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A LOWER BOUND FOR TIME CORRELATION OF LATTICE GASES

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Abstract
The lattice gas model in equilibrium is considered. We give a lower bound of the density-density time correlation for large time, which involves the bulk diffusion matrix in a physically natural way.

1 Introduction

The lattice gas model has been studied by many authors from physical and mathematical point of view. (See Spohn (1991), Part II and the references therein.) In the context of statistical mechanics, the lattice gas model is expected to exhibit various physical properties of interest. In this article we are concerned with diffusive behavior of the lattice gas. The model is described as an infinite particle system with a single conservation law of particle numbers. More precisely, we consider the reversible Markov process on the configuration space $E \equiv \{1,0\}^{\mathbf{Z}^d}$ over the d-dimensional square lattice with the generator of the form

$$\mathcal{L} = \frac{1}{2} \sum_{x,y \in \mathbf{Z}^d} c_{x,y} \nabla_{x,y} \tag{1.1}$$

where $\nabla_{x,y}$ are difference operators defined by

$$\nabla_{x,y} f(\eta) = f(\eta^{x,y}) - f(\eta), \qquad (\eta^{x,y})_z = \begin{cases} \eta_y & \text{if} \quad z = x \\ \eta_x & \text{if} \quad z = y \\ \eta_z & \text{if} \quad z \neq x \text{ and } z \neq y. \end{cases}$$

Each coordinate η_x of a configuration $\eta \in E$ stands for the number of particles at $x \in \mathbf{Z}^d$. Thus the lattice gas model has the conservation law: particles are neither annihilated nor created

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under the time evolution. We suppose that the family $\{c_{x,y}\}$ of rates satisfies the following conditions.

(A.1) *There exists a positive integer R such that*

$$c_{x,y} \equiv 0 \quad \text{whenever} \quad |x-y| > R.$$

(A.2) *The system is nondegenerate in the sense that*

$$c_{x,y}(\eta) > 0 \quad \text{if} \quad |x-y| = 1 \quad \text{and} \quad \eta_x \neq \eta_y.$$

(A.3) *For each pair of sites $x, y \in \mathbf{Z}^d$, $c_{x,y}(\eta)$ is a nonnegative local function of η.*

(A.4) *The system is translation invariant: for all $x, y, a \in \mathbf{Z}^d$ and $\eta \in E$,*

$$c_{x+a,y+a}(\eta) = c_{x,y}(\tau_a \eta),$$

where τ_a is the shift by a, i.e., $(\tau_a \eta)_z = \eta_{z+a}$.

These conditions are enough to ensure the existence and uniqueness of the process $\eta(t) = \{\eta_x(t)\}_{x \in \mathbf{Z}^d}$ associated with $\mathcal{L}$. We also need to assume reversibility of the process. Let $\Phi = \{\Phi(\Lambda)\}_{\Lambda \subset \mathbf{Z}^d}$ be a finite range and translation invariant potential. Then reversibility with respect to Gibbs measures associated with Φ is interpreted as the next condition of detailed balance. (See e.g. Georgii (1979) for more detailes.)

(A.5) *For all $x, y \in \mathbf{Z}^d$ and $\eta \in E$,*

$$c_{x,y}(\eta) = c_{x,y}(\eta^{x,y}) \exp\left[-\nabla_{x,y} H(\eta)\right],$$

where $H(\eta)$ is the Hamiltonian associated with Φ, a formal sum given by

$$H(\eta) = \sum_{\Lambda \subset \mathbf{Z}^d} \Phi(\Lambda) \prod_{x \in \Lambda} \eta_x.$$

Note that $\nabla_{x,y} H(\eta)$ is well-defined since it can be given as the finite sum

$$\nabla_{x,y} H(\eta) = \sum_{\Lambda \cap \{x,y\} \neq \phi} \Phi(\Lambda) \nabla_{x,y} \prod_{z \in \Lambda} \eta_z.$$

We are interested in ergodic properties of the reversible Markov process $\eta(t)$. Because of the conservation law, the system must evolve, in some sense, under very long range interaction. What is expected as quantitative nature of convergence to equilibrium in this model? In Spohn (1991), p.177, a physical conjecture concerning this question is stated as follows. Let ν be a translation invariant Gibbs measure associated with Φ. We simply denote by $\langle \cdot \rangle$ the expectation with respect to ν or the corresponding process in equilibrium. The equilibrium correlation function is then defined by

$$S(x,t) \equiv \langle \eta_x(t) \eta_0(0) \rangle - \rho^2, \tag{1.2}$$

where $\rho = \langle \eta_x(t) \rangle$ is the average density. Due to diffusive property of the lattice gas, it is expected to hold that

$$S(x,t) \cong \chi(\det(4\pi t D))^{-\frac{1}{2}} \exp\left[-\frac{1}{4t}(x, D^{-1}x)\right] \tag{1.3}$$

under a proper rescaling of x and t. Here $\chi \equiv \sum_x S(x,0)$ is the static compressibility by definition and D is a symmetric $d \times d$ matrix (called the bulk diffusion matrix) having the following quadratic form

$$(k,Dk) = \frac{1}{4\chi} \inf_f \sum_y \langle c_{0,y}(\eta)[(k,y)(\eta_0 - \eta_y) + \sum_x \nabla_{0,y}\tau_x f(\eta)]^2\rangle, \tag{1.4}$$

where the infimum is taken over all local functions f on E and $\tau_x f$ are defined by $\tau_x f(\eta) = f(\tau_x \eta)$.
In the case when the lattice gas is of gradient type, Spohn (1991) proved the spatial Fourier transform version of this conjecture. Also, without assuming the gradient condition, Spohn (1991) obtained

$$\liminf_{t\to\infty} S(0,t)t^{\frac{d}{2}} \geq \chi(\det(4\pi\overline{D}))^{-\frac{1}{2}}, \tag{1.5}$$

where $\overline{D}$ is a trivial upper bound (as a matrix) of D with quadratic form

$$(k,\overline{D}k) = \frac{1}{4\chi}\sum_y \langle c_{0,y}(\eta)\,[(k,y)(\eta_0 - \eta_y)]^2\rangle. \tag{1.6}$$

Main result of this paper improves (1.5) by replacing $\overline{D}$ with D, giving more natural lower bound of $S(0,t)$ for large t.

Theorem 1 *Suppose that the family $\{c_{x,y}\}$ satisfies all the conditions from (A.1) to (A.5). We assume that the mixing condition (A.6) below holds for the translation invariant Gibbs measure ν associated with Φ. Then*

$$\liminf_{t\to\infty} S(0,t)t^{\frac{d}{2}} \geq \chi(\det(4\pi D))^{-\frac{1}{2}}. \tag{1.7}$$

Given $\Lambda \subset \mathbf{Z}^d$, let $\mathcal{F}_\Lambda$ be the σ-field generated by coordinates η_x with $x \in \Lambda$ and let $|\Lambda|$ denote the cardinality of Λ. The mixing condition assumed is the following.

(A.6) *Let Λ_1 and Λ_2 be bounded subsets of $\mathbf{Z}^d$. Then for $\mathcal{F}_{\Lambda_1}$-measurable function f and $\mathcal{F}_{\Lambda_2^c}$-measurable function g,*

$$|\langle (f - \langle f\rangle)\,(g - \langle g\rangle)\rangle| \leq C\delta(f)\delta(g)|\Lambda_1|e^{-\alpha\,\mathrm{dist}(\Lambda_1,\Lambda_2^c)},$$

where positive finite constants C and α are independent of f, g, Λ_1 and Λ_2, and $\delta(f)$ is the oscillation of f defined as

$$\delta(f) = \sup\{|f(\eta) - f(\xi)| :\ \eta, \xi \in E\}.$$

Remark 1 It follows from (8.28), (8.32) and (8.33) of Georgii (1988) that if the potential Φ satisfies Dobrushin's uniqueness condition, then the unique Gibbs measure enjoys the property (A.6).
Remark 2 Spohn and Yau (1995) proved positivity of the quadratic form $\chi(k,Dk)$ without any assumption of mixing conditions.

The rest of this paper is organized as follows. The proof of Theorem 1.1 is given in Section 2 by using spatial Fourier transform of $S(x,t)$. So we need a suitable decay of correlations in the variable x. In Section 3, this property is shown to hold under the condition (A.6).

2 Proof of Theorem 1.1

Our proof of Theorem 1.1 is based on estimates for the structure function, i.e., the spatial Fourier transform of the correlation function with respect to $\langle\cdot\rangle$, the translation invariant reversible Gibbs measure for which we suppose to satisfy (A.6). Main ideas used below are taken from Spohn (1991), p.176, except a perturbation argument from the conservation law. We begin with generalizing the notion of the structure function. Let g be a complex-valued local function and denote by $\overline{g}$ the complex conjugate of g. For $x \in \mathbf{Z}^d, k \in \mathbf{R}^d$, and $t \geq 0$, define

$$S^g(x,t) = \langle (g(\eta(0)) - \langle g\rangle)\,(\tau_x \overline{g}(\eta(t)) - \langle \overline{g}\rangle)\rangle$$

and

$$\widehat{S}^g(k,t) = \sum_{x\in\mathbf{Z}^d} e^{i(k,x)} S^g(x,t). \tag{2.1}$$

In special case when $g(\eta) = \eta_0$, set $\widehat{S}(k,t) = \widehat{S}^g(k,t)$. Let $\{T_t\}$ be the semigroup generated by $\mathcal{L}$ with $\{c_{x,y}\}$ satisfying (A.1)-(A.5). By translation invariance (A.4) and uniqueness of the process, we have $T_t(\tau_x\overline{g}) = \tau_x(T_t\overline{g})$. Therefore we can rewrite $S^g(x,t)$ as

$$S^g(x,t) = \langle (g - \langle g\rangle)\,(T_t\tau_x\overline{g} - \langle\overline{g}\rangle)\rangle = \langle (g - \langle g\rangle)\,(\tau_x T_t\overline{g} - \langle\overline{g}\rangle)\rangle,$$

and the sum in (2.1) converges by exponential decay of spatial correlations proved in Section 3 (Lemma 3.1). Some basic properties of $\widehat{S}^g(k,t)$ are shown in the following.

Lemma 2.1 *Let g be a complex-valued local function such that $\langle g\rangle = 0$. Then*

$$\widehat{S}^g(k,t) = \lim_{\Lambda\uparrow\mathbf{Z}^d} \frac{1}{|\Lambda|}\langle \sum_{x\in\Lambda} e^{i(k,x)}\tau_x g(\eta(0)) \sum_{y\in\Lambda} e^{-i(k,y)}\tau_y\overline{g}(\eta(t))\rangle \tag{2.2}$$

and

$$0 \leq \widehat{S}^g(k,t) \leq \widehat{S}^g(k,0), \tag{2.3}$$

where $\Lambda\uparrow\mathbf{Z}^d$ means that Λ runs over hypercubes with center $(0,\cdots,0)\in\mathbf{Z}^d$.

Proof. The equality (2.2) is easily shown by using translation invariance and exponential decay of spatial correlations together. (2.3) is a direct consequence of (2.2). Indeed, fixing a bounded $\Lambda\subset\mathbf{Z}^d$, we observe by the semigroup property and reversibility that

$$\begin{aligned} &\langle \sum_{x\in\Lambda} e^{i(k,x)}\tau_x g(\eta(0)) \sum_{y\in\Lambda} e^{-i(k,y)}\tau_y\overline{g}(\eta(t))\rangle \\ = \;&\langle |T_{t/2}\sum_{x\in\Lambda} e^{i(k,x)}\tau_x g|^2\rangle \geq 0. \end{aligned}$$

The contraction property yields the last inequality of (2.3). ■

The next lemma is our starting point to get a lower bound of $\widehat{S}^g(k,t)$.

Lemma 2.2 *Let f be a complex-valued local function. Assume that $\langle f\rangle = 0$. Then*

$$\langle fT_t\overline{f}\rangle \geq \langle |f|^2\rangle \exp\left[-t\frac{D(f)}{\langle |f|^2\rangle}\right], \tag{2.4}$$

where $D(f)$ is the Dirichlet form given by

$$D(f) = -\langle f \cdot \mathcal{L}\overline{f} \rangle = \frac{1}{4} \sum_{x,y} \langle c_{x,y} | \nabla_{x,y} f|^2 \rangle.$$

Proof. Use the spectral theorem to rewrite the left side of (2.4) in terms of the spectral resolution of (a suitable extension of) $-\mathcal{L}$ and then apply Jensen's inequality. ■

Given a local function g such that $\langle g \rangle = 0$, set for bounded Λ's

$$f_\Lambda = |\Lambda|^{-1/2} \sum_{x \in \Lambda} e^{i(k,x)} \tau_x g.$$

Put $f = f_\Lambda$ in (2.4) and let $\Lambda \uparrow \mathbf{Z}^d$. Then, by (2.2), (2.4) becomes

$$\widehat{S}^g(k,t) \geq \widehat{S}^g(k,0) \exp\left[-t \frac{\widehat{D}_k(g)}{\widehat{S}^g(k,0)} \right] \tag{2.5}$$

where

$$\widehat{D}_k(g) \equiv \lim_{\Lambda \uparrow \mathbf{Z}^d} D(f_\Lambda) = \frac{1}{4} \sum_y \langle c_{0,y} | \sum_x e^{i(k,x)} \nabla_{0,y} \tau_x g|^2 \rangle.$$

In the next step we introduce space-time rescaling of diffusion type and a perturbation from the conservation law (a version of which is expressed as an identity $\widehat{D}_0(\eta_0 - \rho) = 0$):

$$k \mapsto \epsilon k, \quad t \mapsto t/\epsilon^2, \quad g(\eta) = g_\epsilon(\eta) \equiv \eta_0 - \rho - i\epsilon(f(\eta) - \langle f \rangle),$$

where f is an arbitrary real-valued local function and $\epsilon > 0$. Asymptotics of the quantities in (2.5) after the substitution above are described as follows.

Lemma 2.3 *For any $k \in \mathbf{R}^d$ and $t \geq 0$,*

$$\begin{aligned} \widehat{S}^{g_\epsilon}(\epsilon k, t/\epsilon^2) &= \widehat{S}(\epsilon k, t/\epsilon^2) + o(1) \\ \widehat{S}^{g_\epsilon}(\epsilon k, 0) &= \chi + o(1) \end{aligned}$$

and

$$\widehat{D}_{\epsilon k}(g_\epsilon)\epsilon^{-2} = \frac{1}{4} \sum_y \langle c_{0,y}(\eta)[(k,y)(\eta_0 - \eta_y) + \sum_x \nabla_{0,y} \tau_x f(\eta)]^2 \rangle + o(1)$$

as $\epsilon \downarrow 0$.

Proof. (2.2) and Schwarz's inequality together yield

$$\left| \widehat{S}^{g_\epsilon}(\epsilon k, t/\epsilon^2) - \widehat{S}(\epsilon k, t/\epsilon^2) \right| \leq 2\epsilon \sqrt{\widehat{S}^f(\epsilon k, t/\epsilon^2)} \sqrt{\widehat{S}(\epsilon k, t/\epsilon^2)} + \epsilon^2 \widehat{S}^f(\epsilon k, t/\epsilon^2).$$

Since by (2.3) and the dominated convergence theorem

$$0 \leq \widehat{S}^f(\epsilon k, t/\epsilon^2) \leq \widehat{S}^f(\epsilon k, 0) \longrightarrow \widehat{S}^f(0,0)$$

as $\epsilon \downarrow 0$, this proves the first two equalities. The last equality is shown by straightforward calculation. ■

We return to the proof of Theorem 1.1. Combining the three equalities of Lemma 2.3 with (2.5), we get

$$\widehat{S}(\epsilon k, t/\epsilon^2) + o(1) \geq (\chi + o(1)) \exp\left[-t\frac{c_k^*(f)}{\chi + o(1)}\right]$$

as $\epsilon \downarrow 0$, where

$$c_k^*(f) = \frac{1}{4}\sum_y \langle c_{0,y}(\eta)[(k,y)(\eta_0 - \eta_y) + \sum_x \nabla_{0,y}\tau_x f(\eta)]^2\rangle.$$

Since f is arbitrary, this implies

$$\liminf_{\epsilon\downarrow 0} \widehat{S}(\epsilon k, t/\epsilon^2) \geq \chi \exp\left[-t(k, Dk)\right].$$

In the final step, we use Fourier transform and Fatou's lemma to obtain

$$\begin{aligned}
\liminf_{\epsilon\downarrow 0} S(0, 1/\epsilon^2)\epsilon^{-d} &= \liminf_{\epsilon\downarrow 0} \frac{1}{(2\pi)^d}\int_{[-\pi/\epsilon,\pi/\epsilon]^d} dk \widehat{S}(\epsilon k, 1/\epsilon^2) \\
&\geq \frac{\chi}{(2\pi)^d}\int_{\mathbf{R}^d} dk \exp\left[-(k, Dk)\right] \\
&= \chi(\det(4\pi D))^{-\frac{1}{2}},
\end{aligned}$$

which is equivalent to the desired inequality (1.7).

3 Decay of Spatial Correlations

In this section we prove the following lemma used in the previous section. We denote by $\{T_t\}$ the semigroup generated by $\mathcal{L}$ with $\{c_{x,y}\}$ satisfying (A.1)-(A.4). Note that the next lemma requires no assumption of reversibility (A.5).

Lemma 3.1 *Suppose that the mixing condition (A.6) holds for some probability measure $\langle\cdot\rangle$ on E. Let $t \geq 0$. Then for all local functions f and g, there exist positive finite constants C_1 and C_2 such that*

$$|\langle(f - \langle f\rangle)(\tau_x T_t g - \langle\tau_x T_t g\rangle)\rangle| \leq C_1 e^{-C_2|x|} \tag{3.1}$$

is true for all $x \in \mathbf{Z}^d$.

Roughly speaking, this lemma is a consequence of the fact that the following type of quasilocality is preserved under the time evolution. A measurable function f on E is said to be *exponentially quasilocal* if there exist positive finite constants A and a such that for each positive integer l

$$\delta_l(f) \equiv \sup\{|f(\eta) - f(\xi)| : \eta = \xi \text{ on } B_l\} \leq Ae^{-al}, \tag{3.2}$$

where $B_l = \{x \in \mathbf{Z}^d : |x| \leq l\}$. Let us denote by $\mathcal{D}_1$ the class of exponentially quasilocal functions on E. We need the following fact in Spohn (1991)(Theorem 1.4, p.160).

Lemma 3.2 *For all $t > 0$, $T_t\mathcal{D}_1 \subset \mathcal{D}_1$.*

Before proving Lemma 3.1, recall the mixing condition (A.6) we suppose:

$$|\langle (f - \langle f\rangle)\,(g - \langle g\rangle)\rangle| \le C\delta(f)\delta(g)|\Lambda_1|e^{-\alpha\,\mathrm{dist}(\Lambda_1,\Lambda_2^c)} \tag{3.3}$$

Here f is $\mathcal{F}_{\Lambda_1}$-measurable and g is $\mathcal{F}_{\Lambda_2^c}$-measurable. Positive finite constants C and α are independent of f, g, Λ_1 and Λ_2. Further, $\delta(f)$ is the oscillation of f.

Proof of Lemma 3.1. Let f and g be local functions on E. Take a finite subset Λ_1 of $\mathbf{Z}^d$ such that f is $\mathcal{F}_{\Lambda_1}$-measurable. Fix an arbitrary $t \ge 0$. Then by Lemma 3.2, $g_t \equiv T_t g \in \mathcal{D}_1$ and hence for each positive integer l we can choose an $\mathcal{F}_{B_l}$-measurable function g_t^l such that

$$\|g_t^l - g_t\|_\infty \le Ae^{-al} \quad \text{and} \quad \delta(g_t^l) \le \delta(g_t) \le 2\|g\|_\infty \tag{3.4}$$

hold for some A and $a > 0$ which do not depend on l. Set $B_l(x) = \{y \in \mathbf{Z}^d : |y - x| \le l\}$. Since $\tau_x g_t^l$ is $\mathcal{F}_{B_l(x)}$-measurable, by the mixing condition (3.3)

$$|\langle (f - \langle f\rangle)\,(\tau_x g_t^l - \langle \tau_x g_t^l\rangle)\rangle| \le 2C\delta(f)\|g\|_\infty|\Lambda_1|e^{-\alpha\,\mathrm{dist}(\Lambda_1,\Lambda_2^c)}, \tag{3.5}$$

provided that $\Lambda_2 \cap B_l(x) = \phi$. In (3.5), C and α are the same constants as in (3.3) and, in particular, independent of t, l and x.
In the rest of the proof, we will show that (3.1) holds for any $x \in \mathbf{Z}^d$ such that $\mathrm{dist}(\Lambda_1, \{x\}) \ge 6$. Fixing such an x, let l be the smallest integer greater than or equal to $\mathrm{dist}(\Lambda_1, \{x\})/3$. Set

$$\Lambda_2 = \{y \in \mathbf{Z}^d : \mathrm{dist}(\{y\}, \Lambda_1) \le l\}.$$

It is not difficult to see $\Lambda_2 \cap B_l(x) = \phi$. Observe that

$$\begin{aligned}
\mathrm{dist}(\Lambda_1, \Lambda_2^c) \;\ge\; l \;&\ge\; \frac{1}{3}\,\mathrm{dist}(\Lambda_1, \{x\}) \\
&\ge\; \frac{1}{3}\,\mathrm{dist}(\{0\}, \{x\}) - \frac{1}{3}\,\mathrm{diam}(\Lambda_1 \cup \{0\}) \\
&=\; \frac{1}{3}|x| - \frac{1}{3}\,\mathrm{diam}(\Lambda_1 \cup \{0\}),
\end{aligned}$$

where we used notation $\mathrm{diam}(\Lambda) = \sup\{|x - y|; x, y \in \Lambda\}$ for $\Lambda \subset \mathbf{Z}^d$. Using these inequalities and (3.4) and applying (3.5) with this choice of l and Λ_2, we get

$$\begin{aligned}
&|\langle (f - \langle f\rangle)\,(\tau_x g_t - \langle \tau_x g_t\rangle)\rangle| \\
\le\;& 2\delta(f)Ae^{-al} + |\langle (f - \langle f\rangle)\,(\tau_x g_t^l - \langle \tau_x g_t^l\rangle)\rangle| \\
\le\;& 2\delta(f)Ae^{-al} + 2C\delta(f)\|g\|_\infty|\Lambda_1|e^{-\alpha l} \\
\le\;& 2\delta(f)(A + C\|g\|_\infty|\Lambda_1|)e^{-\min\{a,\alpha\}l} \\
\le\;& 2\delta(f)(A + C\|g\|_\infty|\Lambda_1|)e^{\min\{a,\alpha\}\,\mathrm{diam}(\Lambda_1\cup\{0\})/3}e^{-|x|\min\{a,\alpha\}/3}.
\end{aligned}$$

This shows that (3.1) holds with

$$C_1 = 2\delta(f)(A + C\|g\|_\infty|\Lambda_1|)e^{\min\{a,\alpha\}\,\mathrm{diam}(\Lambda_1\cup\{0\})/3}$$

and $C_2 = \min\{a, \alpha\}/3$, provided that $\mathrm{dist}(\Lambda_1, \{x\}) \ge 6$. But this restriction can be removed by modifying the constant C_1 appropriately, and the proof of Lemma 3.1 is completed. ∎

Remark 3 Since the above proof clearly works for $g \in \mathcal{D}_1$, the conclusion of Lemma 3.1 is valid even if $g \in \mathcal{D}_1$.

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