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TESSELLATION OF A TRIANGLE BY REPEATED BARYCENTRIC SUBDIVISION

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Abstract
Under iterated barycentric subdivision of a triangle, most triangles become flat in the sense that the largest angle tends to π. By analyzing a random walk on $SL_2(\mathbb{R})$ we give asymptotics with explicit constants for the number of flat triangles and the degree of flatness at a given stage of subdivision. In particular, we prove analytical bounds for the upper Lyapunov constant of the walk.

1 Introduction

The iterated barycentric subdivision of a triangle Δ is defined as follows. In the first stage the three medians are drawn resulting in six smaller triangles Δ^1 through Δ^6. At stage 2 the process is repeated in each Δ^i, $1 \le i \le 6$, producing in 36 triangles Δ^{ij}, $1 \le i, j \le 6$. In general, on the nth stage the medians are drawn in each triangle produced in the $n-1$st stage, so that 6^n triangles $\Delta^{i_1 \dots i_n}$,$1 \le i_1, \dots, i_n \le 6$, result. We ask: as n grows do the triangles in stage n become "flat", in the sense that they have largest angle approaching π?

It is not hard to show that not all triangles that result from barycentric subdivision become flat: if $\Delta = ABC$ has $m(A) \le m(B) \le m(C)$, with centroid O and X the midpoint of BC, then it is a nice exercise in Euclidean geometry to check that the least angle in COX is at least $\min(m(A), 30°)$; in particular, $\min(m(A), 30°)$ is a lower bound on the maximum least angle appearing among triangles in the nth stage of subdivision, for any n. Nonetheless, Stakhovskii has proposed that as $n \to \infty$ nearly one hundred percent of all triangles at stage n have largest angle near π and in 1996 Bárány, Beardon and Carne [1] gave an elegant proof of this conjecture. They identify iterated barycentric subdivision with a random walk X_n on $SL_2(\mathbb{R})$, the triangles at the nth stage of subdivision being similar to the images of Δ under X_n. Their key observation is that the generators of the random walk generate a dense subgroup of $SL_2(\mathbb{R})$, and so a theorem of Furstenberg [6] implies that there exists $\gamma > 0$ such that for any $v \in \mathbb{R}^2$, $\lim_{n\to\infty} \frac{1}{n} \log \|X_n v\| = \gamma$ with probability 1. (This γ is called the upper Lyapunov constant of the random walk.) It then follows easily that almost all triangles become flat.

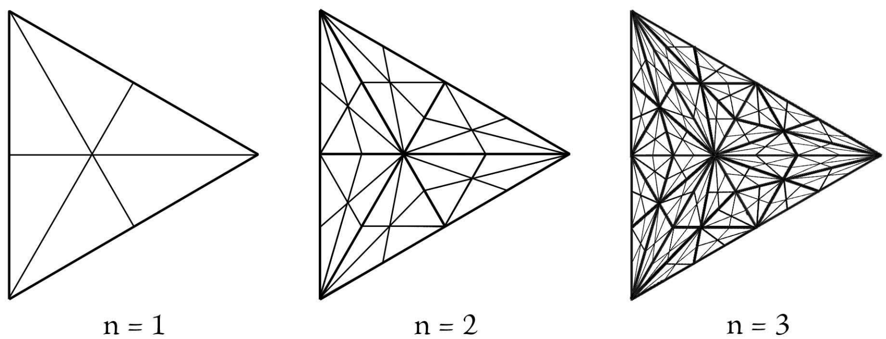

Figure 1: The first three stages of barycentric subdivision for equilateral Δ

While the argument [1] (rediscovered independently in [4] with some generalizations) gives a very satisfactory characterization of barycentric subdivision as $n \to \infty$, it has the disadvantage that the Ergodic Theoretic approach does not readily yield information for "finite" n. Recently Diaconis and Miclo [5] have given another formulation using iterated random function Markov chains that allows them to deduce more information about the limiting shape of triangles after many subdivisions, but this approach also does not translate easily to giving bounds for the number of triangles at a given stage having largest angle of a certain size. The strongest approach in this finite direction is due to David Blackwell [2], who has proved (at least computationally) that barycentric subdivision decreases a certain "pseudo-fatness" metric on average.

In the following I give a hybrid of the methods of [2] and [1] together with a relatively simple geometric construction, which yields an explicit lower bound tending to 1 for the proportion of triangles in stage n having largest angle in a certain range. This range converges to π exponentially quickly with respect to n; in particular, I obtain analytic upper and lower bounds for the constant γ, which, while not matching, can be made to give γ to arbitary precision by working with later stages of subdivision.

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2 Formalism

Embed $\Delta \subset \mathbb{R}^2$ with the standard basis, and let Δ_0 be the equilateral triangle with vertices

$$v_1 = (1, 0), \quad v_2 = (-1/2, \sqrt{3}/2), \quad v_3 = (-1/2, -\sqrt{3}/2).$$

Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be the affine linear map defined by

$$Tv_1=\frac{1}{2}(v_1+v_2),\quad Tv_2=v_2,\quad Tv_3=\frac{1}{3}(v_1+v_2+v_3)$$

that carries Δ_0 onto the uppermost right triangle formed by drawing the medians of Δ_0. For $w\in\mathbb{R}^2$, $Tw=\frac{1}{\sqrt{6}}Aw+T0$ where $A=\begin{bmatrix}\frac{\sqrt{6}}{3} & \frac{\sqrt{2}}{2}\\ 0 & \frac{\sqrt{6}}{2}\end{bmatrix}$ is the linear component of T, normalized to have determinant 1. If $\rho:D_6\to SL_2(\mathbb{R})$ is the representation of the dihedral group

$$D_6=\langle r,s:r^3=s^2=1,rs=sr^{-1}\rangle$$

given by

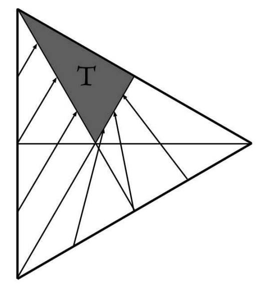

Figure 2: The affine linear transform T.

$$\rho(s)=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix},\qquad \rho(r)=\begin{bmatrix}\frac{-1}{2} & \frac{-\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$$

reflection in the x-axis and rotation by $\frac{2\pi}{3}$ respectively, then writing T^x for $\rho(x)\circ T\circ\rho(x)^{-1}, x\in D_6$, a collection of affine linear maps carrying Δ_0 onto $\Delta_0^1,\dots,\Delta_0^6$ is

$$T^{D_6}=\{T^x:x\in D_6\},$$

and A^{D_6} is the corresponding collection of normalized linear components of these maps.
For affine linear map $Sw=\lambda Bw+v$, $\det B=1$ carrying Δ_0 onto Δ, linearity of barycentric subdivision implies that the six triangles Δ^i are the images under S of the six triangles Δ_0^i. In particular, if we let μ be the uniform distribution on A^{D_6} and define the usual n-fold convolution

$$\mu^{(n)}=\mu\overset{n\text{ times}}{*\cdots*}\mu=\frac{1}{6^n}\sum_{M_1,\dots,M_n\in A^{D_6}}\delta_{M_1\dots M_n},$$

then, up to dilation and translation, the distribution of triangles appearing in Δ_0 after n stages of barycentric subdivision is given by $Y\Delta_0$, where Y is chosen from $SL_2(\mathbb{R})$ according to $\mu^{(n)}$, and the distribution of triangles in Δ is given by $BY\Delta_0$.
The following observations allow us to reformulate statements about barycentric subdivision in terms of operators in $SL_2(\mathbb{R})$.

Observation 1. *Let Δ be a triangle of area $\mathscr{A}$. If the shortest two sides of Δ have length $\ell\le m$, $lm>\frac{4\mathscr{A}}{\sqrt{3}}$ and the largest angle of Δ has measure θ, then $\frac{2\mathscr{A}}{\ell m}\le\pi-\theta\le\frac{2\pi\mathscr{A}}{\ell m}$.*

Observation 2. *With A the linear component of T defined above and $\|\cdot\|$ the operator norm $\|B\|=\sup_{v\in\mathbb{R}^2:\|v\|=1}\|Bv\|$ we have $\alpha:=\|A\|=\sqrt{\frac{4+\sqrt{7}}{3}}$.*

For a proof of these calculations see the notes at the end. Observation 1 shows that the difference between π and the largest angle in $\Delta_0^{i_1\dots i_n}$ is less than a constant factor times $\min\{\|Ys_1\|,\|Ys_2\|,\|Ys_3\|\}^{-2}$ where Y is the corresponding operator chosen from $\mu^{(n)}$ and $s_1=(0,1)$, $s_2=(\frac{\sqrt{3}}{2},\frac{-1}{2})$, $s_3=(\frac{-\sqrt{3}}{2},\frac{-1}{2})$ are unit vectors in the direction of the sides of Δ_0. Meanwhile, observation 1 and

$\|B\|^{-1}\|x\| \le \|Bx\| \le \|B\|\|x\|$ for $B \in SL_2(\mathbb{R})$ and $x \in \mathbb{R}^2$ implies that if $\Delta_0^{i_1\ldots i_n}$ is sufficiently flat, the distances from π of the largest angles in $\Delta_0^{i_1\ldots i_n}$ and $(B\Delta_0)^{i_1\ldots i_n}$ are within a factor of $\pi^2\|B\|^2$, so that henceforth we consider only barcentric subdivision of Δ_0 and restrict our attention to describing the distribution of $\|Yv\|$ for v a unit vector and Y chosen from $\mu^{(n)}$.

Theorem 1. *With μ defined as above, v an arbitrary unit vector in $\mathbb{R}^2$ and for any $\lambda > 0$*

$$\mu^{(n)}\Big(\{M \in SL_2(\mathbb{R}) : \\ 0.05857n - 0.8874\lambda\sqrt{n} < \log\|Mv\| < 0.09461n + 0.8874\lambda\sqrt{n}\Big) \ge 1 - 2e^{-\lambda^2/2}.$$

Here the numerical constants 0.05857 and 0.09461 come from solving a certain algebraic equation pertaining to the dilatation of a circle under the operator A, above, while 0.8874 bounds the step size in a random walk.
Specializing our result to $\lambda = \sqrt{2\log n}$ we have

$$e^{0.05857n-1.255\sqrt{n\log n}} \le \min_{i=1,2,3}\|Y_n s_i\| \le \max_{i=1,2,3}\|Y_n s_i\| \le e^{0.09461n+1.255\sqrt{n\log n}}$$

with probability at least $1 - \frac{6}{n}$. Applying this with observation 1, (and recalling that $\frac{\mathscr{A}}{s^2} = \frac{\sqrt{3}}{4}$ in an equilateral triangle) we obtain that in the nth stage of barycentric subdivision of Δ_0, the proportion of triangles having largest angle θ with

$$\frac{\pi\sqrt{3}}{2}e^{-0.11714n+2.51(n\log n)^{1/2}} > \pi - \theta > \frac{\sqrt{3}}{2}e^{-0.18921n-2.51(n\log n)^{1/2}}$$

is at least $1 - \frac{6}{n}$.

3 Proof of Theorem

Define random walk $Y_0, Y_1, Y_2, \ldots$ on $SL_2(\mathbb{R})$ by $Y_0 = I$, $Y_i = X_i Y_{i-1}$ for $i \ge 1$ where $X_1, X_2, \ldots$ are chosen i.i.d. from μ. Thus, $\mu^{(n)}$ is the measure of Y_n. For v an arbitary unit vector in $\mathbb{R}^2$, the walk Y_n induces a natural random walk $Y_n v$ on $\mathbb{R}^2$. The random difference $Z_n = \log\|Y_n v\| - \log\|Y_{n-1}v\|$ depends only on the direction $\overline{v_{n-1}} = \frac{Y_{n-1}v}{\|Y_{n-1}v\|}$ and is bounded: $-\log\|A\| \le Z_n \le \log\|A\|$. We show that independent of the direction $\overline{v_{n-1}}$, $C \ge \mathbf{E}[Z_n] \ge c > 0$ and deduce the listed bounds for $\|Y_n v\|$ by the method of bounded differences.

Let $w \in \mathbb{R}^2 \setminus \{0\}$ be arbitrary. Given X chosen from A^{D_6} according to μ, we have

$$\mathbf{E}[\log\|Xw\| - \log\|w\|] = \mathbf{E}[\log\|X\overline{w}\|] = \frac{1}{6}\log\prod_{x\in D_6}\|A^x\overline{w}\|,$$

with $\overline{w} = \frac{w}{\|w\|}$. Now A may be expressed as $U_1 D U_2$ where U_1 and U_2 are unitary matrices and D is a diagonal matrix such that $\|D\| = \|A\|$, i.e. we may take $D = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}$ where $\alpha = \|A\|$. Since $\rho(x)$ is unitary for each x we have

$$\mathbf{E}[\log\|X\overline{w}\|] = \frac{1}{6}\log\prod_{i\in\{0,1,2\},j\in\{0,1\}}\|DU_2 r^i s^j \overline{w}\|.$$

Note that the triples of vectors $[U_2\overline{w}, U_2 r\overline{w}, U_2 r^2\overline{w}]$ and $[U_2 s\overline{w}, U_2 rs\overline{w}, U_2 r^2 s\overline{w}]$ each form the sides of an equilateral triangle. Thus by the symmetry $\overline{w} \longleftrightarrow s\overline{w}$ we have the bounds $\frac{1}{6}\log\beta_-^2 \leq \mathbf{E}[\log\|X\overline{w}\|] \leq \frac{1}{6}\log\beta_+^2$ where

$$\beta_- = \min_{\|w\|=1} \|Dw\|\|Drw\|\|Dr^2w\|, \quad \beta_+ = \max_{\|w\|=1} \|Dw\|\|Drw\|\|Dr^2w\|.$$

We now calculate[1] β_- and β_+.

Lemma 2. *We have* $\beta_- = \left(\frac{3}{4}\alpha + \frac{1}{4}\alpha^{-3}\right)$, $\beta_+ = \left(\frac{3}{4}\alpha^{-1} + \frac{1}{4}\alpha^3\right)$. *In particular,* $\beta_-^2 = \frac{172-7\sqrt{7}}{108}$, $\beta_+^2 = \frac{172+7\sqrt{7}}{108}$ *so that* $\delta_- = \frac{1}{6}\log\beta_-^2 > 0.05857$ *and* $\delta_+ = \frac{1}{6}\log\beta_+^2 < 0.09461$.

Proof. Choose coordinates so that D expands the x direction and contracts the y direction. Without loss of generality, w is in the upper half plane, i.e.

$$w = (\cos\theta, \sin\theta), \quad rw = \left(\cos(\theta + \frac{2\pi}{3}), \sin(\theta + \frac{2\pi}{3})\right), \quad r^2 w = \left(\cos(\theta + \frac{4\pi}{3}), \sin(\theta + \frac{4\pi}{3})\right)$$

with $\theta \in [0,\pi]$. Put $x = \cos\theta \in [-1,1]$, $y = \sin\theta \geq 0$. Then

$$\|Dw\|^2\|Drw\|^2\|Dr^2w\|^2 = f(x) = \left(\alpha^2x^2 + \alpha^{-2}y^2\right)\cdot$$
$$\left[\alpha^2\left(\frac{-x}{2} - \frac{\sqrt{3}y}{2}\right)^2 + \alpha^{-2}\left(\frac{\sqrt{3}x}{2} - \frac{y}{2}\right)^2\right]\left[\alpha^2\left(\frac{-x}{2} + \frac{\sqrt{3}y}{2}\right)^2 + \alpha^{-2}\left(\frac{-\sqrt{3}x}{2} - \frac{y}{2}\right)^2\right].$$

This is a degree six polynomial in x. Indeed, $y^2 = 1 - x^2$ so the first term has degree 2. Distributing the two bracketed terms gives

$$\alpha^4\left(\frac{1}{4}x^2 - \frac{3}{4}y^2\right)^2 + \alpha^{-4}\left(\frac{1}{4}y^2 - \frac{3}{4}x^2\right)^2 + \left[\frac{\sqrt{3}}{4}x^2 + \frac{\sqrt{3}}{4}y^2 + xy\right]^2 + \left[\frac{\sqrt{3}}{4}x^2 + \frac{\sqrt{3}}{4}y^2 - xy\right]^2.$$

The two terms in parentheses are degree 4, while the two in brackets sum to $\frac{3}{8} + 2x^2y^2$, which is also degree 4.
Since $f(x)$ is degree 6 in x, $f'(x)$ has at most 5 zeros on $[-1,1]$. Now calculate the maxima and minima another way. Write

$$f(x) = g(\theta) = \prod_{j=0}^{2}\left((\alpha^2 - \alpha^{-2})\cos^2(\theta + \frac{2\pi j}{3}) + \alpha^{-2}\right) \tag{1}$$

and take the logarithmic derivative:

$$\frac{g'(\theta)}{g(\theta)} = -2\sum_{j=0}^{2}\frac{\sin(2(\theta + \frac{2\pi j}{3}))}{\gamma + \cos(2(\theta + \frac{2\pi j}{3}))}, \quad \gamma = 2(\alpha^4 - 1)^{-1} + 1.$$

This expression is $\frac{\pi}{3}$-periodic and vanishes at 0 and $\frac{\pi}{2}$ since in each case one of the sine terms is zero while the other two cancel. Hence $\frac{g'}{g}$ has a zero at $\frac{k\pi}{6}$ for all k. But then $g'(\theta) =$

[1] The qualitative statement $\beta_- > 0$ follows from compactness of the unit circle, together with the fact that among triangles of a given area, the equilateral minimizes the product of the lengths of the sides.

$f'(\cos\theta)\sin\theta$, so we obtain all five zeros of $f'(x)$ from $\theta = \frac{k\pi}{6}$, $1 \le k \le 5$. Since any local maxima or minima of $g(\theta)$, $0 < \theta < \pi$ translates to a local maxima/minima of $f(x)$ on $(-1,1)$, we have found all extreme points of $g(\theta)$ on this interval, hence for all θ by periodicity. It follows that $\beta_-^2 = \min_\theta g(\theta) = \min\{g(0), g(\frac{\pi}{6})\}$ and $\beta_+ = \max\{g(0), g(\frac{\pi}{6})\}$. Substituting this into (1),

$$g(0) = \alpha^2\left(\frac{1}{4}\alpha^2 + \frac{3}{4}\alpha^{-2}\right)^2, \quad g\left(\frac{\pi}{6}\right) = \alpha^{-2}\left(\frac{3}{4}\alpha^2 + \frac{1}{4}\alpha^{-2}\right)^2$$

so that by observation 2

$$g(0) = \frac{172 + 7\sqrt{7}}{108}, \quad g\left(\frac{\pi}{6}\right) = \frac{172 - 7\sqrt{7}}{108}$$

and thus $\beta_-^2 = g(\frac{\pi}{6}) > 1.4211$, $\beta_+^2 = g(0) < 1.7641$ and $\delta_- > 0.05857$, $\delta_+ < 0.09461$ as desired. $\square$

It remains to describe the bounded differences argument proving almost certain exponential growth. Recall that $Z_n = \log\|Y_n v\| - \log\|Y_{n-1}v\|$ is the log-length increment of each step of our walk, and put $W_n = Z_n - \mathbf{E}[Z_n|\overline{v_{n-1}}]$. The sum

$$\sum_{i=1}^{n} W_n = \log\|Y_n v\| - \sum_{i=1}^{n} \mathbf{E}[Z_n|\overline{v_{n-1}}]$$

is a Martingale with increments bounded by

$$|W_n| \le \sup_{\overline{v_{n-1}}} |Z_n| + \sup_{\overline{v_{n-1}}} |\mathbf{E}[Z_n|\overline{v_{n-1}}]| \le \log\|A\| + \delta_+,$$

and hence by Azuma's inequality

$$P\left[\left|\log\|Y_n v\| - \sum_{i=1}^{n} \mathbf{E}[Z_n|\overline{v_{n-1}}]\right| > \lambda(\log\|A\| + \delta_+)\sqrt{n}\right] \le 2e^{-\lambda^2/2}.$$

Now $\sum_{i=1}^{n} \mathbf{E}[Z_n|\overline{v_{n-1}}]$ is itself a random quantity, but in view of Lemma 2,

$$n\delta_- \le \sum_{i=1}^{n} \mathbf{E}[Z_n|\overline{v_{n-1}}] \le n\delta_+.$$

We recover

$$P\left[n\delta_- - \lambda(\log\|A\| + \delta_+)\sqrt{n} \le \log\|Y_n v\| \le n\delta_+ + \lambda(\log\|A\| + \delta_+)\sqrt{n}\right] \ge 1 - e^{-\lambda^2/2},$$

and in view of the bounds on δ_-, δ_+, and $\log\|A\| + \delta_+ < 0.8874$ this proves the theorem.

4 Remarks and related problems

Our constants $\delta_- \approx 0.05857$ and $\delta_+ \approx 0.09461$ give lower and upper bounds for the upper Lyapunov constant $\gamma := \lim_{n\to\infty} \frac{1}{n}\mathbf{E}[\log\|Y_n v\|]$ ($v \neq 0$ arbitrary). We could improve our constants by averaging over the linear operators resulting from later stages of subdivision, but, as it stands, a result of type $\|Y_n v\| = \exp(\gamma n - o(n))$ with probability $1 - o(1)$ is beyond the limitations of our method. Based upon computer calculations, $\gamma \approx 0.071$. For a nice exposition of growth of random walks in matrix groups and Lyapunov constants see [3].

Barycentric subdivision also produces flat simplices in higher dimension (this is done in [4]). In dimension k the corresponding random walk is on $SL_k(\mathbb{R})$ and the analogue of the dihedral group D_6 is the symmetric group S_{k+1}. Furstenberg's theorem applies as before, giving $\lim_n \frac{1}{n} \log \|Y_n v\| = \gamma > 0$. One might again expect to obtain upper and lower bounds for γ by allowing a $(k+1)$-cycle σ to play the role of the rotation r in Lemma 2, although the explicit calculations become increasingly difficult.

Returning to $\mathbb{R}^2$, there remain interesting questions. Put θ_n for the maximum of the least angles among triangles at stage n; in our introduction we sketched an argument that $\theta_n \geq \min(\theta_0, 30°)$. Pictures suggest that more is true, e.g. in stage 3 of subdivision of an equilateral triangle we already have sub-triangles that are almost equilateral. One conjectures that $\theta_n \to 60°$ as $n \to \infty$ and exponentially quickly, although the best argument that I know applies only to subsequences of stages and is polynomial in n.

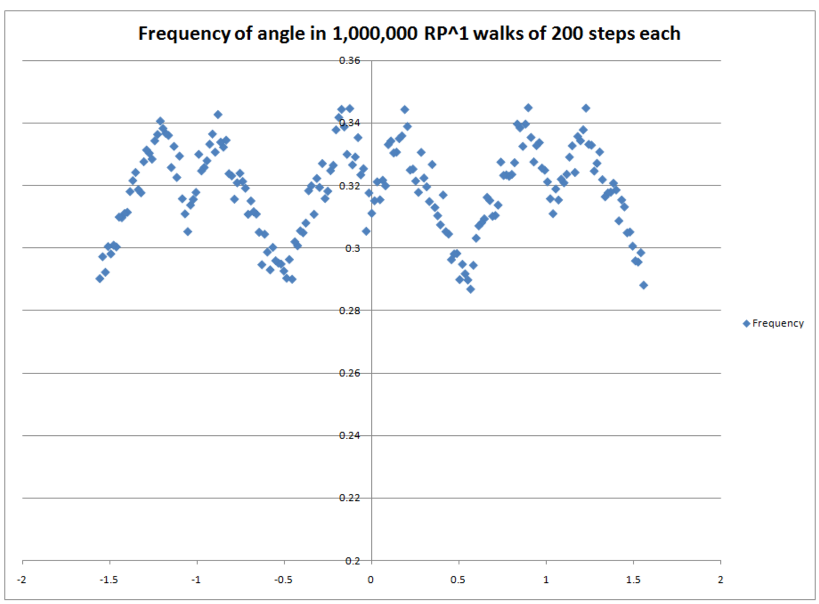

Figure 3: Frequency of angle in 10^6 trials of the $\mathbb{R}P^1$ random walk after 200 steps, starting from the uniform distribution.

Finally, our theorem shows that after many subdivisions most of the resulting triangles are relatively flat, so that they take on the direction of their longest side. One could ask for the distribution of these sides, as $n \to \infty$, which is closely related to stationary distribution of the random walk on $\mathbb{R}P^1$ defined by choosing $\overline{v_0} \in \mathbb{R}P^1$ according to some distribution and putting $\overline{v_n} = \overline{X_n v_{n-1}}$, where the X_n are chosen independently uniformly from A^{D_6}. Such a limiting distribution exists independent of the initial distribution by the application of Furstenberg's work [6] in [1] and [4]. Proving anything else about this distribution, e.g. an explicit description, existence of a density, a rate of convergence, however, seems to be a nice and challenging open problem. A Monte Carlo approximation of the distribution is shown above.

5 End notes

Proof of Observation 1. The largest angle θ of Δ lies opposite the longest side, that is, between the sides of length ℓ and m. Hence $\mathscr{A} = \frac{1}{2}\ell m \sin\theta$. Provided $\ell m > \frac{4\mathscr{A}}{\sqrt{3}}$ we get $\sin\theta < \frac{\sqrt{3}}{2}$ and so $\theta \geq \frac{\pi}{3}$, by virtue of being the largest angle of a triangle, forces $\theta \geq \frac{2\pi}{3}$. The observation then follows from the identity $\sin\theta < \pi - \theta < \pi\sin\theta$, valid for $\frac{\pi}{2} < \theta < \pi$. □

Proof of Observation 2. For $v = (\cos\theta, \sin\theta)$,

$$\|Av\|^2 = \frac{2}{3}\cos^2\theta + 2\sin^2\theta + \frac{2\sqrt{3}}{3}\cos\theta\sin\theta = \frac{4}{3} - \frac{2}{3}\cos 2\theta + \frac{\sqrt{3}}{3}\sin 2\theta.$$

This takes on its extreme values when $\tan 2\theta = \frac{-\sqrt{3}}{2}$, i.e. for $\cos 2\theta = \pm\sqrt{\frac{4}{7}}$, $\sin 2\theta = \mp\sqrt{\frac{3}{7}}$ so we have the desired result. □

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