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THE BARNES G FUNCTION AND ITS RELATIONS WITH SUMS AND PRODUCTS OF GENERALIZED GAMMA CONVOLUTION VARIABLES

ASHKAN NIKEGHBALI
Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland
email: ashkan.nikeghbali@math.uzh.ch

MARC YOR
Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie, et CNRS UMR 7599, 175 rue du Chevaleret F-75013 Paris, France.
email: deaproba@jussieu.fr

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Abstract

We give a probabilistic interpretation for the Barnes G-function which appears in random matrix theory and in analytic number theory in the important moments conjecture due to Keating-Snaith for the Riemann zeta function, via the analogy with the characteristic polynomial of random unitary matrices. We show that the Mellin transform of the characteristic polynomial of random unitary matrices and the Barnes G-function are intimately related with products and sums of gamma, beta and log-gamma variables. In particular, we show that the law of the modulus of the characteristic polynomial of random unitary matrices can be expressed with the help of products of gamma or beta variables. This leads us to prove some non standard type of limit theorems for the logarithmic mean of the so called generalized gamma convolutions.

1 Introduction, motivation and main results

The Barnes G-function, which was first introduced by Barnes in [3] (see also [1]), may be defined via its infinite product representation

$$G(1+z) = (2\pi)^{z/2} \exp\left[-\frac{1}{2}\left[(1+\gamma)z^2 + z\right]\right] \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^n \exp\left[-z+\frac{z^2}{2n}\right] \qquad (1.1)$$

where γ is the Euler constant.
From (1.1), one can easily deduce the following (useful) development of the logarithm of $G(1+z)$

for $|z| < 1$:

$$\log G(1+z) = \frac{z}{2}\left(\log(2\pi) - 1\right) - \frac{1}{2}(1+\gamma)z^2 + \sum_{n=3}^{\infty}(-1)^{n-1}\zeta(n-1)\frac{z^n}{n} \tag{1.2}$$

where ζ denotes the Riemann zeta function.
The Barnes G-function has been prominent in the theory of asymptotics of Toeplitz operators since the 1960's and the work of Szegö, Widom, etc. (see e.g. [7] for historical details and more references) and it has recently occurred as well in the work of Keating and Snaith [16] in their celebrated moments conjecture for the Riemann zeta function. More precisely, they consider the set of unitary matrices of size N, endowed with the Haar probability measure, and they prove the following results.

Proposition 1.1 (Keating-Snaith [16]). *If Z denotes the characteristic polynomial of a generic random unitary matrix of size N, considered at any given point of the unit circle, then the following hold:*

1. *For λ any complex number satisfying $\mathfrak{Re}(\lambda) > -1/2$,*

$$\mathbb{E}_N\left[|Z|^{2\lambda}\right] = \prod_{j=1}^{N}\frac{\Gamma(j)\,\Gamma(j+2\lambda)}{\left(\Gamma(j+\lambda)\right)^2}. \tag{1.3}$$

2. *For $\mathfrak{Re}(\lambda) > -1$*

$$\lim_{N\to\infty}\frac{1}{N^{\lambda^2}}\mathbb{E}_N\left[|Z|^{2\lambda}\right] = \frac{(G(1+\lambda))^2}{G(1+2\lambda)}. \tag{1.4}$$

Then, using a random matrix analogy (now called the "Keating-Snaith philosophy"), they make the following conjecture for the moments of the Riemann zeta function (see [16],[18]):

$$\lim_{T\to\infty}\frac{1}{(\log T)^{\lambda^2}}\frac{1}{T}\int_0^T \mathrm{d}t\left|\zeta\left(\frac{1}{2}+\mathrm{i}t\right)\right|^{2\lambda} = M(\lambda)A(\lambda),$$

where $M(\lambda)$ is the "random matrix factor"

$$M(\lambda) = \frac{(G(1+\lambda))^2}{G(1+2\lambda)}$$

and $A(\lambda)$ is the arithmetic factor

$$A(\lambda) = \prod_{p\in\mathscr{P}}\left[\left(1-\frac{1}{p}\right)^{\lambda^2}\left(\sum_{m=0}^{\infty}\left(\frac{\Gamma(\lambda+m)}{m!\Gamma(\lambda)}\right)^2 p^{-m}\right)\right] \tag{1.5}$$

where, as usual, $\mathscr{P}$ is the set of prime numbers.
Due to the importance of this conjecture, as discussed in several papers in [18], it seems interesting to obtain probabilistic interpretations of the non arithmetic part of the conjecture. More precisely, the aim of this paper is twofold:

- to give a probabilistic interpretation of the "random matrix factor" $M(\lambda)$, and more generally of the Barnes G-function;

- to understand better the nature of the limit theorem (1.4) and its relations with (generalized) gamma variables.

To this end, we first give a probabilistic translation in Theorem 1.2 of the infinite product (1.1) in terms of a limiting distribution involving gamma variables (we note that, although concerning the Gamma function, similar translations have been presented in [12] and [11]). Let us recall that a gamma variable γ_a with parameter $a > 0$ is distributed as

$$\mathbb{P}\left(\gamma_a \in \mathrm{d}t\right) = \frac{t^{a-1}\exp(-t)\,\mathrm{d}t}{\Gamma(a)}, \quad t > 0 \tag{1.6}$$

and has Laplace transform

$$\mathbb{E}\left[\exp\left(-\lambda\gamma_a\right)\right] = \frac{1}{(1+\lambda)^a}, \quad \mathfrak{Re}(\lambda) > -1 \tag{1.7}$$

and Mellin transform

$$\mathbb{E}\left[\left(\gamma_a\right)^s\right] = \frac{\Gamma(a+s)}{\Gamma(a)}, \quad \mathfrak{Re}(s) > -a. \tag{1.8}$$

Theorem 1.2. *If $\left(\gamma_n\right)_{n\geq 1}$ are independent gamma random variables with respective parameters n, then for z such that $\mathfrak{Re}(z) > -1$,*

$$\lim_{N\to\infty} \frac{1}{N^{\frac{z^2}{2}}}\mathbb{E}\left[\exp\left(-z\left(\sum_{n=1}^{N}\left(\frac{\gamma_n}{n}\right) - N\right)\right)\right] = \left(A^z \exp\left(\frac{z^2}{2}\right) G(1+z)\right)^{-1} \tag{1.9}$$

where

$$A = \sqrt{\frac{\mathrm{e}}{2\pi}}. \tag{1.10}$$

The next theorem gives an identity in law for the characteristic polynomial which shall lead to a probabilistic interpretation of the "random matrix factor":

Theorem 1.3. *Let Λ denote the generic matrix of $U(N)$, the set of unitary matrices, fitted with the Haar probability measure, and $Z_N(\Lambda) = \det(I - \Lambda)$. Then the Mellin transform formula (1.3), due to Keating-Snaith, translates in probabilistic terms as*

$$\prod_{j=1}^{N} \gamma_j \overset{law}{=} |Z_N(\Lambda)| \prod_{j=1}^{N} \sqrt{\gamma_j \gamma_j'} \tag{1.11}$$

where all variables in sight are assumed to be independent, and γ_j, γ_j' are gamma random variables with parameter j.

The Barnes G-function now comes into the picture via the following limit results:

Theorem 1.4. *Let $\left(\gamma_n\right)_{n\geq 1}$ be independent gamma random variables with respective parameters n; then the following hold:*

1. *for any λ, with $\mathfrak{Re}(\lambda) > -1$, we have:*

$$\lim_{N\to\infty} \frac{1}{N^{\lambda^2/2}}\mathbb{E}\left[\left(\prod_{j=1}^{N}\gamma_j\right)^{\lambda}\right] \exp\left(-\lambda\sum_{j=1}^{N}\psi(j)\right) = \left(A^{\lambda} G(1+\lambda)\right)^{-1}, \tag{1.12}$$

where $A = \sqrt{\frac{\mathrm{e}}{2\pi}}$ and $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$.

2. *consequently, from (1.12), together with (1.11), we recover the limit theorem (1.4) of Keating and Snaith.*

Remark 1.5. *Let us take* $z = \mathrm{i}u$ *in Theorem 1.2 and* $\lambda = \mathrm{i}u$ *in Theorem 1.4, where* $u \in \mathbb{R}$. *Then both convergences can be interpreted as follows: the characteristic functions of* $\sum_{n=1}^{N}(\frac{\gamma_n}{n} - 1)$ *and* $\sum_{j=1}^{N}(\log \gamma_j - \psi(j))$, *when renormalized by the characteristic function of a Gaussian distribution with mean* 0 *and variance* $\log N$, *converge to some limit functions. This fact has been the starting point of the investigations in [13] about a new mode of convergence in probability theory and number theory, called* mod-Gaussian *convergence.*

Theorem 1.2 can be naturally extended to the more general case of sums of the form

$$S_N = \sum_{n=1}^{N} \left(Y_n - \mathbb{E}\left[Y_n\right]\right) \tag{1.13}$$

where

$$Y_n = \frac{1}{n}\left(y_1^{(n)} + \ldots + y_n^{(n)}\right),$$

and where $\left(y_i^{(n)}\right)_{1\leq i\leq n<\infty}$ are independent, with the same distribution as a given random variable Y, where Y is a generalized gamma convolution variable (in short GGC), that is an infinitely divisible $\mathbb{R}_+$-valued random variable whose Lévy measure is of the form

$$\nu(\mathrm{d}x) = \left(\frac{\mathrm{d}x}{x}\right)\left(\int \mu(\mathrm{d}\xi)\exp(-x\xi)\right), \tag{1.14}$$

where $\mu(\mathrm{d}\xi)$ is a Radon measure on $\mathbb{R}_+$, called the Thorin measure associated to Y. We shall further assume that

$$\int \mu(\mathrm{d}\xi)\frac{1}{\xi^2} < \infty,$$

which, as we shall see is equivalent to the existence of a second moment for Y.
The GGC variables have been studied by Thorin [21] and Bondesson [6], see, e.g., [14] for a recent survey of this topic.

Theorem 1.6. *Let* Y *be a GGC variable, and let* (S_N) *as in (1.13). We note*

$$\sigma^2 = \mathbb{E}\left[Y^2\right] - (\mathbb{E}[Y])^2.$$

Then the following limit theorem for (S_N) *holds: if* $\lambda > 0$,

$$\frac{1}{N^{\frac{\lambda^2\sigma^2}{2}}}\mathbb{E}\left[\exp(-\lambda S_N)\right] \to \mathscr{H}(\lambda),\ \ N \to \infty \tag{1.15}$$

where the function $\mathscr{H}(\lambda)$ *is given by:*

$$\mathscr{H}(\lambda) = \exp\left\{\frac{\lambda^2\sigma^2}{2}(1+\gamma) + \int_0^\infty \frac{\mathrm{d}y}{y}\Sigma_\mu(y)\left(\exp(-\lambda y) - 1 + \lambda y - \frac{\lambda^2 y^2}{2}\right)\right\}, \tag{1.16}$$

with

$$\Sigma_\mu(y) = \int \mu(\mathrm{d}\xi)\frac{1}{\left(2\sinh\left(\frac{\xi y}{2}\right)\right)^2}. \tag{1.17}$$

Limit results such as (1.12) and (1.15) are not standard in Probability theory: they have been the starting point for the authors of [13] to build a little theory for such limit theorems which also appear in number theory (in particular a probabilistic interpretation of the arithmetic factor (1.5) is given in [13]). The rest of the paper is organized as follows:

- in Section 2, we prove Theorems 1.2, 1.3 and 1.4. We also give an interpretation of Theorem 1.2 in terms of Bessel processes as well as an important Lévy-Khintchine type representation for $1/G(1+z)$.

- in Section 3, we shall introduce generalized gamma convolution variables and prove Theorem 1.6 as a natural extension of Theorem 1.2.

2 Proofs of Theorems 1.2, 1.3 and 1.4 and additional probabilistic aspects of the Barnes G-function

2.1 Proof of Theorem 1.2 and interpretation in terms of Bessel processes

2.1.1 Proof of Theorem 1.2

The proof of Theorem 1.2 is rather straightforward. We simply use the fact that for $\mathfrak{Re}(z) > -1$

$$\mathbb{E}\left[\exp\left(-z\frac{\gamma_n}{n}\right)\right] = \frac{1}{\left(1+\frac{z}{n}\right)^n}.$$

Hence, the quantity

$$\mathbb{E}\left[\exp\left(-z\left(\sum_{n=1}^{N}\left(\frac{\gamma_n}{n}\right) - N\right)\right)\right]$$

equals

$$\frac{1}{\prod_{n=1}^{N}\left(\left(1+\frac{z}{n}\right)^n \exp(-z)\right)} \equiv \frac{1}{D_N}.$$

We then write

$$D_N \exp\left(\frac{z^2}{2}\log N\right) = D_N \exp\left(\frac{z^2}{2}\sum_{n=1}^{N}\frac{1}{n}\right)\exp\left(\frac{z^2}{2}\left(\log N - \sum_{n=1}^{N}\frac{1}{n}\right)\right)$$

which from (1.1) converges, as $N \to \infty$, towards

$$G(1+z)(2\pi)^{-z/2}\exp\left(\frac{z+z^2}{2}\right),$$

which proves formula (1.9).

2.1.2 An interpretation of Theorem 1.2 in terms of Bessel processes

Let $\left(R_{2n}(t)\right)_{t\geq 0}$ denote a BES($2n$) process, starting from 0, with dimension $2n$; we need to consider the sequence $\left(R_{2n}\right)_{n=1,2,\ldots}$ of such independent processes. It is well known (see [19] for example) that:

$$R_{2n}^2(t) \overset{\text{law}}{=} (2t)\gamma_n.$$

Thus for fixed $t > 0$,

$$\sum_{n=1}^{N} \frac{R_{2n}^2(t)}{2n} \overset{\text{law}}{=} t\left(\sum_{n=1}^{N} \frac{\gamma_n}{n}\right).$$

Moreover, the process $(R_{2n}^2(t),\ t \geq 0)$ satisfies the following stochastic differential equation, driven by a Brownian Motion $\beta^{(n)}$:

$$\text{for fixed } t > 0,\ R_{2n}^2(t) = 2\int_0^t \sqrt{R_{2n}^2(s)}\mathrm{d}\beta_s^{(n)} + 2nt. \tag{2.1}$$

We now write Theorem 1.2, for $\mathfrak{Re}(z) > -1/t$ as

$$\lim_{N\to\infty} \frac{1}{N^{\frac{t^2z^2}{2}}}\mathbb{E}\left[\exp\left(-z\sum_{n=1}^{N}\frac{R_{2n}^2(t)-2nt}{2n}\right)\right] = \left(A^{tz}\exp\left(\frac{t^2z^2}{2}\right)G(1+tz)\right)^{-1} \tag{2.2}$$

where

$$A = \sqrt{\frac{\mathrm{e}}{2\pi}}.$$

We now wish to write the LHS of (2.2) in terms of functional of a sum of squared Ornstein-Uhlenbeck processes; indeed, if we write, from (2.1):

$$\sum_{n=1}^{N}\frac{R_{2n}^2(t)-2nt}{2n} = 2\sum_{n=1}^{N}\int_0^t \frac{1}{2n}\sqrt{R_{2n}^2(s)}\mathrm{d}\beta_s^{(n)},$$

the RHS appears as a martingale in t, with increasing process

$$\sum_{n=1}^{N}\int_0^t \frac{1}{n^2}R_{2n}^2(s)\,\mathrm{d}s,$$

and we obtain that (2.2) may be written as

$$\lim_{N\to\infty}\frac{1}{N^{\frac{t^2z^2}{2}}}\mathbb{E}^{(z)}\left[\exp\left(\frac{z^2}{2}\sum_{n=1}^{N}\int_0^t \frac{1}{n^2}R_{2n}^2(s)\,\mathrm{d}s\right)\right] = \left(A^{tz}\exp\left(\frac{t^2z^2}{2}\right)G(1+tz)\right)^{-1}$$

where, assuming now z to be a real number, under $\mathbb{P}^{(z)}$ the process $\left(R_{2n}^2(t)\right)_{t\geq 0}$ satisfies, by Girsanov's theorem:

$$\begin{aligned} R_{2n}^2(t) &= 2\int_0^t \sqrt{R_{2n}^2(s)}\left(\mathrm{d}\widetilde{\beta}_s^{(n)} - \frac{z}{n}\sqrt{R_{2n}^2(s)}\mathrm{d}s\right) + 2nt \\ &= 2\int_0^t \sqrt{R_{2n}^2(s)}\mathrm{d}\widetilde{\beta}_s^{(n)} - \frac{2z}{n}\int_0^t R_{2n}^2(s)\,\mathrm{d}s + 2nt. \end{aligned}$$

That is, under $\mathbb{P}^{(z)}$, $\left(R_{2n}^2(t)\right)_{t\geq 0}$ now appears as the square of the norm of a $2n$-dimensional Ornstein-Uhlenbeck process, with parameter $\left(-\frac{z}{n}\right)$.

2.2 Proof of Theorem 1.3

Formula (1.11) follows from the Keating-Snaith formula (1.3) once one recalls formula (1.8) for the Mellin transform of a gamma variable.

2.3 The characteristic polynomial and beta variables

One can use the beta-gamma algebra (see, e.g., Chaumont-Yor [10], p.93-98, and the references therein) and (1.11) to represent (in law) the characteristic polynomial as products of beta variables. More precisely,

Theorem 2.1. *With the notations of Theorem 1.3, we have*

$$|Z_N| \overset{law}{=} 2^N \left(\prod_{j=1}^{N} \sqrt{\beta_{\frac{j}{2},\frac{j}{2}}} \right) \left(\prod_{j=2}^{N} \sqrt{\beta_{\frac{j+1}{2},\frac{j-1}{2}}} \right), \tag{2.3}$$

where all the variables in sight are assumed to be independent beta variables.

Remark 2.2. *The above splitting result was at the origin of the geometric interpretation of the law of Z_N given in [8].*

Proof. a) To deduce from (1.11) the result stated in the theorem, we use the following factorization

$$\gamma_j \overset{\text{law}}{=} \sqrt{\gamma_j \gamma'_j}\xi_j, \tag{2.4}$$

where on the RHS, we have

$$\begin{aligned} &\text{for} \quad j=1, \qquad \xi_1 \overset{\text{law}}{=} 2\sqrt{\beta_{\frac{1}{2},\frac{1}{2}}} \\ &\text{for} \quad j>1, \qquad \xi_j \overset{\text{law}}{=} 2\sqrt{\left(\beta_{\frac{j}{2},\frac{j}{2}}\right)\left(\beta_{\frac{j+1}{2},\frac{j-1}{2}}\right)}. \end{aligned}$$

Thus after simplification on both sides of (1.12) by $\prod_{j=1}^{N} \sqrt{\gamma_j \gamma'_j}$, we obtain

$$|Z_N| \overset{\text{law}}{=} \prod_{j=1}^{N} \xi_j,$$

which yields (2.3).
b) To be complete, it now remains to prove (2.4). First, the duplication formula for the gamma function yields (see e.g. [10] p. p93-98):

$$\gamma_j \overset{\text{law}}{=} 2\sqrt{\gamma_{\frac{j}{2}}\gamma'_{\frac{j+1}{2}}}.$$

Second, we use the beta-gamma algebra to write

$$\gamma_{\frac{j}{2}} \overset{\text{law}}{=} \beta_{\frac{j}{2},\frac{j}{2}}\gamma_j,$$

and for $j > 1$,

$$\gamma'_{\frac{j+1}{2}} \overset{\text{law}}{=} \beta_{\frac{j+1}{2},\frac{j-1}{2}}\gamma'_j.$$

Thus (2.4) follows easily. $\square$

Remark 2.3. *In the above Theorem, the factor 2^N can be explained by the fact that the characteristic polynomial of a unitary matrix is, in modulus, smaller than 2^N. Hence the products of beta variables appearing on the RHS of the formula (2.3) measure how much the modulus of the characteristic polynomial deviates from the largest value it can take.*

2.4 A Lévy-Khintchine type representation of $1/G(1+z)$

In this subsection, we give a Lévy-Khintchine representation type formula for $1/G(1+z)$ which was already discovered by Barnes ([3], p. 309) and which will be used next to prove Theorem 1.4.

Proposition 2.4 ([3], p. 309). *For any $z \in \mathbb{C}$, such that $\mathfrak{Re}(z) > -1$, one has*

$$\frac{1}{G(1+z)} = \exp\left\{ -\frac{1}{2}\left(\log(2\pi) - 1\right) z + \left(1+\gamma\right)\frac{z^2}{2} + \int_0^\infty \frac{\mathrm{d}u \left(\exp(-zu) - 1 + zu - \frac{u^2 z^2}{2}\right)}{u\left(2\sinh(u/2)\right)^2} \right\}. \tag{2.5}$$

Remark 2.5. *Note that formula (2.5) cannot be considered exactly as a Lévy-Khintchine representation. Indeed, the integral featured in (2.5) consists in integrating the function*

$\left(\exp(-zu) - 1 + zu - \frac{u^2z^2}{2}\right)$ *against the measure* $\dfrac{\mathrm{d}u}{u\left(2\sinh(u/2)\right)^2}$, *which is not a Lévy measure, since Lévy measures integrate $(u^2 \wedge 1)$, which is not the case here because of the equivalence: $u\left(2\sinh(u/2)\right)^2 \sim u^3$, when $u \to 0$. Due to this singularity, one cannot integrate $\left(\exp(-zu) - 1 + zu\mathbf{1}_{(u\leq 1)}\right)$ with respect to this measure, and one is forced to "bring in" the companion term $\frac{u^2z^2}{2}$ under the integral sign.*

2.5 Proof of Theorem 1.4

To prove Theorem 1.4, we shall use the following well known lemma (see e.g. Lebedev [17] and Carmona-Petit-Yor [9] where this lemma is also used):

Lemma 2.6. *For any $a > 0$, the random variable $\log\left(\gamma_a\right)$ is infinitely divisible and its Lévy-Khintchine representation is given, for $\mathfrak{Re}(\lambda) > -a$, by*

$$\begin{aligned}\mathbb{E}\left[\gamma_a^\lambda\right] &= \frac{\Gamma(a+\lambda)}{\Gamma(a)} \\ &= \exp\left\{\lambda\psi(a) + \int_0^\infty \frac{\exp(-au)\left(\exp(-\lambda u) - 1 + \lambda u\right)}{u\left(1-\exp(-u)\right)}\mathrm{d}u\right\},\end{aligned}$$

where

$$\psi(a) \equiv \frac{\Gamma'(a)}{\Gamma(a)}.$$

We are now in a position to prove Theorem 1.4. We start by proving the first part, i.e. formula (1.12). Let us write

$$I_N(\lambda) := \frac{1}{N^{\lambda^2/2}}\mathbb{E}\left[\left(\prod_{j=1}^N \gamma_j\right)^\lambda\right].$$

Thus with the help of Lemma 2.6, we obtain

$$N^{\lambda^2/2} I_N(\lambda) = \exp\left\{ \lambda \sum_{j=1}^{N} \psi(j) + \int_0^\infty \frac{\sum_{j=1}^{N} \exp(-ju)}{u(1-\exp(-u))} \left(\exp(-\lambda u) - 1 + \lambda u\right) \mathrm{d}u \right\}. \tag{2.6}$$

Note that

$$\frac{1}{u(1-\exp(-u))} \sum_{j=1}^{N} \exp(-ju) = \frac{\exp(-u)\left(1-\exp(-Nu)\right)}{u\left(1-\exp(-u)\right)^2}$$

and we may now write

$$I_N(\lambda) = \exp\left\{ \lambda \sum_{j=1}^{N} \psi(j) + J_N(\lambda) \right\}, \tag{2.7}$$

where

$$J_N(\lambda) = \left\{ \int_0^\infty \frac{\mathrm{d}u \exp(-u)}{u\left(1-\exp(-u)\right)^2} \left(1-\exp(-Nu)\right)\left(\exp(-\lambda u) - 1 + \lambda u\right) \right\} - \frac{\lambda^2}{2} \log N. \tag{2.8}$$

Next, we shall show that:

$$J_N(\lambda) \xrightarrow[N\to\infty]{} J_\infty(\lambda)$$

together with some integral expression for $J_\infty(\lambda)$, from which it will be easily deduced how $J_\infty(\lambda)$ and $G(1+\lambda)$ are related thanks to Proposition 2.4.
We now write (2.8) in the form

$$\begin{aligned} J_N(\lambda) = \int_0^\infty & \frac{\mathrm{d}u \exp(-u)}{u\left(1-\exp(-u)\right)^2} \left(1-\exp(-Nu)\right) \left(\exp(-\lambda u) - 1 + \lambda u - \frac{\lambda^2 u^2}{2} \right) \\ & + \frac{\lambda^2}{2} \left(\left\{ \int_0^\infty \frac{\mathrm{d}u\, u \exp(-u)}{\left(1-\exp(-u)\right)^2} \left(1-\exp(-Nu)\right) \right\} - \log N \right). \end{aligned} \tag{2.9}$$

Now letting $N \to \infty$, we obtain

$$J_N(\lambda) \to J_\infty(\lambda), \tag{2.10}$$

with

$$\begin{aligned} J_\infty(\lambda) &= \int_0^\infty \frac{\mathrm{d}u \exp(-u)}{u\left(1-\exp(-u)\right)^2} \left(\exp(-\lambda u) - 1 + \lambda u - \frac{\lambda^2 u^2}{2} \right) + \frac{\lambda^2}{2} C \\ &\equiv \widetilde{J}_\infty(\lambda) + \frac{\lambda^2}{2} C, \end{aligned}$$

where

$$C = \lim_{N\to\infty} \left(\left\{ \int_0^\infty \frac{\mathrm{d}u\, u \exp(-u)}{\left(1-\exp(-u)\right)^2} \left(1-\exp(-Nu)\right) \right\} - \log N \right),$$

a limit which we shall show to exist and identify to be $1+\gamma (= C)$ with the following lemma:

Lemma 2.7. *We have*

$$\int_0^\infty \frac{u\exp(-u)}{\left(1-\exp(-u)\right)^2}\left(1-\exp(-Nu)\right)du = \log N + \left(1+\gamma\right) + o(1).$$

Consequently, we have, by letting $N \to \infty$,

$$C = 1+\gamma.$$

Proof.

$$\begin{aligned}\int_0^\infty \frac{u\exp(-u)}{\left(1-\exp(-u)\right)^2}\left(1-\exp(-Nu)\right)du &= \int_0^\infty \frac{u\exp(-u)}{\left(1-\exp(-u)\right)}\left(\sum_{k=0}^{N-1}\exp(-ku)\right)du \\ &= \sum_{k=1}^{N}\int_0^\infty \frac{u}{\left(1-\exp(-u)\right)}\exp(-ku)\,du = \sum_{k=1}^{N}\int_0^\infty du\ \exp(-ku)\,u\sum_{r=0}^{\infty}\exp(-ru) \\ &= \sum_{k=1}^{N}\sum_{r=0}^{\infty}\int_0^\infty du\,u\exp(-(r+k)u) = \sum_{k=1}^{N}\sum_{r=0}^{\infty}\frac{1}{(r+k)^2}.\end{aligned}$$

Now, we write:

$$\begin{aligned}\sum_{k=1}^{N}\sum_{r=0}^{\infty}\frac{1}{(r+k)^2} &= \zeta(2)+\sum_{k=1}^{N-1}\left(\zeta(2)-\sum_{s=1}^{k}\frac{1}{s^2}\right) \\ &= N\zeta(2)-\sum_{k=1}^{N-1}\sum_{s=1}^{k}\frac{1}{s^2} \\ &= N\zeta(2)-\sum_{s=1}^{N}\frac{N-s}{s^2} \\ &= N\sum_{s=N+1}^{\infty}\frac{1}{s^2}+\frac{1}{N}+\sum_{s=1}^{N}\frac{1}{s}-\frac{1}{N}.\end{aligned}$$

The result now follows easily from the facts

$$\begin{aligned}\lim_{N\to\infty} N\sum_{s=N+1}^{\infty}\frac{1}{s^2} &= 1 \\ \sum_{s=1}^{N}\frac{1}{s} &= \log N+\gamma+o(1).\end{aligned}$$

□

We have thus proved so far that:

$$\lim_{N\to\infty}\frac{1}{N^{\lambda^2/2}}\mathbb{E}\left[\left(\prod_{j=1}^{N}\gamma_j\right)^{\lambda}\right]\exp\left(-\lambda\sum_{j=1}^{N}\psi(j)\right) = \exp\left(J_\infty(\lambda)\right), \qquad (2.11)$$

where

$$J_\infty(\lambda) = \int_0^\infty \frac{\mathrm{d}u \exp(-u)}{u\left(1-\exp(-u)\right)^2}\left(\exp(-\lambda u) - 1 + \lambda u - \frac{\lambda^2 u^2}{2}\right) + \frac{\lambda^2}{2}(1+\gamma).$$

We can still rewrite $J_\infty(\lambda)$ as

$$J_\infty(\lambda) = \int_0^\infty \frac{\mathrm{d}u}{u\,(2\sinh(u/2))^2}\left(\exp(-\lambda u) - 1 + \lambda u - \frac{\lambda^2 u^2}{2}\right) + \frac{\lambda^2}{2}(1+\gamma). \tag{2.12}$$

Now comparing (2.5) and (2.12), we obtain

$$\exp\left(J_\infty(\lambda)\right) = \left(A^\lambda G(1+\lambda)\right)^{-1} \tag{2.13}$$

Plugging (2.13) in (2.11) yields the first part of Theorem 1.4:

$$\lim_{N\to\infty} \frac{1}{N^{\lambda^2/2}} \mathbb{E}\left[\left(\prod_{j=1}^N \gamma_j\right)^\lambda\right] \exp\left(-\lambda \sum_{j=1}^N \psi(j)\right) = \left(\sqrt{\frac{2\pi}{\mathrm{e}}}\right)^\lambda \frac{1}{G(1+\lambda)}$$

To prove the second part of Theorem 1.4, we use formula (1.11) together with formula (1.12). Formula (1.11) yields:

$$\frac{1}{N^{2\lambda^2}} \mathbb{E}\left[\left(\prod_{j=1}^N \gamma_j\right)^{2\lambda}\right] = \left(\frac{1}{N^{\lambda^2}} \mathbb{E}_N\left[|Z_N|^{2\lambda}\right]\right)\left(\left(\frac{1}{N^{\lambda^2/2}} \mathbb{E}\left[\left(\prod_{j=1}^N \gamma_j\right)^\lambda\right]\right)^2\right). \tag{2.14}$$

Multiplying both sides by $\exp\left(-2\lambda \sum_{j=1}^N \psi(j)\right)$ and using (1.12) we obtain

$$\lim_{N\to\infty} \frac{1}{N^{\lambda^2}} \mathbb{E}_N\left[|Z|^{2\lambda}\right] = \frac{(G(1+\lambda))^2}{G(1+2\lambda)},$$

which completes the proof of Theorem 1.4.

3 Generalized Gamma Convolutions

3.1 Definition and examples

We recall the definition of a GGC variable; see e.g. [6].

Definition 3.1. *A random variable Y taking values in $\mathbb{R}_+$ is called a GGC variable if it is infinitely divisible with Lévy measure ν of the form:*

$$\nu(\mathrm{d}x) = \frac{\mathrm{d}x}{x}\int \mu(\mathrm{d}\xi)\exp(-\xi x), \tag{3.1}$$

where $\mu(\mathrm{d}\xi)$ is a Radon measure on $\mathbb{R}_+$, called the Thorin measure of Y. That is the Laplace transform of Y is given by

$$\mathbb{E}[\exp(-\lambda Y)] = \exp(-\int \nu(\mathrm{d}x)(1-e^{-\lambda x})),$$

where $\nu(\mathrm{d}x)$ is given by (3.1).

Remark 3.2. *Y is a selfdecomposable random variable because its Lévy measure can be written as $\nu(\mathrm{d}x) = \dfrac{\mathrm{d}x}{x} h(x)$ with h a decreasing function (see, e.g. [20], p.95).*

Remark 3.3. *We shall require Y to have finite first and second moments; these moments can be easily computed with the help of the Thorin measure $\mu(\mathrm{d}\xi)$:*

$$\begin{aligned}\mathbb{E}[Y] &= \mu_{-1} = \int \mu(\mathrm{d}\xi)\frac{1}{\xi} \\ \sigma^2 = \mathbb{E}\left[Y^2\right] - (\mathbb{E}[Y])^2 &= \mu_{-2} = \int \mu(\mathrm{d}\xi)\frac{1}{\xi^2}.\end{aligned}$$

Now we give some examples of GGC variables. Of course, γ_a falls into this category with $\mu(\mathrm{d}\xi) = a\delta_1(\mathrm{d}\xi)$ where $\delta_1(\mathrm{d}\xi)$ is the Dirac measure at 1.
More generally, the next proposition gives a large set of such variables:

Proposition 3.4. *Let f be a nonnegative Borel function such that*

$$\int_0^\infty \mathrm{d}u \log\left(1 + f(u)\right) < \infty,$$

and let (γ_u) denote the standard gamma process. Then the variable Y defined as

$$Y = \int_0^\infty \mathrm{d}\gamma_u f(u) \tag{3.2}$$

is a GGC variable.

Proof. It is easily shown, by approximating f by simple functions that

$$\begin{aligned}\mathbb{E}\left[\exp(-\lambda Y)\right] &= \exp\left(-\int_0^\infty \mathrm{d}u \int_0^\infty \frac{\mathrm{d}x}{x}\exp(-x)\left(1 - \exp\left(-\lambda f(u)x\right)\right)\right) \\ &= \exp\left(-\int_0^\infty \frac{\mathrm{d}y}{y}\left(\int_0^\infty \mathrm{d}u \exp\left(-\frac{y}{f(u)}\right)\right)\left(1 - \exp(-\lambda y)\right)\right)\end{aligned}$$

which yields the result. □

For much more on GGC variables, see [6], [14].

3.2 Proof of Theorem 1.6

We now prove Theorem 1.6, which is a natural extension for Theorem 1.2. Recall that in (1.13), we have defined S_N as

$$S_N = \sum_{n=1}^N \left(Y_n - \mathbb{E}\left[Y_n\right]\right)$$

where

$$Y_n = \frac{1}{n}\left(y_1^{(n)} + \ldots + y_n^{(n)}\right),$$

and where $\left(y_i^{(n)}\right)_{1\leq i\leq n<\infty}$ are independent, with the same distribution as a given GGC variable Y, which has a second moment. For any $\lambda \geq 0$, we have

$$\mathbb{E}\left[\exp\left(-\lambda S_N\right)\right] = \prod_{n=1}^{N}\left(\widetilde{\varphi}\left(\frac{\lambda}{n}\right)\right)^n \tag{3.3}$$

where

$$\widetilde{\varphi}\left(\lambda\right) = \mathbb{E}\left[\exp\left(-\lambda\left(Y - \mathbb{E}\left[Y\right]\right)\right)\right]. \tag{3.4}$$

Now, using the form (3.1) of the Lévy-Khintchine representation for Y, we obtain

$$\begin{aligned}
\prod_{n=1}^{N}\left(\widetilde{\varphi}\left(\frac{\lambda}{n}\right)\right)^n &= \exp\left\{-\sum_{n=1}^{N} n \int \nu\left(\mathrm{d}x\right)\left(1 - \frac{\lambda}{n}x - \exp\left(-\frac{\lambda}{n}x\right)\right)\right\} \\
&= \exp\left(-\int \mu\left(\mathrm{d}\xi\right) I_N\left(\xi,\lambda\right)\right).
\end{aligned} \tag{3.5}$$

where:

$$\begin{aligned}
I_N\left(\xi,\lambda\right) &= \sum_{n=1}^{N} n \int_0^\infty \frac{\mathrm{d}x}{x}\exp\left(-\xi x\right)\left(1 - \frac{\lambda}{n}x - \exp\left(-\frac{\lambda}{n}x\right)\right) \\
&= \sum_{n=1}^{N} n \int_0^\infty \frac{\mathrm{d}y}{y}\exp\left(-n\xi y\right)\left(1 - \lambda y - \exp\left(-\lambda y\right)\right) \\
&= \int_0^\infty \frac{\mathrm{d}y}{y}\left(\sum_{n=1}^{N} n\exp\left(-n\xi y\right)\right)\left(1 - \lambda y - \exp\left(-\lambda y\right)\right).
\end{aligned}$$

Some elementary calculations yield:

$$\sum_{n=1}^{N} n\exp\left(-na\right) = \frac{\exp\left(a\right)\left(1 - \exp\left(-aN\right)\right)}{\left(\exp\left(a\right) - 1\right)^2} - \frac{N\exp\left(-aN\right)}{\left(\exp\left(a\right) - 1\right)}.$$

Consequently, taking $a = \xi y$ in the formula for $I_N\left(\xi,\lambda\right)$, we can write it as

$$I_N\left(\xi,\lambda\right) = J_N\left(\xi,\lambda\right) - R_N\left(\xi,\lambda\right) \tag{3.6}$$

where

$$J_N\left(\xi,\lambda\right) = \int_0^\infty \frac{\mathrm{d}y}{y}\left(1 - \lambda y - \exp\left(-\lambda y\right)\right)\left[\frac{\exp\left(\xi y\right)\left(1 - \exp\left(-\xi yN\right)\right)}{\left(\exp\left(\xi y\right) - 1\right)^2}\right] \tag{3.7}$$

and

$$R_N\left(\xi,\lambda\right) = \int_0^\infty \frac{\mathrm{d}y}{y}\left(1 - \lambda y - \exp\left(-\lambda y\right)\right)\left[\frac{N\exp\left(-\xi yN\right)}{\left(\exp\left(\xi y\right) - 1\right)}\right]. \tag{3.8}$$

It is clear that

$$\lim_{N\to\infty} R_N\left(\xi,\lambda\right) = 0.$$

Now we study $J_N(\xi,\lambda)$ when $N\to\infty$. Let us "bring in" the additional term $\frac{\lambda^2 y^2}{2}$; more precisely, we rewrite $J_N(\xi,\lambda)$ as

$$\begin{aligned}J_N(\xi,\lambda) = \int_0^\infty \frac{\mathrm{d}y}{y}\left(1-\lambda y+\frac{\lambda^2y^2}{2}-\exp(-\lambda y)\right)\left[\frac{\exp(\xi y)\left(1-\exp(-\xi yN)\right)}{\left(\exp(\xi y)-1\right)^2}\right] \\ -\frac{\lambda^2}{2}\int \mathrm{d}y\; y\frac{\exp(\xi y)\left(1-\exp(-\xi yN)\right)}{\left(\exp(\xi y)-1\right)^2}.\end{aligned} \tag{3.9}$$

Hence:

$$J_N(\xi,\lambda) = o(1)+\int_0^\infty \frac{\mathrm{d}y}{y}\frac{1}{\left(2\sinh(\xi y/2)\right)^2}\left(1-\lambda y+\frac{\lambda^2y^2}{2}-\exp(-\lambda y)\right)-\frac{\lambda^2}{2}K_N(\xi), \tag{3.10}$$

where

$$\begin{aligned}K_N(\xi) &= \int \mathrm{d}y\; y\frac{\exp(\xi y)\left(1-\exp(-\xi yN)\right)}{\left(\exp(\xi y)-1\right)^2} \\ &= \frac{1}{\xi^2}\int_0^\infty \frac{u\exp(-u)}{\left(1-\exp(-u)\right)^2}\left(1-\exp(-Nu)\right)du.\end{aligned}$$

Moreover, from Lemma 2.7, we have

$$K_N(\xi) = \frac{1}{\xi^2}\left(\log N+1+\gamma+o(1)\right). \tag{3.11}$$

Now using the above asymptotics, and integrating with respect to $\mu(\mathrm{d}\xi)$ (see (3.5)), we obtain

$$\begin{aligned}&\prod_{n=1}^N \left(\tilde{\varphi}\left(\frac{\lambda}{n}\right)\right)^n = \\ &\quad \exp\left\{o(1)+\frac{\lambda^2}{2}\mu_{-2}\left(\log N+1+\gamma\right)-\int_0^\infty \frac{\mathrm{d}y}{y}\Sigma_\mu(y)\left(1-\lambda y+\frac{\lambda^2y^2}{2}-\exp(-\lambda y)\right)\right\}.\end{aligned} \tag{3.12}$$

Hence we have finally proved that if $\lambda>0$,

$$\frac{1}{N^{\frac{\lambda^2\sigma^2}{2}}}\mathbb{E}\left[\exp\left(-\lambda S_N\right)\right] \xrightarrow[N\to\infty]{} \mathscr{H}(\lambda), \tag{3.13}$$

where the function $\mathscr{H}(\lambda)$ is given by

$$\mathscr{H}(\lambda) = \exp\left\{\frac{\lambda^2\sigma^2}{2}(1+\gamma)+\int_0^\infty \frac{\mathrm{d}y}{y}\Sigma_\mu(y)\left(\exp(-\lambda y)-1+\lambda y-\frac{\lambda^2y^2}{2}\right)\right\}, \tag{3.14}$$

with

$$\Sigma_\mu(y) = \int \mu(\mathrm{d}\xi)\frac{1}{\left(2\sinh\left(\frac{\xi y}{2}\right)\right)^2}, \tag{3.15}$$

which is Theorem 1.6.

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