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SHARP MAXIMAL INEQUALITY FOR MARTINGALES AND STOCHASTIC INTEGRALS

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Abstract

Let $X=(X_t)_{t\geq 0}$ be a martingale and $H=(H_t)_{t\geq 0}$ be a predictable process taking values in $[-1,1]$. Let Y denote the stochastic integral of H with respect to X. We show that

$$||\sup_{t\geq 0} Y_t||_1 \leq \beta_0 ||\sup_{t\geq 0}|X_t|||_1,$$

where $\beta_0=2,0856\ldots$ is the best possible. Furthermore, if, in addition, X is nonnegative, then

$$||\sup_{t\geq 0} Y_t||_1 \leq \beta_0^+ ||\sup_{t\geq 0} X_t||_1,$$

where $\beta_0^+=\frac{14}{9}$ is the best possible.

1 Introduction

Let $(\Omega,\mathscr{F},\mathbb{P})$ be a complete probability space, which is filtered by a nondecreasing right-continuous family $(\mathscr{F}_t)_{t\geq 0}$ of sub-σ-fields of $\mathscr{F}$. Assume that $\mathscr{F}_0$ contains all the events of probability 0. Suppose $X=(X_t)_{t\geq 0}$ is an adapted real-valued right-continuous semimartingale with left limits. Let Y be the Itô integral of H with respect to X,

$$Y_t = H_0X_0 + \int_{(0,t]} H_s dX_s, \quad t\geq 0,$$

where H is a predictable process with values in $[-1,1]$. Let $||Y||_1=\sup_{t\geq 0}||Y_t||_1$ and $X^*=\sup_{t\geq 0} X_t$.

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The main interest of this paper is in the comparison of the sizes of Y^* and $|X|^*$. Let us first describe two related results from the literature. In [4], Burkholder introduced a method of proving maximal inequalities for martingales and obtained the following sharp estimate.

Theorem 1. *If X is a martingale and Y is as above, then we have*

$$||Y||_1 \leq \gamma|| \, |X|^*||_1, \tag{1}$$

where $\gamma = 2,536\ldots$ is the unique solution of the equation

$$\gamma - 3 = -\exp\big(\frac{1-\gamma}{2}\big).$$

The constant is the best possible.

It was then proved by the author in [5], that if X is positive, then the optimal constant γ in (1) equals $2+(3e)^{-1} = 2,1226\ldots$.
We study here a related estimate, with Y replaced by its one-sided supremum:

$$||Y^*||_1 \leq \beta|| \, |X|^*||_1. \tag{2}$$

Let $\beta_0 = 2,0856\ldots$ be the positive solution to the equation

$$2\log\left(\frac{8}{3}-\beta_0\right) = 1-\beta_0$$

and $\beta_0^+ = \frac{14}{9} = 1,555\ldots$. The main result of the paper can be stated as follows.

Theorem 2. *(i) If X is a martingale and Y is as above, then* (2) *holds with $\beta = \beta_0$ and the inequality is sharp.*
(ii) If X is a nonnegative martingale and Y is as above, then (2) *holds with $\beta = \beta_0^+$ and the constant is the best possible.*

As usual, to prove this theorem, it suffices to establish its discrete-time version (by standard approximation argument due to Bichteler [1]; for details, see e.g. [2]). Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, equipped with filtration $(\mathscr{F}_n)_{n\geq 0}$. Let $f = (f_n)_{n\geq 0}$ be an adapted sequence of integrable variables and $g = (g_n)_{n\geq 0}$ be its transform by a predictable sequence $v = (v_n)_{n\geq 0}$ bounded in absolute value by 1. That is, for any $n = 0, 1, 2, \ldots$ we have

$$f_n = \sum_{k=0}^n df_k \quad \text{and} \quad g_n = \sum_{k=0}^n v_k df_k.$$

By predictability of v we mean that v_0 is $\mathscr{F}_0$-measurable (and hence deterministic) and for any $k \geq 1$, v_k is measurable with respect to $\mathscr{F}_{k-1}$. In the special case when each v_k is deterministic and takes values in $\{-1, 1\}$ we will say that g is a ± 1 transform of f. Let $f_n^* = \max_{k\leq n} f_k$ and $f^* = \sup_k f_k$.
A discrete-time version of Theorem 2 is the following.

Theorem 3. *Let f, g, β_0, β_0^+ be as above.*
(i) If f is a martingale, then

$$||g^*||_1 \leq \beta_0|| |f|^*||_1, \tag{3}$$

and the constant β_0 is the best possible.
(ii) If f is a nonnegative martingale, then

$$||g^*||_1 \le \beta_0^+ ||f^*||_1, \tag{4}$$

and the constant β_0^+ is the best possible.

A few words about the organization of the paper. The proof of Theorem 3 is based on Burkholder's technique, which reduces the problem of proving a martingale inequality to finding a certain special function. The description of this technique can be found in Section 2. Then, in the following two sections we provide the special functions corresponding to (3) and (4) and study their properties. In the last section we complete the proofs of Theorem 2 and Theorem 3 by showing that the constants β_0 and β_0^+ can not be replaced by smaller ones.

2 Burkholder's method

Throughout this section we deal with discrete-time setting. Let us start with some standard reductions. Assume f, g are as in the statement of Theorem 3. With no loss of generality we may assume that the process f is *simple*: for any integer n the random variable f_n takes only a finite number of values and there exists a number N such that $f_N = f_{N+1} = \ldots$ with probability 1. Furthermore, it suffices to prove Theorem 3 for ± 1 transforms. To see this, let us consider the following version of the Lemma A.1 from [3]. The proof is identical as in the original setting and hence it is omitted.

Lemma 1. *Let g be the transform of a martingale (resp., nonnegative martingale) f by a real-valued predictable sequence v uniformly bounded in absolute value by* 1. *Then there exist martingales (resp., nonnegative martingales) $F^j = (F_n^j)_{n\ge 0}$ and Borel measurable functions $\phi_j : [-1,1] \to \{-1,1\}$ such that, for $j \ge 1$ and $n \ge 0$,*

$$f_n = F^j_{2n+1}, \quad |f|^* = |F^j|^*,$$

$$g_n = \sum_{j=1}^{\infty} 2^{-j} \phi_j(v_0) G^j_{2n+1},$$

where G^j is the transform of F^j by $\varepsilon = (\varepsilon_k)_{k\ge 0}$ with $\varepsilon_k = (-1)^k$.

Suppose we have established Theorem 3 for ± 1 transforms and let β denote β_0 or β_0^+, depending on whether f is a martingale or nonnegative martingale. Lemma 1 gives us the processes F^j and the functions ϕ_j, $j \ge 1$. Conditionally on $\mathscr{F}_0$, for any $j \ge 1$ the sequence $\phi_j(v_0)G^j$ is a ± 1 transform of F^j and hence we may write

$$||g^*||_1 \le \left|\left| \sum_{j=1}^{\infty} 2^{-j} \sup_n \left(\phi_j(v_0) G^j_{2n+1}\right) \right|\right|_1 \le \sum_{j=1}^{\infty} 2^{-j} \left|\left| \left(\phi_j(v_0) G^j\right)^* \right|\right|_1$$

$$\le \beta \sum_{j=0}^{\infty} 2^{-j} |||F^j|^*||_1 = \beta |||f|^*||_1.$$

The final reduction is that it suffices to prove that for any integer n we have

$$\mathbb{E}\left[g_n^* - \beta |f_n|^*\right] \le 0. \tag{5}$$

To establish the above estimate, consider the following general problem. Let $D=\mathbb{R}\times\mathbb{R}\times(0,\infty)\times\mathbb{R}$ and $V:D\to\mathbb{R}$ be a Borel function. Suppose we want to prove the inequality

$$\mathbb{E}V(f_n,g_n,|f_n|^*,g_n^*)\leq 0 \tag{6}$$

for any integer n, any martingale f and g being its ±1 transform.
The key idea is to study the family $\mathscr{U}$ of all functions $U:D\to\mathbb{R}$ satisfying the following properties.

$$U(1,1,1,1)\leq 0, \tag{7}$$

$$U(x,y,z,w)=U(x,y,|x|\vee z,y\vee w),\quad \text{if } (x,y,z,w)\in D, \tag{8}$$

$$V(x,y,z,w)\leq U(x,y,z,w),\quad \text{if } (x,y,z,w)\in D \tag{9}$$

and, furthermore,

$$\begin{gathered}\alpha U(x+t_1,y+\varepsilon t_1,z,w)+(1-\alpha)U(x+t_2,y+\varepsilon t_2,z,w)\leq U(x,y,z,w),\\ \text{for any } |x|\leq z,\ y\leq w,\ \varepsilon\in\{-1,1\},\ \alpha\in(0,1) \text{ and } t_1,\ t_2\\ \text{with } \alpha t_1+(1-\alpha)t_2=0.\end{gathered} \tag{10}$$

The relation between the class $\mathscr{U}$ and the estimate (6) is described in the following theorem. It is a simple modification of Theorems 2.2 and 2.3 in [4] (see also Section 11 in [2] and Theorem 2.1 in [3]). We omit the proof.

Theorem 4. *The inequality (6) holds for all n and all pairs (f,g) as above if and only if the class $\mathscr{U}$ is nonempty. Furthermore, if $\mathscr{U}$ is nonempty, then there exists the least element in $\mathscr{U}$, given by*

$$U_0(x,y,z,w)=\sup\{\mathbb{E}V(f_\infty,g_\infty,|f|^*\vee z,g^*\vee w)\}. \tag{11}$$

Here the supremum runs over all the pairs (f,g), where f is a simple martingale, $\mathbb{P}((f_0,g_0)=(x,y))=1$ and $dg_k=\pm df_k$ almost surely for all $k\geq 1$.

A similar statement is valid when we want the inequality (6) to hold for any nonnegative martingale f and its ±1 transform g. Let $D^+=[0,\infty)\times\mathbb{R}\times(0,\infty)\times\mathbb{R}$ and let $\mathscr{U}^+$ denote the class of functions $U:D^+\to\mathbb{R}$ satisfying (7), (8), (9) and (10) (with D replaced by D^+ and, in (10), an extra assumption $t_1,\ t_2\geq -x$).

Theorem 5. *The inequality (6) holds for all n and all pairs (f,g) as above if and only if the class $\mathscr{U}^+$ is nonempty. Furthermore, if $\mathscr{U}^+$ is nonempty, then there exists the least element in $\mathscr{U}^+$, given by*

$$U_0^+(x,y,z,w)=\sup\{\mathbb{E}V(f_\infty,g_\infty,|f|^*\vee z,g^*\vee w)\}. \tag{12}$$

Here the supremum runs over all the pairs (f,g), where f is a simple nonnegative martingale, $\mathbb{P}((f_0,g_0)=(x,y))=1$ and $dg_k=\pm df_k$ almost surely for all $k\geq 1$.

Let us now turn to (3) and assume, from now on, that the function V is given by

$$V(x,y,z,w)=V(x,y,|x|\vee z,y\vee w)=y\vee w-\beta(|x|\vee z),$$

where $\beta>0$ is a fixed number. Denote by $\mathscr{U}(\beta)$, $\mathscr{U}^+(\beta)$ the classes $\mathscr{U}$, $\mathscr{U}^+$ corresponding to this choice of V. The purpose of the next two sections is to show that the classes $\mathscr{U}(\beta_0)$ and $\mathscr{U}^+(\beta_0^+)$ are nonempty. This will establish the inequalities (3) and (4).

3 The special function: a general case

We start with the class $\mathscr{U}(\beta_0)$. Let us introduce an auxiliary parameter. The equation

$$2\log\left(2-\frac{2}{3a}\right)=\frac{a-2}{3a},\qquad a>\frac{1}{3}, \tag{13}$$

has a unique solution $a=0.46986\ldots$, related to β_0 by the identity

$$\beta_0=\frac{2a+2}{3a}. \tag{14}$$

Let S denote the strip $[-1,1]\times(-\infty,0]$ and consider the following subsets of S.

$$\begin{aligned} D_1 &= \{(x,y):|x|+y>0\},\\ D_2 &= \{(x,y):0\geq|x|+y>1-\beta_0\},\\ D_3 &= \{(x,y):|x|+y\leq 1-\beta_0\}. \end{aligned}$$

Introduce the special function $u:S\to\mathbb{R}$ by

$$u(x,y)=\begin{cases} a(2|x|-y-2)(1-|x|-y)^{1/2}-3a|x|+y, & \text{if } (x,y)\in D_1,\\ 3a(2-|x|)\exp(\frac{1}{2}(|x|+y))+(1-3a)y-8a, & \text{if } (x,y)\in D_2,\\ \frac{9a^2}{4(3a-1)}(1-|x|)\exp(|x|+y)-\beta_0, & \text{if } (x,y)\in D_3. \end{cases}$$

A function defined on the strip S is said to be *diagonally concave* if it is concave on the intersection of S with any line of slope 1 or -1. We have the following fact.

Lemma 2. *The function u has the following properties.*

$$u(1,\cdot)\ \textit{is convex}, \tag{15}$$

$$u(1,y)\geq-\beta_0, \tag{16}$$

$$u(x,0)\geq-\beta_0, \tag{17}$$

$$u\ \textit{is diagonally concave.} \tag{18}$$

Proof. It is easy to check that u is of class C^1 in the interior of S. Now the condition (15) is apparent and hence so is (16). To see that (17) holds, note that

$$u(x,0)=-a(2(1-|x|)^{3/2}+3|x|),\qquad x\in[-1,1],$$

attains its minimum $-3a>-\beta_0$ at $x\in\{-1,1\}$. Due to the symmetry, it suffices to check the diagonal concavity of u restricted to the set $(0,1)\times(-\infty,0)$. This is obvious on the lines of slope -1. On the remaining lines, fix $(x,y)\in(0,1)\times(-\infty,0)$ and introduce the function F by $F(t)=u(x+t,y+t)$ for t belonging to a certain open interval containing 0. Denoting by A^o the interior of a set A, we easily check that

$$F''(0)=\begin{cases} 3ay(1-x-y)^{-3/2}, & \text{if } (x,y)\in D_1^o,\\ -3ax\exp(\frac{1}{2}(x+y)), & \text{if } (x,y)\in D_2^o,\\ -\frac{9a^2}{3a-1}x\exp(x+y), & \text{if } (x,y)\in D_3^o \end{cases}$$

is nonpositive. This completes the proof. $\square$

Define $U : D \to \mathbb{R}$ by

$$U(x,y,z,w) = y \vee w + (|x| \vee z)u\left(\frac{x}{|x| \vee z}, \frac{y-(y \vee w)}{|x| \vee z}\right). \tag{19}$$

We have

Lemma 3. *The function U belongs to $\mathcal{U}(\beta_0)$.*

Proof. The conditions (7) and (8) follow from the definition of U. The inequality (9) is equivalent to $u \geq -\beta_0$ on the whole strip S, an estimate which follows directly from (16), (17) and (18). The main technical difficulty lies in proving (10). Let us start with some reductions. First, we may assume $\varepsilon = 1$, as $U(x,y,z,w) = U(-x,y,z,w)$. Secondly, by homogeneity, it is enough to show (10) for $z = 1$. Finally, we may set $w = 0$, since $U(x,y,z,w) = U(x,y-w,z,0) + w$. Now fix $(x,y) \in S$ and introduce the function $\Phi : \mathbb{R} \to \mathbb{R}$ by $\Phi(t) = U(x+t, y+t, 1, 0)$. The condition (10) will follow if we show that there exists a concave function Ψ on $\mathbb{R}$ such that $\Phi \leq \Psi$ and $\Phi(0) = \Psi(0)$. The existence will be a consequence of the properties (20) – (24) below.

$$\Phi \text{ is continuous}, \tag{20}$$

$$\Phi \text{ is concave on } [-1-x, 1-x], \tag{21}$$

$$\Phi \text{ is convex on } (-\infty, -1-x] \text{ and on } [1-x, \infty), \tag{22}$$

$$\lim_{t\to-\infty} \Phi'(t) \geq \lim_{t\downarrow -1-x} \Phi'(t), \tag{23}$$

$$\lim_{t\to\infty} \Phi'(t) \leq \lim_{t\uparrow 1-x} \Phi'(t). \tag{24}$$

The property (20) is straightforward to check. If $1-x \leq -y$, then the condition (21) follows from (18). If $1-x > -y$, then (18) gives the concavity only on $[-1-x, -y]$, but for $t \in (-y, 1-x)$ we have

$$\Phi(t) = y + t - a(2(1-|x+t|)^{3/2} + 3|x+t|),$$

which is concave. In addition, one-sided derivatives of Φ match at $-y$ and we are done.
To show (22), fix α_1, $\alpha_2 > 0$ satisfying $\alpha_1 + \alpha_2 = 1$, choose $t_1, t_2 \in (-\infty, -1-x]$ and let $t = \sum \alpha_k t_k$. We have

$$\begin{aligned}\sum \alpha_k \Phi(t_k) &= \sum \alpha_k U(x+t_k, y+t_k, 1, 0)\\ &= \sum \alpha_k \left[(-x-t_k)u\left(-1, \frac{y+t_k}{-x-t_k}\right)\right]\\ &= -(x+t)\sum \frac{\alpha_k(x+t_k)}{x+t}u\left(1, \frac{y+t_k}{-x-t_k}\right).\end{aligned}$$

By (15), this can be bounded from below by

$$-(x+t)u\left(1, \sum \frac{y+t_k}{-x-t_k} \cdot \frac{\alpha_k(x+t_k)}{x+t}\right) = -(x+t)u\left(1, -\frac{y+t}{x+t}\right) = \Phi(t).$$

Hence Φ is convex on $(-\infty, -1-x]$. If $1-x < -y$, then convexity on $[1-x, -y]$ can be established exactly in the same manner. Furthermore, for $t > \max\{1-x, -y\}$ we have

$$\Phi(t) = y - 3ax + (1-3a)t \tag{25}$$

and one-sided derivatives of Φ are equal at $\max\{1-x,-y\}$. Thus (22) follows.
To prove (23), note that the limit on the left equals $-u(1,-1)=1+2a$, while the one on the right equals

$$\begin{array}{ll} 3a-\dfrac{3}{2}a(-y+1+x)^{1/2}, & \text{if } -x+y\geq 0,\\ \dfrac{3a}{2}\exp(\dfrac{1}{2}(y-x)), & \text{if } 0>-x+y\geq 1-\beta_0,\\ \dfrac{9a^2}{4(3a-1)}\exp(y-x), & \text{if } -x+y<1-\beta_0. \end{array}$$

and the estimate is satisfied. Finally, let us turn to (24). The limit on the left is equal to $1-3a$, due to (25). If $-x+y\geq -1-\beta_0$, then the limit on the right is also $1-3a$; for $-x+y\leq -1-\beta_0$ the inequality (24) becomes

$$1-3a\leq -\frac{9a^2}{4(3a-1)}\exp(2-x+y),$$

which is a consequence of the fact that the right hand side is a nonincreasing function of y and both sides are equal for $-x+y=-1-\beta_0$ (see (13) and (14)). □

4 The special function in the nonnegative case

Let S^+ denote the strip $[0,1]\times(-\infty,0]$ and let

$$\begin{array}{rcl} D_1 & = & \{(x,y)\in S^+: x-y>\frac{2}{3},\ x\leq\frac{2}{3}\},\\ D_2 & = & \{(x,y)\in S^+: x+y<\frac{2}{3},\ x>\frac{2}{3}\},\\ D_3 & = & \{(x,y)\in S^+: x+y\geq\frac{2}{3}\},\\ D_4 & = & \{(x,y)\in S^+: x-y\leq\frac{2}{3}\}. \end{array}$$

Introduce the function $u^+: S^+\to\mathbb{R}$ by

$$u^+(x,y)=\begin{cases} x\exp[\frac{3}{2}(-x+y)+1]-\beta_0^+, & \text{if } (x,y)\in D_1,\\ (\frac{4}{3}-x)\exp[\frac{3}{2}(x+y)-1]-\beta_0^+, & \text{if } (x,y)\in D_2,\\ -x+y-\frac{1}{\sqrt{3}}(1-x-y)^{1/2}(2-2x+y), & \text{if } (x,y)\in D_3,\\ x-x\log(\frac{3}{2}(x-y))-\beta_0^+, & \text{if } (x,y)\in D_4. \end{cases}$$

Here is the analogue of Lemma 2.

Lemma 4. *The function u^+ has the following properties.*

$$u^+(1,\cdot)\ \textit{is convex,} \tag{26}$$

$$u^+(1,y)\geq -\beta_0^+\ \textit{for}\ y\leq 0, \tag{27}$$

$$u^+(x,0)\geq -\beta_0^+\ \textit{for}\ x\in[0,1]. \tag{28}$$

$$u^+\ \textit{is diagonally concave}. \tag{29}$$

Proof. It is not difficult to check that u^+ has continuous partial derivatives in the interior of S^+. Now the properties (26) and (27) are easy to see. To show (28) observe that the function $u^+(\cdot,0)$ is concave on $[0,1]$ and $u^+(0,0)=-\beta_0^+<u^+(1,0)$. Finally, it is obvious that u^+ is concave along the lines of slope 1 on $D_1\cup D_4$, and along the lines of slope -1 on $D_2\cup D_3$. For $x\in D_1^o\cup D_4^o$, let $F_-(t)=u(x+t,y-t)$ and derive that

$$F_-''(0)=\begin{cases}(9x-6)\exp[\frac{3}{2}(-x+y)+1], & \text{if } (x,y)\in D_1^o,\\ 4y(x-y)^{-2}, & \text{if } (x,y)\in D_2^o,\end{cases}$$

so $F_-''(0)\leq 0$. Similarly, for $x\in D_2^o\cup D_3^o$, introduce $F_+(t)=u(x+t,y+t)$ and check that

$$F_+''(0)=\begin{cases}(-9x+6)\exp[\frac{3}{2}(x+y)-1], & \text{if } (x,y)\in D_2^o,\\ \sqrt{3}y(1-x-y)^{-3/2}, & \text{if } (x,y)\in D_3^o,\end{cases}$$

which gives $F_+''(0)\leq 0$. This completes the proof. □

Now we define the special function $U^+:D^+\to\mathbb{R}$ by the same formula as in (19), namely

$$U^+(x,y,z,w)=y\vee w+(x\vee z)u^+\left(\frac{x}{x\vee z},\frac{y-(y\vee w)}{x\vee z}\right). \tag{30}$$

The following is the analogue of Lemma 3.

Lemma 5. *The function U^+ belongs to $\mathcal{U}^+(\beta_0^+)$.*

Proof. The approach is essentially the same. The conditions (7) and (8) are immediate, while (9) follows from (27), (28), (29) and the equality $u^+(0,y)=-\beta_0^+$. To show (10), we may assume $z=1$ and $w=0$. Fix $\varepsilon\in\{-1,1\}$, $x\in[0,1]$, $y\in(-\infty,0]$, introduce the function $\Phi(t)=U^+(x+t,y+\varepsilon t,1,0)$ (given for $t\geq -x$) and observe that it suffices to show the existence of a concave function Ψ satisfying $\Psi\geq\Phi$ and $\Psi(0)=\Phi(0)$. Let us only list here the properties of Φ which guarantee the existence, and omit the tedious proof.

$$\Phi \text{ is continuous,} \tag{31}$$

$$\Phi \text{ is concave on } [-x,1-x]. \tag{32}$$

$$\Phi \text{ is convex on } (1-x,\infty), \tag{33}$$

$$\lim_{t\uparrow 1-x}\Phi'(t)\geq\lim_{t\to\infty}\Phi'(t). \tag{34}$$

□

5 Optimality of the constants

In this section we prove that the constants appearing in (3) and (4) are the least possible. This clearly implies that the inequalities in Theorem 2 are also sharp.

The constant β_0 is optimal in (3). Suppose the inequality (5) is valid for all martingales f and their ±1-transforms g. By Theorem 4, the class $\mathcal{U}(\beta)$ is nonempty; let U_0 denote its minimal element. By definition, this function enjoys the following properties.

$$U_0(tx,ty,tz,tw)=tU_0(x,y,z,w),\quad \text{for } t>0,$$

and

$$U_0(x,y,z,w) = U_0(x,y+t,z,w+t) - t, \quad \text{for } t \in \mathbb{R}.$$

Introduce the functions A, $B : (-\infty, 0] \to \mathbb{R}$, $C : [0,1] \to \mathbb{R}$ by

$$A(y) = U_0(0,y,1,0),\ B(y) = U_0(1,y,1,0) = U_0(-1,y,1,0),\ C(x) = U_0(x,0,1,0).$$

For the convenience of the reader, the proof is split into a few parts.

Step 1. Let us start with an estimate which will be used several times. If $y < 0$, $\delta \in (0,1)$ and $t > -y$, then the property (10), with $x = z = 1$, $w = 0$, $t_1 = -\delta$, $t_2 = t$ and $\alpha = t/(t+\delta)$, yields

$$\begin{aligned} B(y) &\geq \frac{t}{t+\delta} U_0(1-\delta, y-\delta, 1, 0) + \frac{\delta}{t+\delta} U_0(1+t, y+t, 1, 0) \\ &= \frac{t}{t+\delta} U_0(1-\delta, y-\delta, 1, 0) + \frac{\delta(1+t)}{t+\delta} \left(\frac{y+t}{1+t} + U_0(1,0,1,0) \right). \end{aligned}$$

Now take $t \to \infty$ to obtain

$$B(y) \geq U_0(1-\delta, y-\delta, 1, 0) + \delta(1 + B(0)). \tag{35}$$

Step 2. For $x \in (0,1]$ and $\delta \in (0,x]$, the property (10), with $y = w = 0$, $z = 1$, $t_1 = 1-x$, $t_2 = -2\delta$ and $\alpha = 2\delta/(1-x+2\delta)$, gives

$$\begin{aligned} C(x) &\geq \frac{2\delta}{1-x+2\delta} U_0(1, x-1, 1, 0) + \frac{1-x}{1-x+2\delta} U_0(x-2\delta, \delta, 1, 0) \\ &= \frac{2\delta}{1-x+2\delta} B(x-1) + \frac{1-x}{1-x+2\delta} (C(x-2\delta) + \delta) \\ &\geq \frac{2\delta}{1-x+2\delta} B(x-1-2\delta) + \frac{1-x}{1-x+2\delta} (C(x-2\delta) + \delta), \end{aligned}$$

where the latter inequality follows from the fact that B is nondecreasing (by the very definition). In an equivalent form, the above reads

$$C(x) - C(x-2\delta) \geq 2\delta \left[\frac{B(x-1-2\delta)}{1-x+2\delta} - \frac{C(x-2\delta)}{1-x+2\delta} \right] + \frac{2\delta(1-x)}{1-x+2\delta}. \tag{36}$$

Furthermore, by (10), for $x \leq 1$ and $\delta \in (0,1)$,

$$U_0(1-\delta, x-1-\delta, 1, 0) \leq \frac{\delta}{1-x+2\delta} C(x-2\delta) + \frac{1-x+\delta}{1-x+2\delta} B(x-1-2\delta).$$

Combining this with (35) yields

$$B(x-1) \geq \delta(1+B(0)) + \frac{\delta}{1-x+2\delta} C(x-2\delta) + \frac{1-x+\delta}{1-x+2\delta} B(x-1-2\delta),$$

or

$$2B(x-1) - 2B(x-1-2\delta) \geq 2\delta \left[\frac{C(x-2\delta)}{1-x+2\delta} - \frac{B(x-1-2\delta)}{1-x+2\delta} \right] + 2\delta(1+B(0)).$$

Adding (36) to the estimate above gives

$$C(x) + 2B(x-1) - C(x-2\delta) - 2B(x-1-2\delta) \geq 2\delta(2+B(0)) - \frac{4\delta^2}{1-x+2\delta}. \tag{37}$$

Now fix an integer n, substitute $\delta = 1/(2n)$, $x = k/n$, $k = 1, 2, \ldots, n$ and sum these inequalities; we get

$$C(1) + 2B(0) - C(0) - 2B(-1) \geq 2 + B(0) - \frac{1}{n^2}\sum_{k=1}^{n} \frac{1}{1 - \frac{k-1}{n}}.$$

Passing to the limit $n \to \infty$ and using the equalities $C(1) = B(0)$, $C(0) = A(0)$ we arrive at

$$2B(0) - A(0) - 2B(-1) \geq 2. \tag{38}$$

Step 3. Now we will show that

$$A(0) \geq B(-1) + 1. \tag{39}$$

To do this, use the property (10) twice to obtain

$$\begin{aligned} A(0) &\geq \frac{\delta}{1+\delta} B(-1) + \frac{1}{1+\delta}(C(\delta) + \delta) \\ &\geq \frac{\delta}{1+\delta} B(-1) + \frac{1}{1+\delta}\big(\delta B(-1) + (1-\delta)(\delta + A(0)) + \delta\big), \end{aligned}$$

or, equivalently, $A(0) \geq B(-1) + 1 - \frac{\delta}{2}$. As δ is arbitrary, (39) follows.

Step 4. The property (10), used twice, yields

$$\begin{aligned} A(y-2\delta) &\geq \frac{\delta}{1+\delta} B(y-2\delta-1) + \frac{1}{1+\delta} U_0(-\delta, y-\delta, 1, 0) \\ &\geq \frac{\delta}{1+\delta} B(y-2\delta-1) + \frac{\delta}{1+\delta} B(y-1) + \frac{1-\delta}{1+\delta} A(y) \end{aligned} \tag{40}$$

if $\delta < 1$ and $y \leq 0$. Moreover, combining (35) for $y-1$ with the following consequence of (10):

$$U_0(1-\delta, y-1-\delta, 1, 0) \geq \delta A(y-2\delta) + (1-\delta) B(y-1-2\delta)$$

gives

$$B(y-1) \geq \delta A(y-2\delta) + (1-\delta) B(y-1-2\delta) + \delta(1 + B(0)). \tag{41}$$

Now multiply (40) by $1+\delta$ and add it to (41) to obtain

$$A(y-2\delta) - B(y-1-2\delta) \geq (1-\delta)(A(y) - B(y-1)) + \delta(1 + B(0)),$$

which, by induction, leads to the estimate

$$A(-2n\delta) - B(-2n\delta - 1) - 1 - B(0) \geq (1-\delta)^n (A(0) - B(-1) - 1 - B(0)),$$

valid for any nonnegative integer n. Fix $y < 0$, $\delta = -y/(2n)$ and let $n \to \infty$ to obtain

$$A(y) - B(y-1) - 1 - B(0) \geq e^{y/2}(A(0) - B(-1) - 1 - B(0)) \geq -B(0)e^{y/2}, \tag{42}$$

where the latter estimate follows from (39).

Step 5. Come back to (41) and write it in equivalent form

$$B(y-1) - B(y-1-2\delta) \geq \delta(A(y-2\delta) - B(y-1-2\delta)) + \delta(1 + B(0)).$$

By (42), we get

$$B(y-1) - B(y-1-2\delta) \geq \delta(-e^{y/2-\delta} B(0) + 2 + 2B(0)).$$

This gives, by induction,

$$B(-1)-B(-2n\delta-1)=\sum_{k=0}^{n-1}[B(-2k\delta-1)-B(-2k\delta-1-2\delta)]$$
$$\geq n\delta(2+2B(0))-\delta B(0)e^{-\delta}\frac{1-e^{-n\delta}}{1-e^{-\delta}}.$$

Now fix $y<0$, take $\delta=-y/(2n)$ and let $n\to\infty$ to obtain

$$B(-1)-B(y-1)\geq -y(1+B(0))-B(0)(1-e^{y/2}). \tag{43}$$

Now, by (38) and (39),

$$B(-1)=\frac{1}{3}B(-1)+\frac{2}{3}B(-1)\leq\frac{1}{3}A(0)+\frac{2}{3}B(-1)+\frac{1}{3}\leq\frac{2}{3}B(0)-1.$$

Furthermore, by the definition of B we have $B(y-1)\geq-\beta$. Plugging these estimates into (43) yields

$$\beta\geq -y(1+B(0))-B(0)(1-e^{y/2})+1-\frac{2}{3}B(0),\ \text{for all } y<0.$$

Note that we have $1+B(0)=U_0(1,1,1,1)\leq 0$, by definition of U_0. Therefore, the right hand side of the inequality above attains its maximum for y satisfying

$$e^{y/2}=\frac{2}{B(0)}+2$$

and we get

$$\beta\geq -2(1+B(0))\log\left(2+\frac{2}{B(0)}\right)+3+\frac{1}{3}B(0).$$

Now, the right hand side, as a function of $B(0)\in(-\infty,-1]$, attains its minimum β_0 at $B(0)=-3a$ (where a is given by (13)). This yields $\beta\geq\beta_0$ and we are done. □

The constant β_0^+ is optimal in (4). Suppose for any nonnegative martingale f and its ±1 transform g we have

$$||g^*||_1\leq\beta||f^*||_1.$$

Then the class $\mathscr{U}^+(\beta)$ is nonempty, so we may consider its minimal element U_0^+. As previously, we have

$$U_0^+(tx,ty,tz,tw)=tU_0^+(x,y,z,w)\quad\text{for } t>0, \tag{44}$$

and

$$U_0^+(x,y,z,w)=U_0^+(x,y+t,z,w+t)-t\quad\text{for } t\in\mathbb{R}.$$

In addition,

$$\text{the function } U_0^+(1,\cdot,1,0) \text{ is nondecreasing.} \tag{45}$$

It is convenient to work with the functions

$$A(y)=U_0^+\big(\frac{2}{3},y,1,0\big),\ B(y)=U_0^+(1,y,1,0),\ C(x)=U_0^+(x,0,1,0).$$

As previously, we divide the proof into a few intermediate steps.

Step 1. First let us note that the arguments leading to (37) are still valid (with the new functions A, B, C defined above) and hence so is this estimate. For a fixed positive integer n, let us write it for $\delta = 1/(6n)$, $x = \frac{2}{3} + 2k\delta$, $k = 1, 2, \ldots, n$ and sum all these inequalities to obtain

$$C(1) + 2B(0) - C\Big(\frac{2}{3}\Big) - 2B\Big(-\frac{1}{3}\Big) \geq \frac{1}{3}(2 + B(0)) - \frac{1}{9n^2}\sum_{k=1}^{n} \frac{1}{\frac{1}{3} - \frac{k-1}{3n}}.$$

Now let $n \to \infty$ and use $C(1) = B(0)$. We get

$$3B(0) \geq C\Big(\frac{2}{3}\Big) + 2B\Big(-\frac{1}{3}\Big) + \frac{1}{3}(2 + B(0)). \tag{46}$$

Step 2. We will show that

$$C\Big(\frac{2}{3}\Big) \geq \frac{2}{3}B\Big(-\frac{1}{3}\Big) + \frac{4}{9} - \frac{\beta}{3}. \tag{47}$$

To this end, note that, using (10) twice, for $\delta < 1/3$,

$$\begin{aligned} C\Big(\frac{2}{3}\Big) &\geq \frac{3\delta}{1+3\delta}B\Big(-\frac{1}{3}\Big) + \frac{1}{1+3\delta}\left[\delta + C\Big(\frac{2}{3} - \delta\Big)\right] \\ &\geq \frac{3\delta}{1+3\delta}B\Big(-\frac{1}{3}\Big) + \frac{1}{1+3\delta}\left\{\delta + \frac{3\delta}{2}(-\beta) + \frac{2-3\delta}{2}\left[\delta + C\Big(\frac{2}{3}\Big)\right]\right\}. \end{aligned}$$

This is equivalent to

$$C\Big(\frac{2}{3}\Big) \geq \frac{2}{3}B\Big(-\frac{1}{3}\Big) + \frac{2}{9}\Big(2 - \frac{3}{2}\delta\Big) - \frac{\beta}{3}$$

and it suffices to let $\delta \to 0$.
Step 3. By (35), we have, for $y < -1/3$,

$$B(y) \geq U_0^+(1-\delta, y-\delta, 1, 0) + \delta(1 + B(0)).$$

Furthermore, again by (10),

$$U_0^+(1-\delta, y-\delta, 1, 0) \geq (1-3\delta)B(y-2\delta) + 3\delta A\Big(y + \frac{1}{3} - 2\delta\Big)$$

and hence

$$B(y) \geq (1-3\delta)B(y-2\delta) + 3\delta A\Big(y + \frac{1}{3} - 2\delta\Big) + \delta(1 + B(0)). \tag{48}$$

Moreover,

$$\begin{aligned} A\Big(y + \frac{1}{3} - 2\delta\Big) &\geq \frac{3\delta}{2+3\delta}U_0^+\Big(0, y - \frac{1}{3} - 2\delta, 1, 0\Big) + \frac{2}{2+3\delta}U_0^+\Big(\frac{2}{3} + \delta, y + \frac{1}{3} - \delta, 1, 0\Big) \\ &\geq \frac{3\delta}{2+3\delta}(-\beta) + \frac{2}{2+3\delta}\left[3\delta B(y) + (1-3\delta)A\Big(y + \frac{1}{3}\Big)\right]. \end{aligned} \tag{49}$$

Step 4. Now we will combine (48) and (49) and use them several times. Multiply (49) by $\gamma > 0$ (to be specified later) and add it to (48). We obtain

$$\begin{aligned} &B(y)\cdot\left(1 - \frac{6\gamma\delta}{2+3\delta}\right) - A\Big(y + \frac{1}{3}\Big)\cdot\frac{(2-6\delta)\gamma}{2+3\delta} \\ &\geq B(y-2\delta)\cdot(1-3\delta) - A\Big(y + \frac{1}{3} - 2\delta\Big)\cdot(\gamma - 3\delta) + \delta\left(1 + B(0) - \frac{3\beta\gamma}{2+3\delta}\right) \\ &\geq B(y-2\delta)\cdot(1-3\delta) - A\Big(y + \frac{1}{3} - 2\delta\Big)\cdot(\gamma - 3\delta) + \delta\left(1 + B(0) - \frac{3\beta\gamma}{2}\right). \end{aligned}$$

Now the choice $\gamma = (5-\sqrt{9-24\delta})/4$ allows to write the inequality above in the form

$$F(y) \geq Q_\delta F(y-2\delta) + \delta\left(1 + B(0) - \frac{3\beta\gamma}{2}\right), \tag{50}$$

where

$$F(y) = B(y)\cdot\left(1 - \frac{6\gamma\delta}{2+3\delta}\right) - A(y+\frac{1}{3})\cdot\frac{(2-6\delta)\gamma}{2+3\delta}$$

and

$$Q_\delta = \frac{1-3\delta}{1-\frac{6\gamma\delta}{2+3\delta}}.$$

The inequality (50), by induction, leads to

$$F(-1/3) \geq Q_\delta^n F(-1/3-2n\delta) + \delta\left(1+B(0)-\frac{3\beta\gamma}{2}\right)\cdot\frac{Q_\delta^n-1}{Q_\delta-1}.$$

Now fix $Y < -1/3$, take $\delta = -(Y+1/3)/(2n)$ and let $n \to \infty$. Then

$$\gamma \to \frac{1}{2}, \quad Q_\delta^n \to \exp\left(\frac{3}{4}(Y+\frac{1}{3})\right)$$

and the estimate yields

$$\begin{aligned} B(-\frac{1}{3}) - \frac{1}{2}A(0) \geq \exp\left(\frac{3}{4}(Y+\frac{1}{3})\right)(B(Y) - \frac{1}{2}A(Y+\frac{1}{3})) \\ -\frac{2}{3}\left(1+B(0)-\frac{3\beta}{4}\right)\left[\exp\left(\frac{3}{4}(Y+\frac{1}{3})\right)-1\right]. \end{aligned}$$

Now we have $B(Y) \geq -\beta$ and $A(Y+\frac{1}{3}) \leq A(0)$. Hence, letting $Y \to -\infty$ yields

$$B(-\frac{1}{3}) - \frac{1}{2}A(0) \geq \frac{2}{3}\left(1+B(0)-\frac{3\beta}{4}\right). \tag{51}$$

Now (46), (47) and (51) imply the desired inequality $\beta \geq \beta_0^+$: indeed, by (46),

$$\frac{8}{3}B(0) \geq A(0) + 2B(-\frac{1}{3}) + \frac{2}{3},$$

which, by (51), can be bounded from below by

$$2A(0) + 2 + \frac{4}{3}B(0) - \beta.$$

Now, (47) and (51) give

$$\frac{2}{3}A(0) \geq \frac{8}{9} + \frac{4}{9}B(0) - \frac{2\beta}{3}$$

and applying the previous estimate yields $\beta \geq 14/9$. The proof is complete. □

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