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CENTRAL LIMIT THEOREM FOR THE EXCITED RANDOM WALK IN DIMENSION $D \geq 2$

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Abstract

We prove that a law of large numbers and a central limit theorem hold for the excited random walk model in every dimension $d \geq 2$.

1 Introduction

An excited random walk with bias parameter $p \in (1/2, 1]$ is a discrete time nearest neighbor random walk $(X_n)_{n \geq 0}$ on the lattice $\mathbb{Z}^d$ obeying the following rule: when at time n the walk is at a site it has already visited before time n, it jumps uniformly at random to one of the $2d$ neighboring sites. On the other hand, when the walk is at a site it has not visited before time n, it jumps with probability p/d to the right, probability $(1-p)/d$ to the left, and probability $1/(2d)$ to the other nearest neighbor sites.

The excited random walk was introduced in 2003 by Benjamini and Wilson [1], motivated by previous works of [5, 4] and [10] on self-interacting Brownian motions. Variations on this model have also been introduced. The excited random walk on a tree was studied by Volkov [15]. The so called *multi-excited random walk*, where the walk gets pushed towards a specific direction upon its first M_x visits to a site x, with M_x possibly being random, was introduced by Zerner in [16] (see also [17] and [9]).

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In [1], Benjamini and Wilson proved that for every value of $p \in (1/2, 1]$ and $d \geq 2$, excited random walks are transient. Furthermore, they proved that for $d \geq 4$,

$$\liminf_{n\to\infty} n^{-1} X_n \cdot e_1 > 0 \quad a.s., \tag{1}$$

where $(e_i : 1 \leq i \leq d)$ denote the canonical generators of the group $\mathbb{Z}^d$. Subsequently, Kozma extended (1) in [7] and [8] to dimensions $d = 3$ and $d = 2$. In this paper, we prove that the biased coordinate of the excited random walk satisfies a law of large numbers and a central limit theorem for every $d \geq 2$ and $p \in (1/2, 1]$.

Theorem 1. *Let $p \in (1/2, 1]$ and $d \geq 2$.*

(i) (Law of large numbers). There exists $v = v(p, d)$, $0 < v < +\infty$ such that a.s.

$$\lim_{n\to\infty} n^{-1} X_n \cdot e_1 = v.$$

(ii) (Central limit theorem). There exists $\sigma = \sigma(p, d)$, $0 < \sigma < +\infty$, such that

$$t \mapsto n^{-1/2}(X_{\lfloor nt \rfloor} \cdot e_1 - v\lfloor nt \rfloor),$$

converges in law as $n \to +\infty$ to a Brownian motion with variance σ^2, with respect to the Skorohod topology on the space of càdlàg functions.

In the recent preprint [14], relying on the lace expansion technique, van der Hofstad and Holmes proved that a weak law of large numbers holds when $d > 5$ and p is close enough (depending on d) to $1/2$, and that a central limit theorem holds when $d > 8$ and and p is close enough (depending on d) to $1/2$.

Our proof is based on the well-known construction of regeneration times for the random walk, the key issue being to obtain good tail estimates for these regeneration times. Indeed, using estimates for the so-called *tan* points of the simple random walk, introduced in [1] and subsequently used in [7, 8], it is possible to prove that, when $d \geq 2$, the number of distinct points visited by the excited random walk after n steps is, with large probability, of order $n^{3/4}$ at least. Since the excited random walk performs a biased random step at each time it visits a site it has not previously visited, the e_1-coordinate of the walk should typically be at least of order $n^{3/4}$ after n steps. Since this number is $o(n)$, this estimate is not good enough to provide a direct proof that the walk has linear speed. However, such an estimate is sufficient to prove that, while performing n steps, the walk must have many independent opportunities to perform a regeneration. A tail estimate on the regeneration times follows, and in turn, this yields the law of large numbers and the central limit theorem, allowing for a full use of the spatial homogeneity properties of the model. When $d \geq 3$, it is possible to replace, in our argument, estimates on the number of tan points by estimates on the number of distinct points visited by the projection of the random walk on the $(e_2, \ldots, e_d)$ coordinates – which is essentially a simple random walk on $\mathbb{Z}^{d-1}$. Such an observation was used in [1] to prove that (1) holds when $d \geq 4$. Plugging the estimates of [6] in our argument, we can rederive the law of large numbers and the central limit theorem when $d \geq 4$ without considering tan points. Furthermore, a translation of the results in [2] and [11] about the volume of the Wiener sausage to the random walk situation considered here, would allow us to rederive our results when $d = 3$, and to improve the tail estimates for any $d \geq 3$.

The regeneration time methods used to prove Theorem 1 could also be used to describe the asymptotic behavior of the configuration of the vertices as seen from the excited random walk. Let $\Xi := \{0,1\}^{\mathbb{Z}^d \setminus \{0\}}$, equipped with the product topology and $\sigma-$algebra. For each time n and site $x \neq X_n$, define $\beta(x,n) := 1$ if the site x was visited before time n by the random walk, while $\beta(x,n) := 0$ otherwise. Let $\zeta(x,n) := \beta(x - X_n, n)$ and define

$$\zeta(n) := (\zeta(x,n);\, x \in \mathbb{Z}^d \setminus \{0\}) \in \Xi.$$

We call the process $(\zeta(n))_{n \in \mathbb{N}}$ the *environment seen from the excited random walk*. It is then possible to show that if $\rho(n)$ is the law of $\zeta(n)$, there exists a probability measure ρ defined on Ξ such that

$$\lim_{n \to \infty} \rho(n) = \rho,$$

weakly.
In the following section of the paper we introduce the basic notation that will be used throughout. In Section 3, we define the regeneration times and formulate the key facts satisfied by them. In Section 4 we obtain the tail estimates for the regeneration times via a good control on the number of tan points. Finally, in Section 5, we present the results of numerical simulations in dimension $d = 2$ which suggest that, as a function of the bias parameter p, the speed $v(p,2)$ is an increasing convex function of p, whereas the variance $\sigma(p,2)$ is a concave function which attains its maximum at some point strictly between $1/2$ and 1.

2 Notations

Let $\mathbf{b} := \{e_1, \ldots, e_d, -e_1, \ldots, -e_d\}$. Let μ be the distribution on $\mathbf{b}$ defined by $\mu(+e_1) = p/d$, $\mu(-e_1) = (1-p)/d$, $\mu(\pm e_j) = 1/2d$ for $j \neq 1$. Let ν be the uniform distribution on $\mathbf{b}$. Let $\mathcal{S}_0$ denote the sample space of the trajectories of the excited random walk starting at the origin:

$$\mathcal{S}_0 := \left\{(z_i)_{i \geq 0} \in (\mathbb{Z}^d)^{\mathbb{N}};\, z_0 = 0,\, z_{i+1} - z_i \in \mathbf{b} \text{ for all } i \geq 0\right\}.$$

For all $k \geq 0$, let X_k denote the coordinate map defined on $\mathcal{S}_0$ by $X_k((z_i)_{i \geq 0}) := z_k$. We will sometimes use the notation X to denote the sequence $(X_k)_{k \geq 0}$. We let $\mathcal{F}$ be the $\sigma-$algebra on $\mathcal{S}_0$ generated by the maps $(X_k)_{k \geq 0}$. For $k \in \mathbb{N}$, the sub-$\sigma-$algebra of $\mathcal{F}$ generated by $X_0, \ldots, X_k$ is denoted by $\mathcal{F}_k$. And we let θ_k denote the transformation on $\mathcal{S}_0$ defined by $(z_i)_{i \geq 0} \mapsto (z_{k+i} - z_k)_{i \geq 0}$. For the sake of definiteness, we let $\theta_{+\infty}((z_i)_{i \geq 0}) := (z_i)_{i \geq 0}$. For all $n \geq 0$, define the following two random variables on $(\mathcal{S}_0, \mathcal{F})$:

$$r_n := \max\{X_i \cdot e_1;\, 0 \leq i \leq n\},$$

$$J_n = J_n(X) := \text{ number of indices } 0 \leq k \leq n \text{ such that } X_k \notin \{X_i;\, 0 \leq i \leq k-1\}.$$

(Note that, with this definition, $J_0 = 1$.)
We now call $\mathbb{P}_0$ the law of the excited random walk, which is formally defined as the unique probability measure on $(\mathcal{S}_0, \mathcal{F})$ satisfying the following conditions: for every $k \geq 0$,

- on $X_k \notin \{X_i;\, 0 \leq i \leq k-1\}$, the conditional distribution of $X_{k+1} - X_k$ with respect to $\mathcal{F}_k$ is μ;
- on $X_k \in \{X_i;\, 0 \leq i \leq k-1\}$, the conditional distribution of $X_{k+1} - X_k$ with respect to $\mathcal{F}_k$ is ν.

3 The renewal structure

We now define the regeneration times for the excited random walk (see [13] for the same definition in the context of random walks in random environment). Define on $(\mathcal{S}_0, \mathcal{F})$ the following $(\mathcal{F}_k)_{k\geq 0}$-stopping times: $T(h) := \inf\{k \geq 1;\, X_k \cdot e_1 > h\}$, and $D := \inf\{k \geq 1;\, X_k \cdot e_1 = 0\}$. Then define recursively the sequences $(S_i)_{i\geq 0}$ and $(D_i)_{i\geq 0}$ as follows: $S_0 := T(0)$, $D_0 := S_0 + D \circ \theta_{S_0}$, and $S_{i+1} := T(r_{D_i})$, $D_{i+1} := S_{i+1} + D \circ \theta_{S_{i+1}}$ for $i \geq 0$, with the convention that $S_{i+1} = +\infty$ if $D_i = +\infty$, and, similarly, $D_{i+1} = +\infty$ if $S_{i+1} = +\infty$. Then define $K := \inf\{i \geq 0;\, D_i = +\infty\}$ and $\kappa := S_K$ (with the convention that $\kappa = +\infty$ when $K = +\infty$).
The key estimate for proving our results is stated in the following proposition.

Proposition 1. *As n goes to infinity,*

$$\mathbb{P}_0(\kappa \geq n) \leq \exp\left(-n^{\frac{1}{19}} + o(1)\right).$$

A consequence of the above proposition is that, under $\mathbb{P}_0$, κ has finite moments of all orders, and also X_κ, since the walk performs nearest-neighbor steps. We postpone the proof of Proposition 1 to Section 4.

Lemma 1. *There exists a $\delta > 0$ such that $\mathbb{P}_0(D = +\infty) > \delta$.*

Proof. This is a simple consequence of two facts. Firstly, in [1] it is established that $\mathbb{P}_0$-a.s, $\lim_{k\to+\infty} X(k) \cdot e_1 = +\infty$. On the other hand, a general lemma (Lemma 9 of [17]) shows that, given the first fact, an excited random walk satisfies $\mathbb{P}_0(D = +\infty) > 0$. □

Lemma 2. *For all $h \geq 0$, $\mathbb{P}_0(T(h) < +\infty) = 1$.*

Proof. This is immediate from the fact that $\mathbb{P}_0$-a.s., $\lim_{k\to+\infty} X(k) \cdot e_1 = +\infty$. □

Now define the sequence of regeneration times $(\kappa_n)_{n\geq 1}$ by $\kappa_1 := \kappa$ and $\kappa_{n+1} := \kappa_n + \kappa \circ \theta_{\kappa_n}$, with the convention that $\kappa_{n+1} = +\infty$ if $\kappa_n = +\infty$. For all $n \geq 0$, we denote by $\mathcal{F}_{\kappa_n}$ the completion with respect to $\mathbb{P}_0-$negligible sets of the $\sigma-$algebra generated by the events of the form $\{\kappa_n = t\} \cap A$, for all $t \in \mathbb{N}$, and $A \in \mathcal{F}_t$.
The following two propositions are analogous respectively to Theorem 1.4 and Corollary 1.5 of [13]. Given Lemma 1 and Lemma 2, the proofs are completely similar to those presented in [13], noting that the process $(\beta(n), X_n)_{n\in\mathbb{N}}$ is strongly Markov, so we omit them, and refer the reader to [13].

Proposition 2. *For every $n \geq 1$, $\mathbb{P}_0(\kappa_n < +\infty) = 1$. Moreover, for every $A \in \mathcal{F}$, the following equality holds $\mathbb{P}_0-a.s.$*

$$\mathbb{P}_0\left(X \circ \theta_{\kappa_n} \in A | \mathcal{F}_{\kappa_n}\right) = \mathbb{P}_0\left(X \in A | D = +\infty\right). \tag{2}$$

Proposition 3. *With respect to $\mathbb{P}_0$, the random variables κ_1, $\kappa_2 - \kappa_1$, $\kappa_3 - \kappa_2, \ldots$ are independent, and, for all $k \geq 1$, the distribution of $\kappa_{k+1} - \kappa_k$ with respect to $\mathbb{P}_0$ is that of κ with respect to $\mathbb{P}_0$ conditional upon $D = +\infty$. Similarly, the random variables X_{κ_1}, $X_{\kappa_2} - X_{\kappa_1}$, $X_{\kappa_3} - X_{\kappa_2}, \ldots$ are independent, and, for all $k \geq 1$, the distribution of $X_{\kappa_{k+1}} - X_{\kappa_k}$ with respect to $\mathbb{P}_0$ is that of X_κ with respect to $\mathbb{P}_0$ conditional upon $D = +\infty$.*

For future reference, we state the following result.

Lemma 3. *On $S_k < +\infty$, the conditional distribution of the sequence $(X_i - X_{S_k})_{S_k \leq i < D_k}$ with respect to $\mathcal{F}_{S_k}$ is the same as the distribution of $(X_i)_{0 \leq i < D}$ with respect to $\mathbb{P}_0$.*

Proof. Observe that between times S_k and D_k, the walk never visits any site that it has visited before time S_k. Therefore, applying the strong Markov property to the process $(\beta(n), X_n)_{n\in\mathbb{N}}$ and spatial translation invariance, we conclude the proof. $\square$

A consequence of Proposition 1 is that $\mathbb{E}_0(\kappa|D = +\infty) < +\infty$ and $\mathbb{E}_0(|X_\kappa||D = +\infty) < +\infty$. Since $\mathbb{P}_0(\kappa \geq 1) = 1$ and $\mathbb{P}_0(X_\kappa \cdot e_1 \geq 1) = 1$, $\mathbb{E}_0(\kappa|D = +\infty) > 0$ and $\mathbb{E}_0(X_\kappa \cdot e_1|D = +\infty) > 0$. Letting $v(p,d) := \frac{\mathbb{E}_0(X_\kappa \cdot e_1|D=+\infty)}{\mathbb{E}_0(\kappa|D=+\infty)}$, we see that $0 < v(p,d) < +\infty$.
The following law of large numbers can then be proved, using Proposition 3, exactly as Proposition 2.1 in [13], to which we refer for the proof.

Theorem 2. *Under $\mathbb{P}_0$, the following limit holds almost surely:*

$$\lim_{n\to+\infty} n^{-1} X_n \cdot e_1 = v(p,d).$$

Another consequence of Proposition 1 is that $\mathbb{E}_0(\kappa^2|D = +\infty) < +\infty$ and $\mathbb{E}_0(|X_\kappa|^2|D = +\infty) < +\infty$. Letting $\sigma^2(p,d) := \frac{\mathbb{E}_0([X_\kappa \cdot e_1 - v(p,d)\kappa]^2|D=+\infty)}{\mathbb{E}_0(\kappa|D=+\infty)}$, we see that $\sigma(p,d) < +\infty$. That $\sigma(p,d) > 0$ is explained in Remark 1 below.
The following functional central limit theorem can then be proved, using Proposition 3, exactly as Theorem 4.1 in [12], to which we refer for the proof.

Theorem 3. *Under $\mathbb{P}_0$, the following convergence in distribution holds: as n goes to infinity,*

$$t \mapsto n^{-1/2}(X_{\lfloor nt \rfloor} \cdot e_1 - v\lfloor nt \rfloor),$$

converges to a Brownian motion with variance $\sigma^2(p,d)$, with respect to the Skorohod topology on the space of càdlàg functions.

Remark 1. *The fact that $\sigma(p,d) > 0$ is easy to check. Indeed, we will prove that the probability of the event $X_\kappa \cdot e_1 \neq v\kappa$ is positive. There is a positive probability that the first step of the walk is $+e_1$, and that $X_n \cdot e_1 > 1$ for all n afterwards. In this situation, $\kappa = 1$ and $X_\kappa \cdot e_1 = 1$. Now, there is a positive probability that the walk first performs the following sequence of steps: $+e_2, -e_2, +e_1$, and that then $X_n \cdot e_1 > 1$ for all n afterwards. In this situation, $\kappa = 3$ and $X_\kappa \cdot e_1 = 1$.*

4 Estimate on the tail of κ

4.1 Coupling with a simple random walk and tan points

We use the coupling of the excited random walk with a simple random walk that was introduced in [1], and subsequently used in [7, 8].
To define this coupling, let $(\alpha_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with uniform distribution on the set $\{1, \dots, d\}$. Let also $(U_i)_{i\geq 1}$ be an i.i.d. family of random variables with uniform distribution on $[0,1]$, independent from $(\alpha_i)_{i\geq 1}$. Call $(\Omega, \mathcal{G}, P)$ the probability space on which these variables are defined. Define the sequences of random variables $Y = (Y_i)_{i\geq 0}$ and $Z = (Z_i)_{i\geq 0}$ taking values in $\mathbb{Z}^d$, as follows. First, $Y_0 := 0$ and $Z_0 := 0$. Then

consider $n \geq 0$, and assume that $Y_0, \dots, Y_n$ and $Z_0, \dots, Z_n$ have already been defined. Let $Z_{n+1} := Z_n + (\mathbf{1}(U_{n+1} \leq 1/2) - \mathbf{1}(U_{n+1} > 1/2))e_{\alpha_{n+1}}$. Then, if $Y_n \in \{Y_i;\, 0 \leq i \leq n-1\}$ or $\alpha_{n+1} \neq 1$, let $Y_{n+1} := Y_n + (\mathbf{1}(U_{n+1} \leq 1/2) - \mathbf{1}(U_{n+1} > 1/2))e_{\alpha_{n+1}}$. Otherwise, let $Y_{n+1} := Y_n + (\mathbf{1}(U_{n+1} \leq p) - \mathbf{1}(U_{n+1} > p))e_1$.
The following properties are then immediate:

- $(Z_i)_{i\geq 0}$ is a simple random walk on $\mathbb{Z}^d$;
- $(Y_i)_{i\geq 0}$ is an excited random walk on $\mathbb{Z}^d$ with bias parameter p;
- for all $2 \leq j \leq d$ and $i \geq 0$, $Y_i \cdot e_j = Z_i \cdot e_j$;
- the sequence $(Y_i \cdot e_1 - Z_i \cdot e_1)_{i\geq 0}$ is non-decreasing.

Definition 1. *If $(z_i)_{i\geq 0} \in \mathcal{S}_0$, we call an integer $n \geq 0$ an (e_1, e_2)–tan point index for the sequence $(z_i)_{i\geq 0}$ if $z_n \cdot e_1 > z_k \cdot e_1$ for all $0 \leq k \leq n-1$ such that $z_n \cdot e_2 = z_k \cdot e_2$.*

The key observation made in [1] is the following.

Lemma 4. *If n is an (e_1, e_2)–tan point index for $(Z_i)_{i\geq 0}$, then $Y_n \notin \{Y_i;\, 0 \leq i \leq n-1\}$.*

Proof. If n is an (e_1, e_2)–tan point index and if there exists an $\ell \in \{0, \dots, n-1\}$ such that $Y_n = Y_\ell$, then observe that, using the fact that $Z_\ell \cdot e_2 = Y_\ell \cdot e_2$ and $Z_n \cdot e_2 = Y_n \cdot e_2$, we have that $Z_\ell \cdot e_2 = Z_n \cdot e_2$. Hence, by the definition of a tan point we must have that $Z_\ell \cdot e_1 < Z_n \cdot e_1$, whence $Y_n \cdot e_1 - Z_n \cdot e_1 < Y_\ell \cdot e_1 - Z_\ell \cdot e_1$. But this contradicts the fact that the coupling has the property that $Y_n \cdot e_1 - Z_n \cdot e_1 \geq Y_\ell \cdot e_1 - Z_\ell \cdot e_1$.

□

Let $H := \{i \geq 1;\, \alpha_i \in \{1, 2\}\}$, and define the sequence of indices $(I_i)_{i\geq 0}$ by $I_0 := 0$, $I_0 < I_1 < I_2 < \cdots$, and $\{I_1, I_2, \ldots\} = H$. Then the sequence of random variables $(W_i)_{i\geq 0}$ defined by $W_i := (Z_{I_i} \cdot e_1, Z_{I_i} \cdot e_2)$ forms a simple random walk on $\mathbb{Z}^2$.
If i and n are such that $I_i = n$, it is immediate to check that n is an (e_1, e_2)–tan point index for $(Z_k)_{k\geq 0}$ if and only if i is an (e_1, e_2)–tan point index for the random walk $(W_k)_{k\geq 0}$.
For all $n \geq 1$, let N_n denote the number of (e_1, e_2)–tan point indices of $(W_k)_{k\geq 0}$ that are $\leq n$.
The arguments used to prove the following lemma are quite similar to the ones used in the proofs of Theorem 4 in [1] and Lemma 1 in [8], which are themselves partly based on estimates in [3].

Lemma 5. *For all $0 < a < 3/4$, as n goes to infinity,*

$$P(N_n \leq n^a) \leq \exp\left(-n^{\frac{1}{3} - \frac{4a}{9} + o(1)}\right).$$

Proof. For all $k \in \mathbb{Z} \setminus \{0\}$, $m \geq 1$, consider the three sets

$$\begin{aligned}
\Gamma(m)_k &:= \mathbb{Z} \times \{2k\lfloor m^{1/2} \rfloor\}, \\
\Delta(m)_k &:= \mathbb{Z} \times ((2k-1)\lfloor m^{1/2} \rfloor, (2k+1)\lfloor m^{1/2} \rfloor), \\
\Theta(m)_k &:= \{v \in \Delta(m)_k;\, |v \cdot e_2| \geq 2k\lfloor m^{1/2} \rfloor\}.
\end{aligned}$$

Let $\chi(m)_k$ be the first time when $(W_i)_{i\geq 0}$ hits $\Gamma(m)_k$, and note that $\chi(m)_k$ is a.s. finite since the simple random walk on $\mathbb{Z}^2$ is recurrent. Let $\phi(m)_k$ be the first time after $\chi(m)_k$ when $(W_i)_{i\geq 0}$ leaves $\Delta(m)_k$. Again, $\phi(m)_k$ is a.s. finite due to the recurrence of the simple random

walk on $\mathbb{Z}^2$. Let $M_k(m)$ denote the number of time indices n that are (e_1, e_2)–tan point indices, and satisfy $\chi(m)_k \leq n \leq \phi(m)_k - 1$ and $W_n \in \Theta(m)_k$.
Two key observations in [1] (see Lemma 2 in [1] and the discussion before its statement) are that:

- the sequence $(M_k(m))_{k \in \mathbb{Z} \setminus \{0\}}$ is i.i.d.;
- there exist $c_1, c_2 > 0$ such that $P(M_1(m) > c_1 m^{3/4}) \geq c_2$.

Now, consider an $\epsilon > 0$ such that $b := 1/3 - 4a/9 - \epsilon > 0$. let $m_n := \lceil (n^a/c_1)^{4/3} \rceil + 1$, and let $h_n := 2(\lceil n^b \rceil + 1)\lfloor m_n^{1/2} \rfloor$. Note that, as $n \to +\infty$, $(h_n)^2 \sim (4c_1^{-4/3}) n^{\frac{2}{3} + \frac{4}{9}a - 2\epsilon}$. Let $R_{n,+}$ and $R_{n,-}$ denote the following events

$$R_{n,+} := \{\text{for all } k \in \{1, \ldots, +\lceil n^b \rceil\},\ M_k(m_n) \leq c_1 m_n^{3/4}\},$$

and

$$R_{n,-} := \{\text{for all } k \in \{-\lceil n^b \rceil, \ldots, -1\},\ M_k(m_n) \leq c_1 m_n^{3/4}\}.$$

From the above observations, $P(R_{n,+} \cup R_{n,-}) \leq 2(1 - c_2)^{\lceil n^b \rceil}$.
Let $q_n := \lfloor n(h_n)^{-2} \rfloor$, and let V_n be the event

$$V_n := \{\text{for all } i \in \{0, \ldots, n\},\ -h_n \leq W_i \cdot e_2 \leq +h_n\}.$$

By Lemma 6 below, there exists a constant $c_3 > 0$ such that, for all large enough n, all $-h_n \leq y \leq +h_n$, and $x \in \mathbb{Z}$, the probability that a simple random walk on $\mathbb{Z}^2$ started at (x, y) at time zero leaves $\mathbb{Z} \times \{-h_n, \ldots, +h_n\}$ before time h_n^2, is larger than c_3. A consequence is that, for all $q \geq 0$, the probability that the same walk fails to leave $\mathbb{Z} \times \{-h_n, \ldots, +h_n\}$ before time qh_n^2 is less than $(1 - c_3)^q$. Therefore $P(V_n) \leq (1 - c_3)^{q_n}$.
Observe now that, on V_n^c,

$$n \geq \max(\phi(m_n)_k;\ 1 \leq k \leq \lceil n^b \rceil) \text{ or } n \geq \max(\phi(m_n)_k;\ -\lceil n^b \rceil \leq k \leq -1).$$

Hence, on V_n^c,

$$N_n \geq \sum_{k=1}^{\lceil n^b \rceil} M_k(m_n) \text{ or } N_n \geq \sum_{k=-1}^{-\lceil n^b \rceil} M_k(m_n).$$

We deduce that, on $R_{n,+}^c \cap R_{n,-}^c \cap V_n^c$, $N_n \geq c_1 m_n^{3/4} > n^a$.
As a consequence, $\mathbb{P}_0(N_n \leq n^a) \leq \mathbb{P}_0(R_{n,+} \cup R_{n,-}) + \mathbb{P}_0(V_n)$, so that $\mathbb{P}_0(N_n \leq n^a) \leq 2(1 - c_2)^{\lceil n^b \rceil} + (1 - c_3)^{q_n}$
Noting that, as n goes to infinity, $q_n \sim nh_n^{-2} \sim (4c_1^{-4/3})^{-1} n^{1/3 - 4a/9 + 2\epsilon}$, the conclusion follows.
□

Lemma 6. *There exists a constant $c_3 > 0$ such that, for all large enough h, all $-h \leq y \leq +h$, and $x \in \mathbb{Z}$, the probability that a simple random walk on $\mathbb{Z}^2$ started at (x, y) at time zero leaves $\mathbb{Z} \times \{-h, \ldots, +h\}$ before time h^2, is larger than c_3.*

Proof. Consider the probability that the e_2 coordinate is larger than h at time h^2. By standard coupling, this probability is minimal when $y = -h$, so the central limit theorem applied to the walk starting with $y = -h$ yields the existence of c_3.
□

Lemma 7. *For all* $0 < a < 3/4$*, as* n *goes to infinity,*

$$\mathbb{P}_0(J_n \le n^a) \le \exp\left(-n^{\frac{1}{3} - \frac{4a}{9} + o(1)}\right).$$

Proof. Observe that, by definition, I_k is the sum of k i.i.d. random variables whose distribution is geometric with parameter $2/d$. By a standard large deviations bound, there is a constant c_6 such that, for all large enough n, $P(I_{\lfloor nd^{-1} \rfloor} \ge n) \le \exp(-c_6 n)$. Then observe that, if $I_{\lfloor nd^{-1} \rfloor} \le n$, we have $J_n(Y) \ge N_{\lfloor nd^{-1} \rfloor}$ according to Lemma 4 above. (Remember that, by definition, $J_n(Y)$ is the number of indices $0 \le k \le n$ such that $Y_k \notin \{Y_i;\, 0 \le i \le k-1\}$.)
Now, fix $0 < a < 3/4$. According to Lemma 5 above, we have that, for all $a < a' < 3/4$, as n goes to infinity,

$$P(N_{\lfloor nd^{-1} \rfloor} \le \lfloor nd^{-1} \rfloor^{a'}) \le \exp\left(-\lfloor nd^{-1} \rfloor^{\frac{1}{3} - \frac{4a'}{9} + o(1)}\right),$$

from which it is easy to deduce that, as n goes to infinity, $P(N_{\lfloor nd^{-1} \rfloor} \le n^a) \le \exp\left(-n^{\frac{1}{3} - \frac{4a}{9} + o(1)}\right)$.
Now we deduce from the union bound that $P(J_n(Y) \le n^a) \le P(I_{\lfloor nd^{-1} \rfloor} \ge n) + P(N_{\lfloor nd^{-1} \rfloor} \le n^a)$. The conclusion follows.

□

4.2 Estimates on the displacement of the walk

Lemma 8. *For all* $1/2 < a < 3/4$*, as* n *goes to infinity,*

$$\mathbb{P}_0(X_n \cdot e_1 \le n^a) \le \exp\left(-n^{\psi(a)+o(1)}\right),$$

where $\psi(a) := \min\left(\frac{1}{3} - \frac{4a}{9}, 2a - 1\right)$.

Proof. Let $\gamma := \frac{2p-1}{2d}$. Let $(\varepsilon_i)_{i \ge 1}$ be an i.i.d. family of random variables with common distribution μ on $\mathbf{b}$, and let $(\eta_i)_{i \ge 1}$ be an i.i.d. family of random variables with common distribution ν on $\mathbf{b}$ independent from $(\varepsilon_i)_{i \ge 1}$. Let us call $(\Omega_2, \mathcal{G}_2, Q)$ the probability space on which these variables are defined.
Define the sequence of random variables $(\xi_i)_{i \ge 0}$ taking values in $\mathbb{Z}^d$, as follows. First, set $\xi_0 := 0$. Consider then $n \ge 0$, assume that $\xi_0, \ldots, \xi_n$ have already been defined, and consider the number $J_n(\xi)$ of indices $0 \le k \le n$ such that $\xi_k \notin \{\xi_i;\, 0 \le i \le k-1\}$. If $\xi_n \notin \{\xi_i;\, 0 \le i \le n-1\}$, set $\xi_{n+1} := \xi_n + \varepsilon_{J_n(\xi)}$. Otherwise, let $\xi_{n+1} := \xi_n + \eta_{n - J_n(\xi)+1}$. It is easy to check that the sequence $(\xi_n)_{n \ge 0}$ is an excited random walk on $\mathbb{Z}^d$ with bias parameter p.
Now, according to Lemma 7, for all $1/2 < a < 3/4$, $Q(J_n \le n^a) \le \exp\left(-n^{\frac{1}{3} - \frac{4a}{9} + o(1)}\right)$. It is easy to deduce that, for all $1/2 < a < 3/4$, $Q(J_{n-1} \le 2\gamma^{-1} n^a) \le \exp\left(-n^{\frac{1}{3} - \frac{4a}{9} + o(1)}\right)$.
Now observe that, by definition, for $n \ge 1$,

$$\xi_n = \sum_{i=1}^{J_{n-1}} \varepsilon_i + \sum_{i=1}^{n - J_{n-1}} \eta_i. \tag{3}$$

Now, there exists a constant c_4 such that, for all large enough n, and every $2\gamma^{-1}n^a \leq k \leq n$,

$$Q\left(\sum_{i=1}^{k} \varepsilon_i \cdot e_1 \leq (3/2)n^a\right) \leq Q\left(\sum_{i=1}^{k} \varepsilon_i \cdot e_1 \leq \tfrac{3}{4}\gamma k\right) \leq \exp\left(-c_4 n^a\right),$$

by a standard large deviations bound for the sum $\sum_{i=1}^{k} \varepsilon_i \cdot e_1$, whose terms are i.i.d. bounded random variables with expectation $\gamma > 0$. By the union bound, we see that

$$Q\left(\sum_{i=1}^{J_{n-1}} \varepsilon_i \cdot e_1 \leq (3/2)n^a\right) \leq n \exp\left(-c_4 n^a\right) + \exp\left(-n^{\frac{1}{3} - \frac{4a}{9} + o(1)}\right). \tag{4}$$

Now, there exists a constant c_5 such that, for all large enough n, and for every $1 \leq k \leq n$, $Q\left(\sum_{i=1}^{k} \eta_i \cdot e_1 \leq -(1/2)n^a\right) \leq \exp\left(-c_5 n^{2a-1}\right)$, by a standard moderate deviations bound for the simple symmetric random walk on $\mathbb{Z}$. By the union bound again, we see that

$$Q\left(\sum_{i=1}^{n-J_{n-1}} \eta_i \cdot e_1 \leq -(1/2)n^a\right) \leq n \exp\left(-c_5 n^{2a-1}\right). \tag{5}$$

Noting that, by (3), the event $\{\xi_n \cdot e_1 < n^a\}$ is included in the event $\{\sum_{i=1}^{J_{n-1}} \varepsilon_i \cdot e_1 \leq (3/2)n^a\} \cup \{\sum_{i=1}^{n-J_{n-1}} \eta_i \cdot e_1 \leq -(1/2)n^a\}$, the conclusion now follows by (4) and (5), and the union bound, using the fact that, for $1/2 < a < 3/4$, $a \geq \psi(a)$.

□

Lemma 9. *As n goes to infinity,*

$$\mathbb{P}_0(n \leq D < +\infty) \leq \exp\left(-n^{1/11+o(1)}\right).$$

Proof. Consider $1/2 < a < 3/4$, and write $\mathbb{P}_0(n \leq D < +\infty) = \sum_{k=n}^{+\infty} \mathbb{P}_0(D = k) \leq \sum_{k=n}^{+\infty} \mathbb{P}_0(X_k \cdot e_1 = 0) \leq \sum_{k=n}^{+\infty} \mathbb{P}_0(X_k \cdot e_1 \leq k^a)$. Now, according to Lemma 8, $\mathbb{P}_0(X_k \cdot e_1 \leq k^a) \leq \exp\left(-k^{\psi(a)+o(1)}\right)$. It is then easily checked that $\sum_{k=n}^{+\infty} \exp\left(-k^{\psi(a)+o(1)}\right) \leq \exp\left(-n^{\psi(a)+o(1)}\right)$. As a consequence, $\mathbb{P}_0(n \leq D < +\infty) \leq \exp\left(-n^{\psi(a)+o(1)}\right)$. Choosing a so as to minimize $\psi(a)$, the result follows.

□

4.3 Proof of Proposition 1

Let a_1, a_2, a_3 be positive real numbers such that $a_1 < 3/4$ and $a_2 + a_3 < a_1$. For every $n > 0$, let $u_n := \lfloor n^{a_1} \rfloor$, $v_n := \lfloor n^{a_2} \rfloor$, $w_n := \lfloor n^{a_3} \rfloor$. In the sequel, we assume that n is large enough so that $v_n(w_n + 1) + 2 \leq u_n$. Let

$$A_n := \{X_n \cdot e_1 \leq u_n\};\ B_n := \bigcap_{k=0}^{v_n} \{D_k < +\infty\};\ C_n := \bigcup_{k=0}^{v_n} \{w_n \leq D_k - S_k < +\infty\}.$$

(With the convention that, in the definition of C_n, $D_k - S_k = +\infty$ whenever $D_k = +\infty$.) We shall prove that $\{\kappa \geq n\} \subset A_n \cup B_n \cup C_n$, then apply the union bound to $\mathbb{P}_0(A_n \cup B_n \cup C_n)$, and then separately bound the three probabilities $\mathbb{P}_0(A_n)$, $\mathbb{P}_0(B_n)$, $\mathbb{P}_0(C_n)$.

Assume that $A_n^c \cap B_n^c \cap C_n^c$ occurs. Our goal is to prove that this assumption implies that $\kappa < n$.

Call M the smallest index k between 0 and v_n such that $D_k = +\infty$, whose existence is ensured by B_n^c. By definition, $\kappa = S_M$, so we have to prove that $S_M < n$. For notational convenience, let $D_{-1} = 0$. By definition of M, we know that $D_{M-1} < +\infty$. Now write $r_{D_{M-1}} = \sum_{k=0}^{M-1}(r_{D_k} - r_{S_k}) + (r_{S_k} - r_{D_{k-1}})$, with the convention that $\sum_{k=0}^{-1} = 0$. Since the walk performs nearest-neighbor steps, we see that for all $0 \leq k \leq M-1$, $r_{D_k} - r_{S_k} \leq D_k - S_k$. On the other hand, by definition, for all $0 \leq k \leq M-1$, $r_{S_k} - r_{D_{k-1}} = 1$. Now, for all $0 \leq k \leq M-1$, $D_k - S_k \leq w_n$, due to the fact that C_n^c holds and that $D_k < +\infty$. As a consequence, we obtain that $r_{D_{M-1}} \leq M(w_n+1) \leq v_n(w_n+1)$. Remember now that $v_n(w_n+1)+2 \leq u_n$, so we have proved that, $r_{D_{M-1}} + 2 \leq u_n$. Now observe that, on A_{n^c}, $X_n \cdot e_1 > u_n$. As a consequence, the smallest i such that $X_i \cdot e_1 = r_{D_{M-1}} + 1$ must be $< n$. But S_M is indeed the smallest i such that $X_i \cdot e_1 = r_{D_{M-1}} + 1$, so we have proved that $S_M < n$ on $A_n^c \cap B_n^c \cap C_n^c$.

The union bound then yields the fact that, for large enough n, $\mathbb{P}_0(\kappa \geq n) \leq \mathbb{P}_0(A_n) + \mathbb{P}_0(B_n) + \mathbb{P}_0(C_n)$.

Now, from Lemma 8, we see that $\mathbb{P}_0(A_n) \leq \exp(-n^{\psi(a_1)+o(1)})$. By repeatedly applying Lemma 2 and the strong Markov property at the stopping times S_k for $k = 0, \ldots, v_n$ to the process $(\beta(n), X_n)_{n\in\mathbb{N}}$, we see that $\mathbb{P}_0(B_n) \leq \mathbb{P}_0(D < +\infty)^{v_n}$. Hence, from Lemma 1, we know that $\mathbb{P}_0(B_n) \leq (1-\delta)^{v_n}$.

From the union bound and Lemma 3, we see that $\mathbb{P}_0(C_n) \leq (v_n+1)\mathbb{P}_0(w_n \leq D < +\infty)$, so, by Lemma 9, $\mathbb{P}_0(C_n) \leq (v_n+1)\exp(-n^{a_3/11+o(1)})$.

Using Lemma 8, we finally obtain the following estimate:

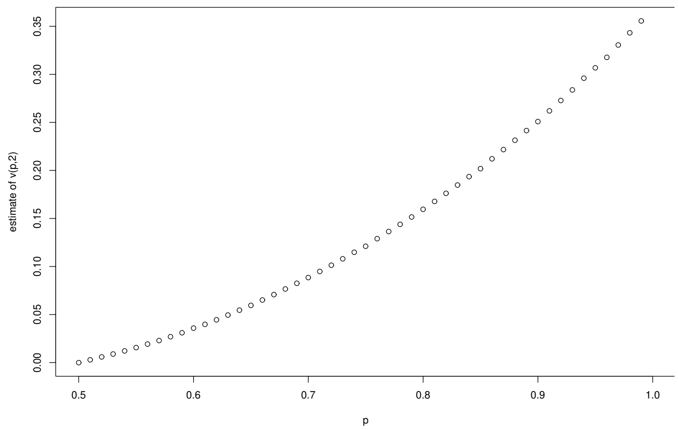
$$\mathbb{P}_0(\kappa \geq n) \leq (1-\delta)^{\lfloor n^{a_2} \rfloor} + (\lfloor n^{a_2} \rfloor + 1)\exp\left(-n^{a_3/11+o(1)}\right) + \exp\left(-n^{\psi(a_1)+o(1)}\right).$$

Now, for all ϵ small enough, choose $a_1 = 12/19$, $a_2 = 1/19$, $a_3 = 11/19 - \epsilon$. This ends the proof of Proposition 1.

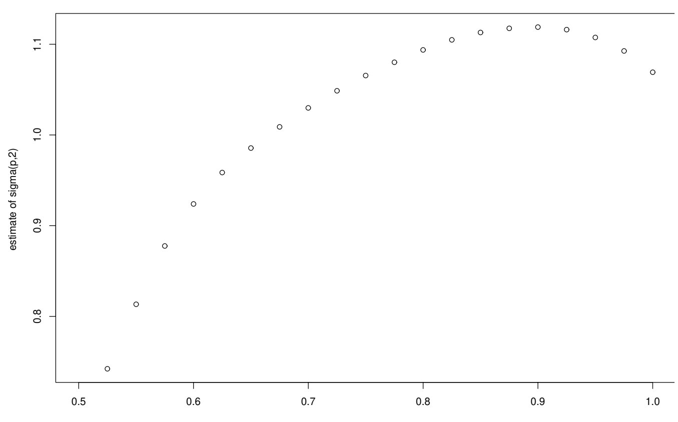
5 Simulation results

We have performed simulations of the model in dimension $d = 2$, using a C code and the Gnu Scientific Library facilities for random number generation.

The following graph is a plot of an estimate of $v(p, 2)$ as a function of p. Each point is the average over 1000 independent simulations of $(X_{10000} \cdot e_1)/10000$.



The following graph is a plot of an estimate of $\sigma(p, 2)$ as a function of p. Each point is the standard deviation over 1000000 independent simulations of $(X_{10000} \cdot e_1)/(10000)^{1/2}$ (obtaining a reasonably smooth curve required many more simulations for σ than for v).



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