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MARTINGALE REPRESENTATION AND A SIMPLE PROOF OF LOGARITHMIC SOBOLEV INEQUALITIES ON PATH SPACES

MIREILLE CAPITAINE
Département de Mathématiques, Laboratoire de Statistique et Probabilités associé au C.N.R.S., Université Paul-Sabatier, 31062, Toulouse (France)
ELTON P. HSU[1]
Department of Mathematics, Northwestern University, Evanston, Illinois 60208 (U.S.A.)
MICHEL LEDOUX
Département de Mathématiques, Laboratoire de Statistique et Probabilités associé au C.N.R.S., Université Paul-Sabatier, 31062, Toulouse (France)

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Abstract
We show how the Clark-Ocone-Haussmann formula for Brownian motion on a compact Riemannian manifold put forward by S. Fang in his proof of the spectral gap inequality for the Ornstein-Uhlenbeck operator on the path space can yield in a very simple way the logarithmic Sobolev inequality on the same space. By an appropriate integration by parts formula the proof also yields in the same way a logarithmic Sobolev inequality for the path space equipped with a general diffusion measure as long as the torsion of the corresponding Riemannian connection satisfies Driver's total antisymmetry condition.

1 Introduction

Let $\omega = (\omega_t)_{t\geq 0}$ be a standard Brownian motion starting from the origin with values in $\mathbb{R}^n$ and denote by $W_0(\mathbb{R}^n)$ the path space of continuous functions from $[0,1]$ to $\mathbb{R}^n$ starting from the origin. We denote further by $\mathbb{E}$ expectation with respect to the law μ (the Wiener measure) of ω on $W_0(\mathbb{R}^n)$. Gross [G] proved the following logarithmic Sobolev inequality holds:

$$\mathbb{E}(F^2 \log F^2) - \mathbb{E}(F^2)\log \mathbb{E}(F^2) \leq 2\,\mathbb{E}(|DF|^2_{\mathbb{H}}) \tag{1}$$

where $D : \mathrm{L}^2(\mu) \mapsto \mathrm{L}^2(\mu; \mathbb{H})$ is the (Malliavin) gradient operator on $W_0(\mathbb{R}^n)$ and $\mathbb{H}$ the Cameron-Martin Hilbert space. Logarithmic Sobolev inequalities were introduced to study

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hypercontractivity properties of Markov semigroups and they imply spectral gap inequalities such as

$$\mathbb{E}(F^2) - \big(\mathbb{E}(F)\big)^2 \le \mathbb{E}\big(|DF|^2_{\mathbb{H}}\big) \tag{2}$$

in the present case.

Thanks to the linear structure of $W_0(\mathbb{R}^n)$, the proof of the logarithmic Sobolev inequality (1) may be reduced to the case of finite dimensional Gaussian measures, for which rather elementary semigroup arguments may be used (cf. [Ba]). Such semigroup arguments can actually be formulated in terms of stochastic calculus on Brownian paths, which may then be shown to work easily in the infinite dimensional setting as well. To illustrate the purpose of this paper, let us first describe a proof of (1) and (2) along these lines. This proof is known to a number of people although we were unable to trace it back in the literature[2]. The starting point is the so-called Clark-Ocone-Haussmann formula (see e.g. [N]) which indicates that, for every functional F in the domain of the gradient operator D,

$$F - \mathbb{E}(F) = \int_0^1 \langle \mathbb{E}\big((DF)^{\cdot}_t \,|\mathcal{B}_t\big), d\omega_t\rangle \tag{3}$$

where $(\mathcal{B}_t)_{t\ge 0}$ is the filtration of ω. Taking the L^2-norm of both sides of (1) already yields the spectral gap inequality (2), since

$$\begin{aligned}
\mathbb{E}\big(\big|F - \mathbb{E}(F)\big|^2\big) &= \mathbb{E}\bigg(\int_0^1 \langle \mathbb{E}\big((DF)^{\cdot}_t \,|\mathcal{B}_t\big), d\omega_t\rangle\bigg)^2 \\
&= \mathbb{E}\bigg(\int_0^1 \big|\mathbb{E}\big((DF)^{\cdot}_t \,|\,\mathcal{B}_t\big)\big|^2 dt\bigg) \\
&\le \mathbb{E}\bigg(\int_0^1 \big|(DF)^{\cdot}_t\big|^2 dt\bigg) = \mathbb{E}\big(|DF|^2_{\mathbb{H}}\big).
\end{aligned}$$

Now, the same idea can be used to establish the logarithmic Sobolev inequality. Namely, what (3) tells us is that the martingale $M_t = \mathbb{E}(F\,|\,\mathcal{B}_t)$, $0 \le t \le 1$, is such that

$$dM_t = \langle \mathbb{E}\big((DF)^{\cdot}_t \,|\mathcal{B}_t\big), d\omega_t\rangle.$$

For simplicity we may assume that F is in the domain of D and $F \ge \varepsilon$ for some $\varepsilon > 0$. The latter can be removed afterwards by letting ε tend to 0. Applying Itô's formula to $M_t \log M_t$, we get

$$\mathbb{E}(M_1 \log M_1) - \mathbb{E}(M_0 \log M_0) = \frac{1}{2}\,\mathbb{E}\bigg(\int_0^1 \frac{1}{M_t}\big|\mathbb{E}\big((DF)^{\cdot}_t \,|\,\mathcal{B}_t\big)\big|^2 dt\bigg)$$

that is,

$$\mathbb{E}(F\log F) - \mathbb{E}(F)\log\mathbb{E}(F) = \frac{1}{2}\,\mathbb{E}\bigg(\int_0^1 \frac{1}{\mathbb{E}(F\,|\,\mathcal{B}_t)}\big|\mathbb{E}\big((DF)^{\cdot}_t \,|\,\mathcal{B}_t\big)\big|^2 dt\bigg).$$

Replace now F by F^2. By the Cauchy-Schwarz inequality,

$$\big|\mathbb{E}\big((DF^2)^{\cdot}_t \,|\,\mathcal{B}_t\big)\big|^2 = 4\big|\mathbb{E}\big(F(DF)^{\cdot}_t \,|\,\mathcal{B}_t\big)\big|^2 \le 4\,\mathbb{E}(F^2\,|\,\mathcal{B}_t)\,\mathbb{E}\big(\big|(DF)^{\cdot}_t\big|^2\,|\,\mathcal{B}_t\big).$$

[2]The third author learned it several years ago from B. Maurey. Recently, J. Neveu also mentioned this proof to him.

Hence

$$\mathbb{E}(F^2 \log F^2) - \mathbb{E}(F^2) \log \mathbb{E}(F^2) \leq 2\,\mathbb{E}\bigg(\int_0^1 \big|(DF)_t\big|^2 dt\bigg) = 2\,\mathbb{E}\big(|DF|_{\mathbb{H}}^2\big),$$

which is the desired logarithmic Sobolev inequality (1).
Recently, E. P. Hsu [H2], [H3] and S. Aida and K. D. Elworthy [A-K] established a logarithmic Sobolev inequality for the law of Brownian motion on a compact Riemannian manifold (or more generally for complete Riemannian manifolds with bounded Ricci curvature). The method of [H3] is based on logarithmic Sobolev inequalities for the heat kernel measures together with a Markovian tensorization and Bismut's formula to control the spatial derivative of the heat kernel. S. Aida and K. D. Elworthy embbed the manifold into an Euclidean space and then use the logarithmic Sobolev inequality (1) on flat space. As a result, their logarithmic Sobolev constant also depends on the embedding rather than only on the bound on the Ricci curvature as in [H3]). This logarithmic Sobolev inequality improves upon the previous spectral gap inequality due to S. Fang [F1]. Now, S. Fang's beautiful proof is based on a version of the Clark-Ocone-Haussmann formula for Brownian motion on a manifold. This representation formula is presented in a handy way in [H4] and simply involves an additional curvature term in (3). The aim of this note is then simply to observe that, together with this representation formula, the preceding simple proof of the logarithmic Sobolev inequality for Brownian motion in $\mathbb{R}^n$ yields in exactly the same way the logarithmic Sobolev inequality for the law of Brownian motion on a complete Riemannian manifold with bounded Ricci curvature. This result is presented in Section 2. In Section 3 we extend the argument by an appropriate integration by parts formula to general diffusion processes with generators of the form $\frac{1}{2}\Delta + V$ over a complete Riemannian manifold. The manifold is assumed to be equipped with a connection compatible with the Riemannian metric, and its torsion satisfies Driver's total antisymmetry condition. The resulting logarithmic Sobolev inequality extends simultaneously the recent results of F.-Y. Wang [W] and S. Fang [F2]. In the last section we show how the previous stochastic calculus argument may be applied to yield similarly the isoperimetric inequality on path spaces in [B-L].

2 Logarithmic Sobolev Inequality for Brownian Motion on a Manifold

We first recall Fang's martingale representation formula as presented in [H4]. We follow the exposition of [H4] and refer to it for further details. Let M be a complete and connected Riemannian manifold of dimension n equipped with the Levi-Civita connection ∇. Denote by Δ the Laplace-Beltrami operator on M. Fix a point x_0 in M and let $W_{x_0}(M)$ be the space of (pinned) continuous paths from $[0,1]$ to M starting at x_0. Let $O(M)$ be the bundle of orthonormal frames and let $\pi : O(M) \mapsto M$ be the canonical projection. Each frame $u \in O(M)$ is a linear isometry $u : \mathbb{R}^n \mapsto T_{\pi(u)}(M)$, the tangent space at $\pi(u)$. Let $\{H_i\,; 1 \leq i \leq n\}$ be the canonical horizontal vector fields on $O(M)$. Fix an orthonormal frame u_0 at x_0 and let U be the diffusion process solution of the Stratonovich stochastic differential equation

$$dU_t = \sum_{i=1}^n H_i(U_t) \circ d\omega_t^i, \quad U_0 = u_0,$$

where $\omega = (\omega_t)_{t \geq 0}$ is a standard Brownian motion on $\mathbb{R}^n$ starting from the origin. The diffusion process U has $\frac{1}{2}\sum_{i=1}^n H_i^2$ as its generator and is called horizontal Brownian motion.

The projection $\gamma = \pi(U)$ is a Brownian motion on M starting from x_0 whose law is the Wiener measure ν on $W_{x_0}(M)$ with the generator $\frac{1}{2}\Delta$. The map $J : W_0(\mathbb{R}^n) \mapsto W_{x_0}(M)$ given by $J\omega = \gamma$ is called the Itô map.
Each h in the Cameron-Martin Hilbert space $\mathbb{H}$ determines a vector field D_h as follows. For a typical Brownian path γ, the map $U_t : \mathbb{R}^n \mapsto T_{\gamma_t}(M)$ is an isometry. Hence $U_t h_t$ is a vector at γ_t. The vector field D_h is defined by $D_h(\gamma_t) = U_t h_t$. Let F be a cylindrical function on $W_{x_0}(M)$ of the form $F(\gamma) = \phi(\gamma_{t_1}, \ldots, \gamma_{t_\ell})$ where $0 \leq t_1 < \cdots < t_\ell \leq 1$ and ϕ is a smooth real-valued function on M^ℓ. For such an F,

$$D_h F(\gamma) = \sum_{i=1}^{\ell} \langle \nabla^{(i)} F(\gamma), U_{t_i} h_{t_i} \rangle_{T_{\gamma_{t_i}}},$$

where $\nabla^{(i)} F$ is the (usual) gradient on M of ϕ with respect to the i-th variable. The directional derivative operator D_h is closable in $L^2(\nu)$, and so is the gradient operator $D : L^2(\nu) \mapsto L^2(\nu; \mathbb{H})$ defined by $\langle DF, h\rangle_{\mathbb{H}} = D_h F$. We denote by $\mathrm{Dom}(D)$ its domain in $L^2(\nu)$.
We now present Fang's version Clark-Ocone-Haussmann formula for the Brownian motion γ in M, following [H4]. We denote by Ric the Ricci tensor on M and write more precisely Ric_u, $u \in O(M)$, for the linear symmetric transformation $v \mapsto \mathrm{Ric}_u(v)$ from $T_{\pi(u)}(M) \cong \mathbb{R}^n$ into itself. We assume throughout this section that M has bounded curvature, that is

$$\sup\big\{ \|\mathrm{Ric}_u\| \, ; u \in O(M) \big\} = K < \infty,$$

where $\|\cdot\|$ is the operator norm on $\mathbb{R}^n$. Let $(A_t)_{0 \leq t \leq 1}$ be the matrix-valued process (Ricci flow) satisfying the differential equation

$$\frac{dA_t}{dt} - \frac{1}{2} A_t \mathrm{Ric}_{U_t} = 0, \quad A_0 = I.$$

Then, for every cylindrical function F in the domain of D,

$$F - \mathbb{E}(F) = \int_0^1 \Big\langle E\Big((DF)^{\cdot}_t - \frac{1}{2} A_t^* \int_t^1 (A_s^*)^{-1} \, \mathrm{Ric}_{U_s} (DF)^{\cdot}_s \, ds \,\Big|\, \mathcal{B}_t \Big), d\omega_t \Big\rangle. \tag{4}$$

The identity (4), which extends the flat case (3), relies on the Bismut-Driver integration by parts formula (see [Bi], [D], [H4] and the references therein) to the effect that for any $h \in \mathbb{H}$, the adjoint D_h^* of D_h with respect to ν is given by

$$D_h^* = -D_h + \int_0^1 \langle \dot{h}_t + \frac{1}{2} \, \mathrm{Ric}_{U_t} h_t, d\omega_t \rangle. \tag{5}$$

Provided with this formula, the representation (4) is established from this formula through rather standard arguments (cf. [H4]).
We can now prove the logarithmic Sobolev inequality for γ following almost exactly the previous proof in the flat case. Let F be a cylindrical function in $\mathrm{Dom}(D)$ such that $F \geq \varepsilon$ for some $\varepsilon > 0$. By (4), the martingale $M_t = \mathbb{E}(F \,|\mathcal{B}_t)$, $0 \leq t \leq 1$, satisfies

$$dM_t = \langle H_t, d\omega_t \rangle$$

where

$$H_t = E\Big((DF)^{\cdot}_t - \frac{1}{2} A_t^* \int_t^1 (A_s^*)^{-1} \, \mathrm{Ric}_{U_s} (DF)^{\cdot}_s \, ds \,\Big|\, \mathcal{B}_t \Big).$$

Applying Itô's formula to $M_t \log M_t$, we get

$$\mathbb{E}(M_1 \log M_1) - \mathbb{E}(M_0 \log M_0) = \frac{1}{2}\,\mathbb{E}\bigg(\int_0^1 \frac{1}{M_t}\,|H_t|^2 dt\bigg).$$

In other words,

$$\mathbb{E}(F \log F) - \mathbb{E}(F)\log \mathbb{E}(F) = \frac{1}{2}\,\mathbb{E}\bigg(\int_0^1 \frac{1}{\mathbb{E}(F\,|\,\mathcal{B}_t)}\,|H_t|^2 dt\bigg).$$

Replace now F by F^2 and set for simplicity $j_t = |(DF^2)^{\cdot}_t|$. We have,

$$|H_t| \le \mathbb{E}\bigg(j_t + \frac{1}{2}\int_t^1 \big\|A_t^*(A_s^*)^{-1}\big\|\,\|\mathrm{Ric}_{U_s}\|\,j_s\,ds\,\Big|\,\mathcal{B}_t\bigg).$$

Since for every $s \ge t$, $A_s^{-1}A_t = -\frac{1}{2}\int_t^s \mathrm{Ric}_{U_r} A_r^{-1}A_t\,dr$, by Gronwall's lemma we have $\|A_s^{-1}A_t\| \le \mathrm{e}^{K(s-t)/2}$. Thus,

$$|H_t| \le \mathbb{E}\bigg(j_t + \frac{1}{2}\,K\int_t^1 \mathrm{e}^{K(s-t)/2} j_s ds\,\Big|\,\mathcal{B}_t\bigg).$$

Since $j_t = 2|F||(DF)^{\cdot}_t|$, it follows from the Cauchy-Schwarz inequality that,

$$|H_t|^2 \le 4\,\mathbb{E}(F^2\,|\,\mathcal{B}_t)\,\mathbb{E}\bigg(\Big(\big|(DF)^{\cdot}_t\big| + \frac{1}{2}\,K\int_t^1 \mathrm{e}^{K(s-t)/2}\big|(DF)^{\cdot}_s\big|\,ds\Big)^2\,\Big|\,\mathcal{B}_t\bigg).$$

Therefore,

$$\mathbb{E}\bigg(\int_0^1 \frac{1}{\mathbb{E}(F^2\,|\,\mathcal{B}_t)}\,|H_t|^2 dt\bigg) \le 4\int_0^1 \mathbb{E}\bigg(\Big(|(DF)^{\cdot}_t| + \frac{1}{2}\,K\int_t^1 \mathrm{e}^{K(s-t)/2}\big|(DF)^{\cdot}_s\big|\,ds\Big)^2\bigg)dt.$$

Now we have

$$\begin{aligned}\bigg(\int_t^1 \mathrm{e}^{K(s-t)/2}\big|(DF)^{\cdot}_s\big|\,ds\bigg)^2 &\le \int_t^1 \mathrm{e}^{K(s-t)}ds\int_0^1 \big|(DF)^{\cdot}_s\big|^2 ds\\ &= \frac{1}{K}\Big(\mathrm{e}^{K(1-t)}-1\Big)|DF|^2_{\mathbb{H}}.\end{aligned}$$

It then follows easily that

$$\mathbb{E}\bigg(\int_0^1 \frac{1}{\mathbb{E}(F^2\,|\,\mathcal{B}_t)}\,|H_t|^2 dt\bigg) \le 4\,c(K)\,\mathbb{E}\big(|DF|^2_{\mathbb{H}}\big),$$

where

$$c(K) = 1 + \frac{1}{4}\big(\mathrm{e}^K - 1 - K\big) + \sqrt{\mathrm{e}^K - 1 - K} \;\le\; \mathrm{e}^K.$$

We thus have established a logarithmic Sobolev inequality for Brownian motion on $W_{x_0}(M)$ in the form

$$\mathbb{E}(F^2\log F^2) - \mathbb{E}(F^2)\log\mathbb{E}(F^2) \le 2\,\mathrm{e}^K\,\mathbb{E}\big(|DF|^2_{\mathbb{H}}\big) \tag{6}$$

where we recall that K is the uniform bound on the Ricci curvature of M. The extension to all functions F in the domain of D follows in a standard way. In particular, when $K = 0$ ($M = \mathbb{R}^n$ or $\mathbb{R}^n/\mathbb{Z}^n$ for example), we recover (1).

3 Diffusion Processes and Connections with Torsion

In this section, we extend the preceding logarithmic Sobolev inequality to the law of a general diffusion process with generator $L = \frac{1}{2}\Delta + V$ on the path space over a Riemannian manifold M equipped with a connection compatible with the Riemannian metric but not necessarily torsion-free. We refer to [D] and [H1] for the general setting, notation and geometric consideration.
Let M be a complete, connected Riemannian manifold equipped now with a connection ∇ compatible with the Riemannian metric. We assume that the torsion $\Theta = \{\Theta_u; u \in O(M)\}$ satisfies Driver's total antisymmetry condition i.e., $\langle \Theta_u(x,y), z\rangle_{\mathbb{R}^n}$ is alternating in all three variables (x,y,z). Let V be some smooth vector field on M. The main step of the proof is to get an integration by parts formula for the connection ∇ and the diffusion measure generated by L. To this aim, one may either redo parts of [H1], or use the integration by parts formula of Driver [D] for $\frac{1}{2}\Delta$ to derive that of $L = \frac{1}{2}\Delta + V$ via Girsanov's theorem. We follow here the second route.
We first recall Driver's integration by parts formula for the Wiener measure ν on M [D]. Namely, the adjoint D_h^* of D_h with respect to ν is given by (cf. [D], [H1] and the notation therein)

$$D_h^* = -D_h + \ell_h$$

where

$$\ell_h = \int_0^1 \langle \dot{h}_t + \frac{1}{2}\,\widehat{\Theta}_{U_t} h_t + \frac{1}{2}\,\mathrm{Ric}_{U_t} h_t, d\omega_t\rangle. \tag{7}$$

Here, the map $\widehat{\Theta}_u : \mathbb{R}^n \mapsto \mathbb{R}^n$ is given by

$$\widehat{\Theta}_u(v) = \sum_{i=1}^n H_i \Theta_u(e_i, v), \quad v \in \mathbb{R}^n,$$

where $\{e_i\,; 1 \le i \le n\}$ are the coordinate unit vectors in $\mathbb{R}^n$ and $\{H_i\,; 1 \le i \le n\}$ are the canonical horizontal vector fields on $O(M)$.
Let η be the diffusion measure on $W_{x_0}(M)$ for $L = \frac{1}{2}\Delta + V$. For $u \in O(M)$ set $\overline{V}(u) = u^{-1}V(\pi(u)) \in \mathbb{R}^n$ and $\overline{\nabla V}(u) = u^{-1}\nabla V(\pi(u)) \in \mathbb{R}^n \times \mathbb{R}^n$; namely, $\overline{V}$ and $\overline{\nabla V}$ are the scalarization of the tensors V and ∇V, respectively. Set finally

$$db_t = d\omega_t - \overline{V}(U_t)dt.$$

The process ω is not a Brownian motion under η, but b is. We have the following integration by parts formula.

Proposition 1 *For any h in $\mathbb{H}$, the adjoint $\widetilde{D}_h^*$ of D_h with respect to η is given by*

$$\widetilde{D}_h^* = -D_h + \int_0^1 \langle \dot{h}_t + \frac{1}{2}\,\widehat{\Theta}_{U_t} h_t + \frac{1}{2}\,\mathrm{Ric}_{U_t} h_t - \Theta_{U_t}(\overline{V}(U_t), h_t) - \overline{\nabla V}(U_t) h_t, db_t\rangle.$$

We outline a proof of this proposition using (7) and the techniques of [H1]. It is enough to show that $\widetilde{D}_h^* 1$ is given by the second term on the right-hand side of the above formula. First of all it is easy to check that

$$\widetilde{D}_h^* 1 = \Big(\frac{d\eta}{d\nu}\Big)^{-1} D_h^*\Big(\frac{d\eta}{d\nu}\Big).$$

By Girsanov's theorem,

$$\log \frac{d\eta}{d\nu}(\gamma) = \int_0^1 \langle \overline{V}(U_t), d\omega_t \rangle - \frac{1}{2}\int_0^1 |\overline{V}(U_t)|^2 dt.$$

We thus need to compute

$$D_h \int_0^1 \langle \overline{V}(U_t), d\omega_t \rangle \quad \text{and} \quad D_h \int_0^1 \left|\overline{V}(U_t)\right|^2 dt.$$

Now, by Theorem 2.1 of [H1],

$$(D_h\omega)_t = h_t - \int_0^t \Theta_{U_s}(\circ d\omega_s, h_s) - \int_0^t K_s \circ d\omega_s$$

where $K = K_h$ is defined as

$$K_t = \int_0^t \Omega_{U_s}(\circ d\omega_s, h_s),$$

Ω being the curvature tensor (cf. [H1]). Furthermore (cf. [H1, (2.4) and (2.8)]),

$$D_h\overline{V} = \overline{\nabla V}h - K\overline{V}.$$

Hence,

$$\begin{aligned} D_h \int_0^1 \langle \overline{V}(U_t), d\omega_t \rangle \quad &= \quad \int_0^1 \langle \overline{\nabla V}(U_t)h_t - K_t\overline{V}(U_t), d\omega_t \rangle \\ &\quad + \int_0^1 \langle \overline{V}(U_t), \dot{h}_t dt - \Theta_{U_t}(\circ d\omega_t, h_t) - K_t \circ d\omega_t \rangle. \end{aligned}$$

Since K is antisymmetric, we have $\langle \overline{V}, K\overline{V} \rangle = 0$. Therefore,

$$\frac{1}{2} D_h \int_0^1 \left|\overline{V}(U_t)\right|^2 dt = \int_0^1 \langle \overline{V}(U_t), \overline{\nabla V}(U_t)h_t \rangle dt.$$

Putting everything together, and recalling (7), we get

$$\begin{aligned} \widetilde{D}_h^* 1 \quad &= \quad -D_h \log\Big(\frac{d\eta}{d\nu}\Big) + \ell_h \\ &= \quad -\int_0^1 \langle \overline{\nabla V}(U_t)h_t - K_t\overline{V}(U_t), d\omega_t \rangle \\ &\qquad - \int_0^1 \langle \overline{V}(U_t), \dot{h}_t dt - \Theta_{U_t}(\circ d\omega_t, h_t) - K_t \circ d\omega_t \rangle \\ &\qquad + \int_0^1 \langle \overline{V}(U_t), \overline{\nabla V}(U_t)h_t \rangle dt + \int_0^1 \langle \dot{h}_t + \frac{1}{2}\widehat{\Theta}_{U_t}h_t + \frac{1}{2}\mathrm{Ric}_{U_t}h_t, d\omega_t \rangle. \end{aligned}$$

We now convert Stratonovich integrals into Itô's integrals using the relations

$$\Theta_{U_t}(\circ d\omega_t, h_t) = \Theta_{U_t}(d\omega_t, h_t) + \frac{1}{2}\widehat{\Theta}_{U_t}h_t\, dt$$

and

$$K_t \circ d\omega_t = K_t d\omega_t - \frac{1}{2}\,\mathrm{Ric}_{U_t} h_t dt,$$

which follow from the definitions of $\widehat{\Theta}$ and K. Driver's condition on the torsion form Θ implies that $\langle \overline{V}, \Theta(d\omega, h)\rangle = -\langle \Theta(\overline{V}, h), d\omega\rangle$ and $\langle \Theta(\overline{V}, h), \overline{V}\rangle = 0$.
Hence we easily get

$$\begin{aligned}\widetilde{D}_h^* 1 &= \int_0^1 \langle \dot{h}_t + \frac{1}{2}\,\widehat{\Theta}_{U_t} h_t + \frac{1}{2}\,\mathrm{Ric}_{U_t} h_t - \Theta_{U_t}\big(\overline{V}(U_t), h_t\big) - \overline{\nabla V}(U_t) h_t, d\omega_t\rangle \\ &\quad - \int_0^1 \langle \dot{h}_t + \frac{1}{2}\,\widehat{\Theta}_{U_t} h_t + \frac{1}{2}\,\mathrm{Ric}_{U_t} h_t - \Theta_{U_t}\big(\overline{V}(U_t), h_t\big) - \overline{\nabla V}(U_t) h_t, \overline{V}(U_t) dt\rangle.\end{aligned}$$

Recalling that $db_t = d\omega_t - \overline{V}(U_t)dt$, we obtain the desired result immediately.
Given the preceding integration by parts formula, the proof of the logarithmic Sobolev inequality for the diffusion measure η on a manifold with torsion Θ is entirely similar to the proof in Section 2. As was indicated there, one deduces from the proposition with standard arguments (following e.g. [H4]) a representation formula for a smooth cylindrical functional F which takes the form (with $\mathbb{E}$ for integration with respect to η)

$$F - \mathbb{E}(F) = \int_0^1 \Big\langle E\Big((DF)^{\cdot}_t - \frac{1}{2} A_t^* \int_t^1 (A_s^*)^{-1} M_s (DF)^{\cdot}_s \, ds \,\Big|\, \mathcal{B}_t\Big), db_t\Big\rangle$$

where the flow $(A_t)_{0\le t\le 1}$ now satisfies

$$\frac{dA_t}{dt} - M_t A_t = 0, \quad A_0 = I,$$

and

$$M_t = \frac{1}{2}\,\widehat{\Theta}_{U_t} + \frac{1}{2}\,\mathrm{Ric}_{U_t} - \Theta_{U_t}\big(\overline{V}(U_t), \cdot\big) - \overline{\nabla V}(U_t).$$

Therefore, following the arguments of Section 2, if

$$K = \sup\big\{\big\|\widehat{\Theta}_u + \mathrm{Ric}_u - 2\Theta_u\big(\overline{V}(u), \cdot\big) - 2\overline{\nabla V}(u)\big\|; u \in O(M)\big\} < \infty,$$

then, for any F in the domain of D, we have

$$\mathbb{E}(F^2 \log F^2) - \mathbb{E}(F^2) \log \mathbb{E}(F^2) \le 2\,\mathrm{e}^K\, \mathbb{E}\big(|DF|^2_{\mathbb{H}}\big).$$

We summarize our conclusions in the following theorem, which covers, with its simple proof, both Wang's result on the case $\Theta = 0$, $V \neq 0$ [W] and Fang's result on the case $\Theta \neq 0$, $V = 0$ [F2].

Theorem 1 *Consider a diffusion process with generator $L = \frac{1}{2}\Delta + V$ on a complete Riemannian manifold with a connection compatible with the Riemannian metric whose torsion form Θ satisfies Driver's total antisymmetry condition. Then, if*

$$K = \sup\big\{\big\|\widehat{\Theta}_u + \mathrm{Ric}_u - 2\Theta_u\big(\overline{V}(u), \cdot\big) - 2\overline{\nabla V}(u)\big\|; u \in O(M)\big\} < \infty,$$

for any functional F in the domain of D,

$$\mathbb{E}(F^2 \log F^2) - \mathbb{E}(F^2) \log \mathbb{E}(F^2) \le 2\,\mathrm{e}^K\, \mathbb{E}\big(|DF|^2_{\mathbb{H}}\big).$$

4 Isoperimetric Inequalities on Path Spaces

In this last section, we show how the preceding stochastic calculus approach may be used similarly to recover and extend the isoperimetric inequality on path spaces proved in [B-L]. For the sake of comparison, let us first recall a functional form of the isoperimetric inequality proved in [B-L]: on $W_0(\mathbb{R}^n)$, for any smooth functional F with values in $[0,1]$,

$$\mathcal{U}(\mathbb{E}(F)) \le \mathbb{E}\Big(\sqrt{\mathcal{U}^2(F) + |DF|^2_{\mathbb{H}}}\Big), \tag{9}$$

where $\mathcal{U} = \varphi \circ \Phi^{-1}$, $\varphi(x) = (2\pi)^{-1/2}\mathrm{e}^{-x^2/2}$, $\Phi(x) = \int_{-\infty}^x \varphi(t)dt$, $x \in \mathbb{R}$. We refer to [B-L] for further details and comments on this inequality and on its isoperimetric content. Note in particular that it always implies the corresponding logarithmic Sobolev inequalities (cf. Proposition 3.2 in [B-L]). The isoperimetric inequality for the law of Brownian motion on a manifold with bounded Ricci curvature was established in [B-L] using a Markovian tensorization property together with the same argument as the one developed in [H3] for the logarithmic Sobolev inequality. We apply here the simple martingale representation techniques of the preceding sections to extend this result to the generality of Theorem 1. That is, we have

Theorem 2 *Under the notation and hypotheses of Theorem 1, for any F in the domain of D and with values in* $[0,1]$,

$$\mathcal{U}(\mathbb{E}(F)) \le \mathbb{E}\Big(\sqrt{\mathcal{U}^2(F) + \mathrm{e}^K |DF|^2_{\mathbb{H}}}\Big).$$

For simplicity we only give the proof in the flat case (9). The arguments presented in the preceding sections for logarithmic Sobolev inequality show clearly that the general result will follow almost identically. Fix F, say smooth and cylindrical, with values in $[0,1]$. Consider as before $M_t = \mathbb{E}(F\,|\mathcal{B}_t)$, $0 \le t \le 1$, and besides the $\mathbb{H}$-valued martingale $N_t = \mathbb{E}(DF\,|\mathcal{B}_t)$, $0 \le t \le 1$. For an element h in $\mathbb{H}$, and $0 \le t \le 1$, define $h^t \in \mathbb{H}$ by $h^t_s = h_{t\wedge s}$. We will apply Itô's formula to the (Hilbert space valued) semimartingale $\Psi(M_t, N_t, t)$ between $t=0$ and $t=1$ with

$$\Psi(x,y,t) = \sqrt{\mathcal{U}^2(x) + |y^t|^2_{\mathbb{H}}}, \quad x \in [0,1], \quad y \in \mathbb{H}, \quad t \in [0,1]..$$

Note that, using the basic relation $\mathcal{U}\mathcal{U}'' = -1$, we have

$$\begin{aligned}
\partial^2_x \Psi &= \frac{1}{\Psi^3}\mathcal{U}'^2 |y^t|^2_{\mathbb{H}} - \frac{1}{\Psi},\\
\partial^2_y \Psi &= \frac{1}{\Psi^3}\Big[\big(\mathcal{U}^2 + |y^t|^2\big)\langle(\cdot)^t, (\cdot)^t\rangle_{\mathbb{H}} - y^t \otimes y^t(\cdot,\cdot)\Big] \quad (\in \mathbb{H}^* \otimes \mathbb{H}^*),\\
\partial^2_{xy}\Psi &= -\frac{1}{\Psi^3}\mathcal{U}\mathcal{U}' y^t \quad (\in \mathbb{H}^*)\\
\partial_t \Psi &= \frac{1}{2\Psi}\big|\dot{y}(t)\big|^2.
\end{aligned}$$

In the notation of the present proof, the Clark-Ocone-Haussman formula can be written as $d\langle M\rangle_t = |\dot{N}_t(t)|^2 dt$. Applying Itô's formula we obtain (with the obvious notational simplifications),

$$\mathbb{E}\Big(\sqrt{\mathcal{U}^2(F) + |DF|^2_{\mathbb{H}}}\Big)$$

$$= \mathcal{U}(\mathbb{E}(F)) + \frac{1}{2}\mathbb{E}\Big(\int_0^1 \frac{1}{\Psi^3}\sum_{i=1}^n \Big(\big|(N_t)^t\big|_{\mathbb{H}}^2\big|(S_t^i)^t\big|_{\mathbb{H}}^2 - \langle (N_t)^t, S_t^i\rangle_{\mathbb{H}}^2\Big)\,dt\Big)$$
$$+ \frac{1}{2}\mathbb{E}\Big(\int_0^1 \frac{1}{\Psi^3}\sum_{i=1}^n \Big(\big|(N_t)^t\big|_{\mathbb{H}}^2 (R_t^i)^2\mathcal{U}'^2 - 2R_t^i\langle (N_t)^t, S_t^i\rangle_{\mathbb{H}}\mathcal{U}\mathcal{U}' + \big|(S_t^i)^t\big|_{\mathbb{H}}^2\mathcal{U}^2\Big)\,dt\Big),$$

where we denote for simplicity

$$d\langle M\rangle_t = \sum_{i=1}^n (R_t^i)^2 dt \quad (= |\dot{N}_t(t)|^2 dt), \quad d\langle N\rangle_t = \sum_{i=1}^n S_t^i \otimes S_t^i\, dt$$

and

$$d\langle M, N\rangle_t = \sum_{i=1}^n R_t^i S_t^i\, dt.$$

Since $\big|(N_t)^t\big|_{\mathbb{H}}^2\big|(S_t^i)^t\big|_{\mathbb{H}}^2 - \langle (N_t)^t, S_t^i\rangle_{\mathbb{H}}^2 \geq 0$ and

$$\big|(N_t)^t\big|_{\mathbb{H}}^2 (R_t^i)^2\mathcal{U}'^2 - 2R_t^i\langle (N_t)^t, S_t^i\rangle_{\mathbb{H}}\mathcal{U}\mathcal{U}' + \big|(S_t^i)^t\big|_{\mathbb{H}}^2\mathcal{U}^2 = \big|(N_t)^t R_t^i\mathcal{U}' - (S_t^i)^t\mathcal{U}\big|_{\mathbb{H}}^2 \geq 0,$$

the conclusion immediately follows.

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