to show, for example by a simple coupling argument, that $T_t : \mathcal{C}_b \to \mathcal{C}_b$, where $\mathcal{C}_b$ is the space of bounded continuous functions on E. (Note T_t does not send $\mathcal{C}_0$, the space of continuous functions vanishing at infinity, into itself.)

For a (Borel) probability ρ on E we write $T_t^*\rho$ for the push forward under T_t, that is the law satisfying $(T_t^*\rho)(f) = \int_E T_t f(x)\rho(dx)$. Then an entrance measure is a family of probabilities $\rho = (\rho_t : t > 0)$ on E satisfying $\rho_{s+t} = T_t^*\rho_s$ for $s, t > 0$. For any fixed $t > 0$ the tightness estimate in Section 3.1 above shows that the laws $(\rho_t^{(n)} : n \geq 1)$ are tight. We want to construct limits for each t that mesh together to form an entrance law. Choose times $s_k > 0$ decreasing to 0. By a diagonal argument we can find a subsequence n' so that $\rho_{s_k}^{(n')}$ converge, with a limit that we call ρ_{s_k}, for all $k \geq 1$. Note that $T_t^*\rho_{s_k}^{(n')} = \rho_{s_l}^{(n')}$ when $t = s_k - s_l > 0$. Passing to the limit, using $T_t : \mathcal{C}_b \to \mathcal{C}_b$, we find that $T_t^*\rho_{s_k} = \rho_{s_l}$. Now finally fill in the other values ρ_t by setting $\rho_t = T_{t-s_k}^*\rho_{s_k}$ for any $t \geq s_k$. This produces an entrance law $\rho = (\rho_t : t > 0)$. Moreover, the same argument shows that $\rho_t^{(n')} \to \rho_t$ for any $t > 0$. One can construct a law $\mathbf{P}_\rho$ on the canonical space $D((0,\infty), E)$ so that the canonical variables $((X_t, W_t) : t > 0)$ are the particle system started according to the entrance law. We do this without further comment (as we did in the statement of the main Theorem 1).

Passing to the limit in (8) gives

$$\mathbf{P}_f\left[U_t \equiv 1\right] = \mathbf{E}_\rho\left[\prod_i f(X_t^i)\right]. \tag{10}$$

This formula determines the law of the particle positions $(X_t^i : i \leq N_t)$ under $\mathbf{P}_\rho$. For continuous f, ϕ the map $(x, w) \to H_\phi(f, (x, w))$ is continuous and passing to the limit in (4) produces

$$\mathbf{E}_f\left[\exp\left(-U_t(\phi)\right)\right] = \mathbf{E}_\rho\left[\prod_i \left(1 - (1 - e^{-W_t^i(\phi)}) f(X_t^i)\right)\right]. \tag{11}$$

Unfortunately we do not believe that the duality formulae (10) and (11) determine the law of $((X_t^i, W_t^i) : i \leq N_t)$ under $\mathbf{P}_\rho$. However we will show in Section 3.4 that there is actually a unique entrance law $\rho = (\rho_t : t > 0)$ that satisfies (10) and (11) at all times $t > 0$ simultaneously and, moreover, that $\rho_t^{(n)} \to \rho_t$ for all $t > 0$.
We can deduce the law of W_t^Λ under $\mathbf{P}_\rho$ for any fixed Λ and t. As before, a monotone class argument shows that (11) holds for all measurable f and ϕ. By taking $f = \mathbf{I}(\Lambda)$ we obtain the Laplace transform of W_t^Λ under $\mathbf{P}_\rho$:

$$\mathbf{E}_{\mathbf{I}(\Lambda)}\left[\exp\left(-U_t(\phi)\right)\right] = \mathbf{E}_\rho\left[\exp\left(-W_t^\Lambda(\phi)\right)\right]. \tag{12}$$

Thus $W_t^\Lambda(dx)$ under $\mathbf{P}_\rho$ has the same distribution as $U_t(x)dx$ under $\mathbf{P}_{\mathbf{I}(\Lambda)}$. In particular we have that $W_t^{\mathcal{T}}(dx) = dx$ for all $t \geq 0$, almost surely.

3.3 Small time behaviour

When the particle density is high the behaviour should be of mean field type, that is almost deterministic. We state two indications of this as a lemma.

Lemma 3 *(a) Under the entrance measure ρ the random measure on $\mathcal{T}$ defined by*

$$\mu_t^{(1)} = \frac{\lambda t}{2} \sum_i \delta_{X_t^i}$$

converges in probability to Lebesgue measure as $t \downarrow 0$.

(b) For $\phi, \psi : \mathcal{T} \to \mathbf{R}$ write $\phi \otimes \psi$ for the function $\phi(x)\psi(y)$ on $\mathcal{T}^2$. Define a random measure on $\mathcal{T}^2$ by

$$\mu_t^{(2)}(\phi \otimes \psi) = \sum_i W_t^i(\phi)\psi(X_t^i).$$

Then, under the entrance measure, $\mu_t^{(2)}$ converges in probability as $t \to 0$ to normalised uniform measure on the diagonal $\{(x, y) \in \mathcal{T}^2 : x = y\}$.

Remark. In particular the number of particles N_t at time t satisfies $(\lambda t/2)N_t \to 1$ in probability as $t \to 0$. Compare this with the non-mean field behaviour of instantaneously coalescing particles established by Donnelly et al. [2].
Proof. Replacing f in (10) by $\exp(-\theta\lambda t f/2)$ for $\theta > 0$, we obtain

$$\mathbf{P}_{1-\exp(-\theta\lambda t f/2)}\left[U_t \equiv 0\right] = \mathbf{E}_\rho\left[\exp\left(-\theta\mu_t^{(1)}(f)\right)\right].$$

Now use the upper and lower bounds on this extinction probability found in Lemma 2. They show that this Laplace functional converges to $\exp(-\theta f(1))$ as $t \to 0$ which implies the desired convergence.

For part (b) recall that $\mu_t^{(2)}(\phi \otimes \mathbf{I}(\Lambda)) = W_t^\Lambda(\phi)$ has the same law as $U_t(\phi)$ under $\mathbf{P}_{I_\Lambda}$ and so, as $t \to 0$, converges to $\int_\Lambda \phi \, dx$ for any continuous ϕ. Thus $\mu_t^{(2)}(h) \to \int_{\mathcal{T}} h(x,x)\, dx$ in probability for $h(x,y) = \sum_{i=1}^n \phi_i(x)\, \mathbf{I}(y \in \Lambda_i)$. An approximation argument, recalling that $\mu_t^{(2)}(1) = 1$, shows the same holds for all continuous $h : \mathcal{T}^2 \to \mathbf{R}$, which implies the desired result. ∎

3.4 Uniqueness of the entrance law

Let $\hat\rho_s$ be the law on E formed by taking the law of the particle positions $(X_s^i : i \le N_s)$ under $\mathbf{P}_\rho$ and attaching the labels $(\lambda s/2)\delta_{X_s^i}$. Then we claim that

$$T^*_{t-s}\hat\rho_s \to \rho_t \quad \text{as } s \to 0. \tag{13}$$

Since the law of the particle positions under $\mathbf{P}_\rho$ is determined, so is the law $T^*_{t-s}\hat\rho_s$ and this claim characterizes the entrance law ρ. Moreover it implies that $\rho_t^{(n)} \to \rho_t$ for each $t > 0$. Indeed, arguing as in Section 3.2, for any subsequence n' there exists a sub-subsequence n'' along which $\rho_s^{(n'')} \to \rho_s$ for all $s > 0$. Now apply the subsequence principle.

To establish the claim (13) we form a coupling of one system of labelled particles $((X_t^i, W_t^i) : i \le N_t)$ with law ρ_t and another $((X_{s,t}^i, W_{s,t}^i) : i \le N_{s,t})$ with law $T^*_{t-s}\hat\rho_s$, constructed as follows. Let the particle positions $(X_{s,r}^i : r \ge s)$ follow those of $(X_r^i : r \ge s)$ under the entrance measure ρ, but define the labels $W_{s,s}^i$ at time s to be $(\lambda s/2)\delta_{X_s^i}$. Then the evolution of the labels $(W_{s,r}^i : r \ge s)$ is determined by the motion of the particle positions, in that they evolve by addition at the same set of collision events as $(X_r^i : r \ge s)$. For this coupling we have the following lemma.

Lemma 4 *For measurable Λ and ϕ let $W_{s,t}^\Lambda(\phi) = \sum_i W_{s,t}^i(\phi)\mathbf{I}(X_{s,t}^i \in \Lambda)$. Then $W_{s,t}^\Lambda(\phi) \to W_t^\Lambda(\phi)$ in probability as $s \to 0$.*

Proof. Define $\Omega_s = \{sN_s \le 4/\lambda\}$. Lemma 3 implies that $\mathbf{P}_\rho[\Omega_s] \to 1$ as $s \to 0$. We will show that

$$\mathbf{E}\left[\left|W_{s,t}^\Lambda(\phi) - W_t^\Lambda(\phi)\right|^2 ; \Omega_s\right] \to 0 \quad \text{as } s \to 0$$

which then implies the result

Both sets of labels $r \to W_r^i$ and $r \to W_{s,r}^i$ follow the same set of coalescences over $r \ge s$ and we may think of each of the differences $W_r^i - W_{s,r}^i$ as a single label that is a signed measure. Note that the duality, and hence the moment formulae, still hold when the labels are signed measures. Thus, conditioning on the information at time s and applying the second moment formula (7), we have

$$\begin{aligned}
&\mathbf{E}\left[\left|W_t^\Lambda(\phi) - W_{s,t}^\Lambda(\phi)\right|^2 ; \Omega_s\right] \\
&= \mathbf{E}\left[\sum_{i,j}\left(W_s^i(\phi) - (\lambda s/2)\phi(X_s^i)\right)\left(W_s^j(\phi) - (\lambda s/2)\phi(X_s^j)\right) U_{t-s}(X_s^i)U_{t-s}(X_s^j) \,;\, \Omega_s\right] \\
&\quad + \mathbf{E}\left[\sum_i \left(W_s^i(\phi) - (\lambda s/2)\phi(X_s^i)\right)^2 U_{t-s}(X_s^i)(1 - U_{t-s}(X_s^i)) \,;\, \Omega_s\right] \\
&= I + II.
\end{aligned}$$

The expectation here is over the weighted particles (X^i_s, W^i_s) under the entrance measure and an independent solution U_t to the stochastic PDE started at $\mathbf{I}_\Lambda$. We rewrite the first term, using the notation from Lemma 3, as

$$I = \mathbf{E}\left[\left(\mu_s^{(2)}(\phi \otimes U_{t-s}) - \mu_s^{(1)}(\phi\, U_{t-s}) \right)^2 ; \Omega_s \right].$$

Lemma 3 and the uniform continuity of $U_t(x)$ imply that both $\mu_s^{(1)}(\phi U_{t-s})$ and $\mu_s^{(2)}(\phi \otimes U_{t-s})$ converge to $U_t(\phi)$ as $s \to 0$. The term $\mu_s^{(2)}(\phi \otimes U_{t-s})$ is bounded by $\|\phi\|_\infty$ and the term $\mu_s^{(1)}(\phi U_{t-s})$ is bounded by 2 on Ω_s, so dominated convergence shows that the first term vanishes as $s \to 0$. We can bound the second term by

$$\begin{aligned} II &\leq 2\mathbf{E}\left[\sum_i (W^i_s(\phi))^2 \right] + (\lambda^2 s^2/2)\mathbf{E}\left[\sum_i \phi^2(X^i_s) \, ; \, \Omega_s \right] \\ &\leq C(\lambda, \phi)\left(\mathbf{E}\left[\sum_i (W^i_s(1))^2 \right] + \mathbf{E}\left[s^2 N_s \, ; \, \Omega_s \right] \right) = IIa + IIb. \end{aligned}$$

Note that $s^2 N_s \leq 4s/\lambda$ on Ω_s which shows that $IIb \to 0$ as $s \to 0$. Set $I^M_j = [(j-1)/M, j/M)$. Then, for any $M \in \mathbf{N}$,

$$\mathbf{E}\left[\sum_i (W^i_s(1))^2 \right] \leq \sum_{j=1}^M \mathbf{E}\left[(W_s^{I^M_j}(1))^2 \right]. \tag{14}$$

Recall that $W^{I^M_j}(1)$ has the same distribution as $U_s(1)$ under $\mathbf{P}_{\mathbf{I}(I^M_j)}$. The first moment $\mathbf{P}_f[U_t(1)] = f(1)$, and an Itô's formula calculation yields

$$\mathbf{E}_{\mathbf{I}(I^M_j)}\left[U_s^2(1) \right] = M^{-2} + \int_0^s \int_{\mathcal{T}} \mathbf{E}_{\mathbf{I}(I^M_j)}\left[U_r(x)(1 - U_r(x)) \right] dx dr \leq M^{-2} + sM^{-1}.$$

Using this in (14) shows that $IIa \to 0$ as $s \to 0$ and completes the proof. ∎

The lemma implies that the law of $\mu_t^{(2)}(h)$ under $\mathbf{P}_\rho$ is uniquely determined when $h(x,y) = \sum_{i=1}^n \phi(x)\mathbf{I}(y \in \Lambda_i)$. This in turn ensures that the law of $\mu_t^{(2)}$ under $\mathbf{P}_\rho$ is uniquely determined. The map $(x,w) \to \mu_t^{(2)}$ is continuous from E to $\mathcal{M}(\mathcal{T}^2)$ and injective on the open subset $E' := \{(x,w) : x_i \neq x_j \text{ for all } i \neq j\}$. A theorem of Kuratowski (see [3] Appendix 10) guarantees the inverse is measurable. Moreover the law ρ_t is concentrated on E' (since finitely many Brownian particles will have disjoint positions at a fixed time), hence is uniquely determined on E. Finally the family $(T^*_{t-s}\hat{\rho}_s : s \geq 0)$ is tight, by exactly the calculations from Section 3.1. Then Lemma 4 guarantees that any limit point must be ρ_t and the convergence claim (13) follows.

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