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CORNERS AND RECORDS OF THE POISSON PROCESS IN QUADRANT

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Abstract

The scale-invariant spacings lemma due to Arratia, Barbour and Tavaré establishes the distributional identity of a self-similar Poisson process and the set of spacings between the points of this process. In this note we connect this result with properties of a certain set of extreme points of the unit Poisson process in the positive quadrant.

1 Introduction

For fixed $k > 0$ let $\mathcal{T}^{(k)}$ be the *self-similar* (or *scale-invariant*) Poisson point process on $\mathbb{R}_+$, with intensity function k/t. Let $\mathcal{S}^{(k)}$ be the point process of *spacings* in $\mathcal{T}^{(k)}$, meaning that the generic point of $\mathcal{S}^{(k)}$ is a difference $t - s$, where $t > s$ are some consequitive points of $\mathcal{T}^{(k)}$ (so $[s,t] \cap \mathcal{T}^{(k)} = \{s,t\}$). The *scale-invariant spacings lemma* due to Arratia, Barbour and Tavaré [2, Lemma 7.1] asserts that

$$\text{the ABT lemma}: \qquad \mathcal{S}^{(k)} =_d \mathcal{T}^{(k)}. \tag{1}$$

In this note we re-derive this remarkable result from the perspective of the theory of records, and connect it with the circle of ideas around Ignatov's theorem [7, 10, 11, 15, 17]. We choose the framework of the Poisson point process in the positive quadrant since this setting is very geometric and allows us to exploit various symmetry properties of the Lebesgue measure. The connection between k-records and k-corners in Proposition 6 and the intensity formula (5) are new.

See [2, 3, 13, 14] for other occurrences of the self-similar Poisson process in combinatorial probability.

2 Corners and records

Let $\mathcal{P}$ be the Poisson point process in $\mathbb{R}_+^2$ with unit intensity. All point processes considered here have no multiple points, a feature which enables us to treat these processes as random sets rather than counting measures. We shall interpret an atom $a = (t, x) \in \mathcal{P}$ as the value x observed at time t. With probability one no two atoms of $\mathcal{P}$ lie on the same vertical or horizontal line, hence there is a one-to-one correspondence between observation times and observed values. The coordinate projections will be denoted $\tau(a) = t$ and $\xi(a) = x$. The process $\mathcal{P}$ is locally finite, however this does not apply to its projections: for every interval $]s, t[$ $(0 < s < t)$ there are infinitely many atoms with $\tau(a) \in]s, t[$, and for every interval $]x, y[$ $(0 < x < y)$ there are infinitely many atoms with $\xi(a) \in]x, y[$.
For k a positive integer, a point $a \in \mathbb{R}_+^2$ is said to be a *k-corner* of $\mathcal{P}$ if

(I) either $a \in \mathcal{P}$ and there are $k - 1$ points $b \in \mathcal{P}$ strictly south-west from a,

(II) or $a \notin \mathcal{P}$, there are $k - 2$ points $b \in \mathcal{P}$ strictly south-west from a (i.e. in the open south-west quadrant with apex a), a point $c \in \mathcal{P}$ strictly west from a and a point $d \in \mathcal{P}$ strictly south from a.

To interpret the definition geometrically, suppose a light source allocated at point $a \in \mathbb{R}_+^2$ illuminates the area south-west from a including the edges. Generate a rectangular grid, dense in the quadrant, by drawing all vertical and horizontal lines through atoms of $\mathcal{P}$. The k-corners are the points a of the grid which illuminate exactly k atoms of $\mathcal{P}$.
We denote $\mathcal{C}^{(k)}$ the set of k-corners and denote $\mathcal{R}^{(k)}$ its subset defined by the condition (I) alone. Obviously, $\mathcal{C}^{(1)} = \mathcal{R}^{(1)}$, but for $k > 1$ the inclusion $\mathcal{R}^{(1)} \subset \mathcal{C}^{(1)}$ is strict almost surely. Following [15] we call the points $a \in \mathcal{R}^{(k)}$ *k-records*. For $a \in \mathcal{P}$ the *initial rank* of a is one bigger the number of atoms $b \in \mathcal{P}$ strictly south-west from a, hence a k-record is an observation of initial rank k.
Notably, $\tau(\mathcal{C}^{(k)}) =_d \xi(\mathcal{C}^{(k)})$ and $\tau(\mathcal{R}^{(k)}) =_d \xi(\mathcal{R}^{(k)})$. This is seen from the fact that the reflection $(t, x) \mapsto (x, t)$ about the bisectrix preserves both the coordinatewise partial order and the Lebesgue measure, hence preserves the law of $\mathcal{P}$.
Let $M_t^{(k)}$ be the k-th smallest value observed before t, which is the k-th minimal point of the Poisson process $\{\xi(a) : a \in \mathcal{P} \cap ([0, t] \times [0, \infty[)\}$. It is easily seen that $(M_t^{(k)}, t > 0)$ is a nonincreasing piecewise-constant càdlàg process, whose flats start at the k-corners of $\mathcal{P}$. Indeed, $M_t^{(k)} < M_{t-}^{(k)}$ means that there is an atom $(t, x) \in \mathcal{P}$ with $t \in \tau(\cup_{i \leq k} \mathcal{R}^{(i)})$; then $x = M_t^{(k)}$ if (t, x) is a k-record, and $M_t^{(k)} = M_{t-}^{(k-1)}$ if the initial rank of (t, x) is less than k. Furthermore, if $(t, x) \in \mathcal{P}$ is a j-record for $j < k$ then $M_t^{(k)} > x$ and $M_s^{(k)} = x$ for s the time of the $(k - j)$th observation in $]t, \infty[\times [0, x]$. It follows that $\xi(\cup_{j \leq k} \mathcal{R}^{(j)}) = \xi(\mathcal{C}^{(k)})$, and by symmetry also that $\tau(\cup_{j \leq k} \mathcal{R}^{(j)}) = \tau(\mathcal{C}^{(k)})$. However, despite the coincidence of projections, the point processes $\mathcal{C}^{(k)}$ and $\cup_{j \leq k} \mathcal{R}^{(k)}$ are very different.

Remark 1. The term 'k-record' in the existing literature is ambiguous. By some authors (see e.g. [16]) a k-record is a new value of the kth minimum caused by an observation of the initial rank at most k, and this corresponds to the historically first usage of the term in [9]. By other authors (especially in the work on Ignatov's theorem, see [7] for a survey) a k-record is an observation of the initial rank exactly k. According to [1], these are k-records of types 2 and 1, respectively. Looking in the earlier work on the order properties of multivariate samples [4], the k-records (in the sense of [7], or of type 1 in [1]) correspond to 'the kth layer 3rd quadrant admissible points'. Thus our k-records are as in [7, 17] (hence type 1 in [1]), while our k-corners are the 'k-records' in the sense of [16] (hence type 2 in [1]).

3 Projections and intensity

The process $(M_t^{(k)}, t > 0)$ is Markovian, with a familiar kind of dynamics [5, 6, 10, 11, 12]. Given $M_t = x$, the residual life-time in x is E/x and the new state when the transition occurs is Bx, where the random variables E and B are independent, E is exponential(1), and B is beta$(k,1)$ with density

$$\mathbb{P}(B \in \mathrm{d}z) = kz^{k-1}\mathrm{d}z \quad z \in [0,1]. \tag{2}$$

The marginal distributions have gamma densities

$$\mathbb{P}(M_t^{(k)} \in \mathrm{d}x) = \frac{e^{-tx}(tx)^{k-1}t\mathrm{d}x}{\Gamma(k)}\,, \qquad x > 0. \tag{3}$$

A self-similarity property

$$(cM_t^{(k)}, t > 0) =_d (M_{t/c}^{(k)}, t > 0), \qquad c > 0 \tag{4}$$

follows from the invariance of the Lebesgue measure under the hyperbolic shifts $(t, x) \mapsto (t/c, cx)$. The process 'enters from infinity', i.e. has the asymptotic initial value $M_{0+}^{(k)} = \infty$, and has the asymptotic terminal value $M_{\infty-}^{(k)} = 0$.
In the following result, the assertion about $\tau(\mathcal{R}^{(1)})$ is an instance [17, Proposition 4.9], while the last claim is a specialisation of Ignatov's theorem in the form of [15, Corollary 5.1].

Proposition 2. *The point processes $\tau(\mathcal{R}^{(k)})$ for $k = 1, 2, \dots$ are iid Poisson, each with intensity $1/t$. The process $\tau(\mathcal{C}^{(k)})$ is Poisson with intensity k/t. The analogous facts are true for $\xi(\mathcal{R}^{(k)})$ and $\xi(\mathcal{C}^{(k)})$.*

Proof. Fix t and let $a_1, a_2, \dots$ be the points of $\mathcal{P} \cup ([0,t] \times [0, \infty[)$ labelled by increase of their x-values $M_t^{(1)}, M_t^{(2)}, \dots$. Because the initial rank of a_k is equal to $\#\{i : i \le k, \tau(a_i) \le \tau(a_k)\}$, the processes $\tau(\mathcal{R}^{(k)}) \cap [0,t]$ for $k = 1, 2, \dots$ are completely determined by the time projections $(\tau(a_k), k \in \mathbb{N})$, hence they are jointly independent of the x-projections $(M_t^{(k)}, k \in \mathbb{N})$. On the other hand, the initial ranks of observations after t depend on $\mathcal{P} \cap ([0,t] \times [0, \infty[)$ only through $(M_t^{(k)}, k \in \mathbb{N})$, from which follows that the *multivariate* point process $(\tau(\mathcal{R}^{(k)}), k \in \mathbb{N})$ on $\mathbb{R}_+$ has independent increments, meaning that its restrictions to disjoint intervals are independent (a property also called *complete independence* in [8]). The intensity of each $\tau(\mathcal{R}^{(k)})$ is readily identified as $1/t$ since an observation of initial rank k occurs in $[t - \mathrm{d}t, t]$ precisely when a_k arrives on this interval, and since the law of $\tau(a_k)$ is uniform$[0,t]$. It follows that each $\tau(\mathcal{R}^{(k)})$ is a self-similar Poisson process with intensity $1/t$. The multivariate process $(\tau(\mathcal{R})^{(k)}, k \in \mathbb{N})$ is *simple*, hence by a standard result from the theory of point processes [8, p. 205] the component processes $\tau(\mathcal{R}^{(k)})$'s are jointly independent.
By symmetry about the bisectrix the above is extended to k-record values. Superposing k iid Poisson processes yields the result about $\tau(\mathcal{C}^{(k)}) = \cup_{j \le k} \tau(\mathcal{R}^{(j)})$, and finally this is extended to $\xi(\mathcal{C}^{(k)})$ by symmetry. □

Remark 3. Ignatov's theorem in its classical form asserts that the point processes of k-record values, derived from an iid sequence (with some distribution function F) are iid. By application of the probability integral transform, the case of arbitrary continuous F is reducible to the instance of F being uniform$[0,1]$. In its turn, the uniform case is readily covered by Proposition 2, because the values of the observations in the strip $\mathcal{P} \cap ([0,\infty[\times[0,x])$, arranged in their time-order, are iid uniform$[0,1]$. For continuous F, this argument for Ignatov's theorem seems to be the shortest known. Note that the symmetry between record values and record times is lost if the Lebesgue measure $\mathrm{d}t\mathrm{d}x$ in the quadrant is replaced by any other $\mathrm{d}t \cdot \nu(\mathrm{d}x)$ in $\mathbb{R}_+ \times \mathbb{R}$ with nonatomic, sigma-finite ν such that $\nu[-\infty, x] < \infty$ for $x \in \mathbb{R}$.

Lemma 4. *Conditionally given $(M_s^{(k)}, s \le t)$, the vector $(M_t^{(1)}, \dots, M_t^{(k-1)})$ is independent of the observation times $(\tau(\mathcal{R}^{(j)}) \cap [0,t], j \in \mathbb{N})$ and has the same law as the vector of $k-1$ minimal order statistics sampled from the uniform distribution on $[0, M_t^{(k)}]$.*

Proof. Condition on the location of the last k-corner before t, say (u,x). We have $\mathcal{P} \cap (\,]u,t[\times[0,x]) = \varnothing$. The rectangular grid spanned on k points involved in the definition of (u,x) is distributed like the product grid generated by $k-1$ order statistics from uniform$[0,u]$ and independent $k-1$ order statistics from uniform$[0,x]$. The grid and the processes $\mathcal{P} \cap (]t,\infty] \times [0,x])$, $\mathcal{P} \cap ([0,t]\times]x,\infty])$ are jointly independent, which readily yields the result, because $(M_s^{(k)}, s \le t)$ is determined by $\mathcal{P} \cap ([0,u]\times]x,\infty])$, and $(\tau(\mathcal{R}^{(j)}) \cap [0,t], j \in \mathbb{N})$ is determined by $\mathcal{P} \cap ([0,u]\times]x,\infty])$ and the τ-projection of the grid. $\square$

Remark 5. The kth sample from the uniform distribution hits each of the k spacings generated by $k-1$ order statistics with probability $1/k$. Thus Lemma 4 implies that each $\tau(\mathcal{R}^i)$ is a pointwise Bernoulli thinning with probability $1/k$ of the process $\tau(\mathcal{C}^{(k)})$, for any $k \ge i$. Once the independence of increments of the superposition process $\tau(\mathcal{C}^{(k)}) = \cup_{i \le k}\tau(\mathcal{R}^{(i)})$ is acquired, these facts can be used to avoid the most subtle part of our proof of Proposition 2: the reference to the general result that the independence of increments of a multivariate process and simplicity imply independence of the marginal point processes.

We compute next the density $p_m(a_1, \dots, a_m)$ of the event that there are m k-corners at locations $a_1 = (t_1, x_1), \dots, a_m = (t_m, x_m)$, with $t_1 < \dots < t_m,\ x_1 > \dots > x_m$, and no further k-corners occur between t_1 and t_m. This event occurs when $M_{t_1-} = x$ for some $x > x_1$ and the process at time t_1 decrements to x_1, then spends the time $t_2 - t_1$ at x_1, then decrements to x_2 and so on, hence the infinitesimal probability of the event in focus is

$$\begin{aligned}\frac{1}{\Gamma(k)}\int_{x_1}^{\infty}(t_1x)^{k-1}e^{-t_1x}t_1\mathrm{d}x\,(x\mathrm{d}t_1)\,k\left(\frac{x_1}{x}\right)^{k-1}\frac{\mathrm{d}x_1}{x}&\\ x_1e^{-(t_2-t_1)x_1}\mathrm{d}t_2\,k\left(\frac{x_2}{x_1}\right)^{k-1}\frac{\mathrm{d}x_2}{x_1}\cdots&\\ x_{m-1}e^{-(t_m-t_{m-1})x_{m-1}}\mathrm{d}t_m\,k\left(\frac{x_m}{x_{m-1}}\right)^{k-1}\frac{\mathrm{d}x_m}{x_{m-1}},&\end{aligned}$$

which after massive cancellation results in

$$p_m(a_1, \dots, a_m) = \frac{k^m}{\Gamma(k)}(t_1x_m)^{k-1}\exp\left[-x_1t_1 - (t_2-t_1)x_1 - \dots - (t_m - t_{m-1})x_m)\right]. \tag{5}$$

The expression in the right-hand side is invariant under the substitution $t_1 \leftrightarrow x_m, \dots, t_m \leftrightarrow x_1$, which is equivalent to our observation that the law of $\mathcal{C}^{(k)}$ is preserved by the reflection about the bisectrix.
The $m = 1$ instance of (5) is the intensity of $\mathcal{C}^{(k)}$,

$$p_1(t, x) = \frac{k}{\Gamma(k)}(tx)^{k-1}e^{-tx}, \tag{6}$$

which being compared with the (obvious) intensity function $(tx)^{k-1}e^{-tx}/\Gamma(k)$ of the k-record process $\mathcal{R}^{(k)}$ makes us wonder where the factor k is coming from. The structure of the processes $\mathcal{R}^{(k)}$ (which are neither independent nor identically distributed) is apparently more complex than that of $\mathcal{C}^{(k)}$'s. In particular, the process $(R_t^{(k)}, t > 0)$ of the value of the last k-record observed before t is not even Markovian for $k > 1$: the law of the life-time at value x is that of the (infinite-mean) random sum $x^{-1}(E_1 + \ldots + E_N)$, where all variables are independent, E_j's are unit exponential, and N is the first success time in a series of Bernoulli trials with 'harmonic' success probabilities $1/k, 1/(k + 1), \ldots$ (explicitly, $\mathbb{P}(N = n) = 1/((n + 1)(n + 2))$ for $k = 2$, but no simple formula exists for $k > 2$). Still, $\mathcal{R}^{(k)}$ can be accessed through $\mathcal{C}^{(k)}$:

Proposition 6. *The law of $\mathcal{R}^{(k)}$ is that of a pointwise Bernoulli thinning of $\mathcal{C}^{(k)}$ with probability $1/k$.*

Proof. Like the thinning argument in Remark 5, this is a consequence of Lemma 4. □

This explains, of course, the factor k in (6).

4 An argument for the ABT lemma

By Proposition 2 we can identify $\mathcal{T}^{(k)}$ with $\tau(\mathcal{C}^{(k)})$ and $\mathcal{S}^{(k)}$ with the set of life-times of $(M_t^{(k)}, t > 0)$. Let $\mathcal{J}^{(k)}$ be a planar process having points (s, x) where $x \in \xi(\mathcal{C}^{(k)})$ and s is the life-time of $(M_t^{(k)}, t > 0)$ at x. Because the projection $\xi(\mathcal{C}^{(k)})$ is a Poisson process, we conclude that $\mathcal{J}^{(k)}$ is a marked Poisson process (same as the one denoted $\mathcal{R}$ in [2, p. 47]) with intensity $(k/x)(xe^{-sx}) = ke^{-sx}$; therefore by symmetry of the intensity $\tau(\mathcal{J}^{(k)}) =_d \xi(\mathcal{J}^{(k)}) =_d \xi(\mathcal{C}^{(k)})$, and (1) now follows from Proposition 2.
A novelty of this argument is in exploiting the distributional identity of two projections of $\mathcal{C}^{(k)}$. This allowed us to avoid the computational part of the proof in [2], where one needed to derive the Poisson character of the process $\mathcal{T}^{(k)}$ from its definition by partial summations over $\mathcal{J}^{(k)}$.
Our proof of (1) based on the k-corners of the unit Poisson process in $\mathbb{R}_+^2$ works only for integer k. For general $k > 0$ one can argue by interpolation from the integer values, since all distributions involved depend on the parameter k analytically.
An alternative approach for arbitrary $k > 0$ is to directly define a self-similar Markov process $(M_t^{(k)}, t > 0)$ with transitions as described in Section 3 and marginal distributions (3), see [6] (to fit exactly in the framework of [6], one should consider the reciprocal process $1/M_t^{(k)}$ which 'enters from 0+'). The consistency of such definition amounts to checking, by means of a moments formula (see e.g. [12, Equation 8]), that beta$(k, 1)$ distribution for B indeed corresponds to the gamma marginals (3). From this description of the process, one can derive (5) and from this conclude about the $t \leftrightarrow x$ symmetry of the graph of $t \mapsto M_t^{(k)}$.

Analogous self-similar processes can be constructed for more general distributions of the stick-breaking factor like our B. The case of beta distribution (2) is, in fact, very special in that only for this distribution of B the set of jump-times (and the range) of a self-similar process is Poisson, see [12, Proposition 8].

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