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A NOTE ON ERGODIC TRANSFORMATIONS OF SELF-SIMILAR VOLTERRA GAUSSIAN PROCESSES

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Abstract
We derive a class of ergodic transformations of self-similar Gaussian processes that are Volterra, i.e. of type $X_t = \int_0^t z_X(t,s)dW_s$, $t \in [0,\infty)$, where z_X is a deterministic kernel and W is a standard Brownian motion.

1 Introduction

Let $(X_t)_{t\in[0,\infty)}$ be a continuous *Volterra* Gaussian process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. This means that

$$X_t \;=\; \int_0^t z_X(t,s)dW_s,\; a.s.,\; t \in [0,\infty), \tag{1.1}$$

where the kernel $z_X \in L^2_{\mathrm{loc}}\left([0,\infty)^2\right)$ is *Volterra*, i.e. $z_X(t,s) = 0,\ s \geq t$, and $(W_t)_{t\in[0,\infty)}$ is a standard Brownian motion. Clearly, X is centered and

$$R^X(s,t) \;:=\; \mathrm{Cov}_{\mathbb{P}}\left(X_s, X_t\right) \;=\; \int_0^s z_X(s,u)z_X(t,u)du,\; 0 \leq s \leq t < \infty.$$

We assume that X is *β-self-similar* for some $\beta > 0$, i.e.

$$\left(X_{at}\right)_{t\in[0,\infty)} \;\stackrel{d}{=}\; \left(a^\beta X_t\right)_{t\in[0,\infty)},\; a > 0, \tag{1.2}$$

where $\stackrel{d}{=}$ denotes equality of finite-dimensional distributions. Furthermore, we assume that z_X is *non-degenerate* in the sense that the family $\{z_X(t,\cdot)\,|\,t \in (0,\infty)\}$ is linearly independent and generates a dense subspace of $L^2\left([0,\infty)\right)$. Then

$$\Gamma_t(X) \;:=\; \overline{\mathrm{span}\{X_s\,|\,s \in [0,t]\}} \;=\; \Gamma_t(W),\; t \in (0,\infty), \tag{1.3}$$

where the closure is in $L^2(\mathbb{P})$, or equivalently,

$$\mathbb{F}^X \;=\; \mathbb{F}^W,$$

where $\mathbb{F}^X := \left(\mathcal{F}_t^X\right)_{t\in[0,\infty)}$ denotes the completed natural filtration of X.

We assume implicitly that $(\Omega, \mathcal{F}, \mathbb{P})$ is the *coordinate space of* X, which means that $\Omega = \{\omega : [0,\infty) \to \mathbb{R} \,|\, \omega \text{ is continuous}\}$, $\mathcal{F} = \mathcal{F}_\infty^X := \sigma(X_t \,|\, t \in [0,\infty))$ and $\mathbb{P}$ is the probability measure with respect to which the *coordinate process* $X_t(\omega) = \omega(t)$, $\omega \in \Omega$, $t \in [0,\infty)$, is a centered Gaussian process with covariance function R^X. Recall that a measurable map

$$\begin{array}{rcl} \mathcal{Z} \;:\; (\Omega,\mathcal{F},\mathbb{P}) & \to & (\Omega,\mathcal{F},\mathbb{P}) \\ X(\omega) & \mapsto & \mathcal{Z}(X(\omega)) \end{array}$$

is a *measure-preserving transformation*, or *endomorphism, on* $(\Omega, \mathcal{F}, \mathbb{P})$ if $\mathbb{P}^{\mathcal{Z}} = \mathbb{P}$, or equivalently, if $\mathcal{Z}(X) \overset{d}{=} X$. If $\mathcal{Z}$ is also bijective and $\mathcal{Z}^{-1}$ is measurable, then it is an *automorphism.*

Processes of the above type are a natural generalization of the nowadays in connection with finance and telecommunications extensively studied *fractional Brownian motion with Hurst index* $H \in (0,1)$, or H*-fBm.* The H-fBm, denoted by $\left(B_t^H\right)_{t\in[0,\infty)}$, is the continuous, centered Gaussian process with covariance function

$$R^{B^H}(s,t) \;=\; \frac{1}{2}\left(s^{2H} \;+\; t^{2H} \;-\; |s-t|^{2H}\right),\; s,t \in [0,\infty).$$

For $H = \frac{1}{2}$, fBm is standard Brownian motion. H-fBm is H-self-similar and has stationary increments. The non-degenerate Volterra kernel is given by

$$z_{B^H}(t,s) \;=\; c(H)(t-s)^{H-\frac{1}{2}} \cdot {}_2F_1\left(\frac{1}{2}-H, H-\frac{1}{2}, H+\frac{1}{2}, 1-\frac{t}{s}\right),\; 0<s<t<\infty,$$

where $c(H) := \left(\frac{2H\Gamma\left(\frac{3}{2}-H\right)}{\Gamma\left(H+\frac{1}{2}\right)\Gamma(2-2H)}\right)^{\frac{1}{2}}$ with Γ denoting the Gamma function, and ${}_2F_1$ is the Gauss hypergeometric function. In 2003, Molchan (see [8]) showed that the transformation

$$\mathcal{Z}_t\left(B^H\right) \;:=\; B_t^H \;-\; 2H\int_0^t \frac{B_s^H}{s}ds,\; t\in[0,\infty), \tag{1.4}$$

is measure-preserving and satisfies

$$\Gamma_T\left(\mathcal{Z}\left(B^H\right)\right) \;=\; \Gamma_T\left(Y^H\right),\; T>0, \tag{1.5}$$

where

$$Y_t^H \;:=\; M_t^H \;-\; \frac{t}{T}\xi_T^H,\; t\in[0,T]. \tag{1.6}$$

Here, $M_t^H := \sqrt{2-2H}\int_0^t s^{\frac{1}{2}-H}dW_s$, $t \in [0,\infty)$, is the *fundamental martingale of* B^H and $\xi_T^H := 2H\int_0^T \left(\frac{s}{T}\right)^{2H-1} dM_s^H$.

In this work, we present a class of measure-preserving transformations (on the coordinate space) of X, which generalizes this result. Moreover, we show that these measure-preserving transformations are ergodic.

2 Ergodic transformations

First, we introduce the class of measure-preserving transformations:

Theorem 2.1. *Let $\alpha > \frac{-1}{2}$. Then the transformation*

$$\mathcal{Z}_t^{\alpha}(X) := X_t - (2\alpha+1)t^{\beta-\alpha-\frac{1}{2}} \int_0^t s^{\alpha-\beta-\frac{1}{2}} X_s ds, \ t \in [0,\infty), \tag{2.1}$$

is an automorphism on the coordinate space of X. The inverse is given by

$$\mathcal{Z}_t^{\alpha,-1}(X) = X_t - (2\alpha+1)t^{\alpha+\beta+\frac{1}{2}} \int_t^{\infty} X_s s^{-\beta-\alpha-\frac{3}{2}} ds, \ a.s., \ t \in [0,\infty). \tag{2.2}$$

The integrals on the right-hand sides are $L^2(\mathbb{P})$-limits of Riemann sums.

Proof. First, note that R^X is continuous. Furthermore, by combining Hölder's inequality and (1.2), we have that

$$\left|R^X(s,t)\right| \leq \mathrm{E}_{\mathbb{P}}(X_1)^2 s^{\beta} t^{\beta}, \ s,t \in (0,\infty).$$

Hence, the double Riemann integrals

$$\int_0^t \int_0^t (us)^{\alpha-\beta-\frac{1}{2}} R^X(u,s) du ds$$

and

$$\int_t^{\infty} \int_t^{\infty} (us)^{-\beta-\alpha-\frac{3}{2}} R^X(u,s) du ds$$

are finite. Thus, the integrals in (2.1) and (2.2) are well-defined (see [5], section 1).
Second, we show that $\mathcal{Z}^{\alpha}$ is a measure-preserving transformation. Let $Y_t := \exp(-\beta t) X_{\exp(t)}$ and $Y_t^{\alpha} := \exp(-\beta t)\mathcal{Z}^{\alpha}_{\exp(t)}(X)$, $t \in \mathbb{R}$, denote the Lamperti transforms of X and $\mathcal{Z}^{\alpha}(X)$, respectively. Hence, the process $(Y_t)_{t\in\mathbb{R}}$ is stationary. By substituting $v := \ln(s)$, we obtain that

$$\begin{aligned}
Y_t^{\alpha} &= \exp(-\beta t)\left(X_{\exp(t)} - (2\alpha+1)\exp\left(\left(\beta-\alpha-\frac{1}{2}\right)t\right)\int_0^{\exp(t)} s^{\alpha-\beta-\frac{1}{2}} X_s ds\right) \\
&= Y_t - (2\alpha+1)\exp\left(\left(-\alpha-\frac{1}{2}\right)t\right)\int_{-\infty}^{t} \exp\left(v\left(\alpha-\beta+\frac{1}{2}\right)\right) X_{\exp(v)} dv \\
&= Y_t - (2\alpha+1)\exp\left(\left(-\alpha-\frac{1}{2}\right)t\right)\int_{-\infty}^{t} \exp\left(v\left(\alpha+\frac{1}{2}\right)\right) Y_v dv \\
&= \int_{-\infty}^{\infty} h^{\alpha}(t-v) Y_v dv, \ a.s., \ t \in \mathbb{R},
\end{aligned}$$

where

$$h^{\alpha}(x) := \delta_0(x) - (2\alpha+1)1_{(0,\infty)}(x)\exp\left(-\left(\alpha+\frac{1}{2}\right)x\right), \ x \in \mathbb{R}.$$

Thus, Y^{α} is a linear, non-anticipative, time-invariant transformation of Y. The spectral distribution function of Y^{α} is given by (see [13], p. 151)

$$dF^{\alpha}(\lambda) = \left|H^{\alpha}(\lambda)\right|^2 dF(\lambda), \ \lambda \in \mathbb{R},$$

where $H^\alpha(\lambda) := \int_{\mathbb{R}} \exp(-i\lambda x)h^\alpha(x)dx$ denotes the Fourier transform of h^α and F is the spectral distribution function of Y. We have that (see [3], p. 14 and p. 72)

$$\left|H^\alpha(\lambda)\right| = \left|1-(2\alpha+1)\left(\frac{\alpha+\frac{1}{2}-i\lambda}{\left(\frac{1}{2}+\alpha\right)^2+\lambda^2}\right)\right| = \left|\frac{-\frac{1}{2}-\alpha+i\lambda}{\frac{1}{2}+\alpha+i\lambda}\right| = 1,\ \lambda\in\mathbb{R},$$

i.e. $F^\alpha \equiv F$. It follows from this that $(Y_t^\alpha)_{t\in\mathbb{R}} \overset{d}{=} (Y_t)_{t\in\mathbb{R}}$, or equivalently, $(\mathcal{Z}_t^\alpha(X))_{t\in[0,\infty)} \overset{d}{=} (X_t)_{t\in[0,\infty)}$.
Third, by splitting integrals and using Fubini's theorem, we obtain that

$$\mathcal{Z}_t^{\alpha,-1}\left(\mathcal{Z}^\alpha(X)\right) = X_t = \mathcal{Z}_t^\alpha\left(\mathcal{Z}^{\alpha,-1}(X)\right),\ a.s.,\ t\in[0,\infty). \quad \square$$

Remark 2.2. Theorem 2.1 generalizes (1.4). In fact, $\mathcal{Z}^{H-\frac{1}{2}}\left(B^H\right) = \mathcal{Z}\left(B^H\right)$, $H\in(0,1)$.

Remark 2.3. Theorem 2.1 holds true for general continuous centered β-self-similar Gaussian processes.

Next, we present two auxiliary lemmas concerning the structure of z_X:

Lemma 2.4. *Let $(X_t)_{t\in[0,\infty)}$ be a Volterra Gaussian process with a non-degenerate Volterra kernel z_X. Then the following are equivalent:*
1. X is β-self-similar, i.e.

$$\int_0^s z_X(at,au)z_X(as,au)du = a^{2\beta-1}\int_0^s z_X(t,u)z_X(s,u)du,\ 0<s\le t<\infty,\ a>0.$$

2. It holds that

$$z_X(at,as) = a^{\beta-\frac{1}{2}}z_X(t,s),\ 0<s<t<\infty,\ a>0.$$

3. There exists $F_X\in L^2\left((0,1),(1-x)^{2\beta-1}dx\right)$ such that

$$z_X(t,s) = (t-s)^{\beta-\frac{1}{2}}F_X\left(\frac{s}{t}\right),\ 0<s<t<\infty.$$

Proof. $1\Rightarrow 2$: For $a>0$, let $z_{Y(a)}(t,s) := a^{\frac{1}{2}-\beta}z_X(at,as)$, $0<s<t<\infty$, and let $Y_t(a) := \int_0^t z_{Y(a)}(t,s)dW_s$, $t\in[0,\infty)$. Clearly, $z_{Y(a)}$ is non-degenerate. From (1.3), we obtain that $\Gamma_t(Y(a)) = \Gamma_t(W)$, $t\in(0,\infty)$. From part 1, it follows that $X \overset{d}{=} Y(a)$. Thus, the process $W'_t := \int_0^t z_X^*(t,s)dY_s(a)$, $t\in[0,\infty)$, where z_X^* is the reciprocal of z_X and the integral is an abstract Wiener integral, is a standard Brownian motion with $\Gamma_t(W') = \Gamma_t(Y(a))$, $t\in(0,\infty)$. Hence, $\Gamma_t(W) = \Gamma_t(W')$, i.e. W and W' are indistinguishable. Therefore, $Y_t(a) = \int_0^t z_X(t,s)dW_s$, a.s., $t\in(0,\infty)$, i.e. $Y_t(a) = X_t$, a.s., $t\in[0,\infty)$. In particular, $0 = \mathrm{E}_{\mathbb{P}}(Y_t(a)-X_t)^2 = \int_0^t\left(z_{Y(a)}(t,s)-z_X(t,s)\right)^2ds$, $t\in(0,\infty)$. Thus, $z_X(t,\cdot)\equiv z_{Y(a)}(t,\cdot)$, $t\in(0,\infty)$.
$2\Rightarrow 3$: Let $G_X(t,s) := (t-s)^{\frac{1}{2}-\beta}z_X(t,s)$, $0<s<t<\infty$. From part 2, it follows that $G_X(at,as) = G_X(t,s)$, $0<s<t<\infty$, $a>0$. Hence, for every (t,s), $s<t$, the function G_X is constant on the line $\{(at,as)\,|\,a\in(0,\infty)\}$, which depends only on the slope $\frac{s}{t}$. Thus, $G_X(t,s) = F_X\left(\frac{s}{t}\right)$, $0<s<t<\infty$, for some $F_X\in L^2\left((0,1),(1-x)^{2\beta-1}dx\right)$.
$3\Rightarrow 1$: This is trivial. $\square$

Lemma 2.5. *Let $\alpha > \frac{-1}{2}$. Then we have that*

$$t^{\beta-\alpha-\frac{1}{2}}\int_s^t u^{\alpha-\beta-\frac{1}{2}}z_X(u,s)du = s^\alpha\int_s^t z_X(t,u)u^{-\alpha-1}du,\ 0<s<t<\infty.$$

Proof. From Lemma 2.4, it follows that the Volterra kernel of X can be written as

$$z_X(t,s) \;=\; (t-s)^{\beta-\frac{1}{2}} F_X\left(\frac{s}{t}\right),\; 0<s<t<\infty,$$

for some function F_X. By substituting first $x := \frac{s}{u}$ and then $v := tx$, we obtain that

$$\begin{aligned}
t^{\beta-\alpha-\frac{1}{2}} \int_s^t u^{\alpha-\beta-\frac{1}{2}} z_X(u,s)du &= t^{\beta-\alpha-\frac{1}{2}} \int_s^t u^{\alpha-\beta-\frac{1}{2}} (u-s)^{\beta-\frac{1}{2}} F_X\left(\frac{s}{u}\right) du \\
&= t^{\beta-\alpha-\frac{1}{2}} s^{\alpha} \int_{\frac{s}{t}}^1 x^{-\alpha-1}(1-x)^{\beta-\frac{1}{2}} F_X(x)dx \\
&= s^{\alpha} \int_s^t (t-v)^{\beta-\frac{1}{2}} F_X\left(\frac{v}{t}\right) v^{-\alpha-1}dv \\
&= s^{\alpha} \int_s^t z_X(t,v)v^{-\alpha-1}dv. \quad \square
\end{aligned}$$

The next lemma is the key result for deriving the ergodicity of the measure-preserving transformations:

Lemma 2.6. *Let $\alpha > \frac{-1}{2}$. Then*

$$\mathcal{Z}_t^{\alpha}(X) \;=\; \int_0^t z_X(t,s)d\mathcal{Z}_s^{\alpha}(W),\; a.s.,\; t\in[0,\infty).$$

Proof. By combining (1.1) and the stochastic Fubini theorem, using Lemma 2.5, again the stochastic Fubini theorem, and finally using partial integration, we obtain that

$$\begin{aligned}
\mathcal{Z}_t^{\alpha}(X) &= X_t \;-\; (2\alpha+1)\int_0^t \left(t^{\beta-\alpha-\frac{1}{2}} \int_u^t s^{\alpha-\beta-\frac{1}{2}} z_X(s,u)ds\right) dW_u \\
&= X_t \;-\; (2\alpha+1)\int_0^t \left(u^{\alpha} \int_u^t z_X(t,s)s^{-\alpha-1}ds\right) dW_u \\
&= X_t \;-\; (2\alpha+1)\int_0^t z_X(t,s)s^{-\alpha-1} \int_0^s u^{\alpha}dW_u ds \\
&= X_t \;-\; (2\alpha+1)\int_0^t z_X(t,s) \left((-\alpha)s^{-\alpha-1}\int_0^s u^{\alpha-1}W_u du ds + s^{-1}W_s ds\right) \\
&= \int_0^t z_X(t,s)d\mathcal{Z}_s^{\alpha}(W),\; a.s.,\; t\in(0,\infty). \quad \square
\end{aligned}$$

In the following, let $\mathcal{Z}^{\alpha,n} := (\mathcal{Z}^{\alpha})^n$ denote the n-th iterate of $\mathcal{Z}^{\alpha}$, $n \in \mathbb{Z}$. Also, let

$$\Gamma_\infty(X) \;:=\; \overline{\operatorname{span}\{X_t \mid t\in[0,\infty)\}}.$$

For $\alpha > -\frac{1}{2}$, let

$$N_t^{\alpha} \;:=\; \int_0^t s^{\alpha}dW_s,\; t\in[0,\infty).$$

Clearly, N^{α} is an $\left(\alpha+\frac{1}{2}\right)$-self-similar $\mathbb{F}^X$-martingale. From Lemma 2.6, it follows that

$$\mathcal{Z}_t^{\alpha}(X) \;=\; \int_0^t z_X(t,s)s^{-\alpha}d\mathcal{Z}_s^{\alpha}\left(N^{\alpha}\right),\; a.s.,\; t\in(0,\infty). \tag{2.3}$$

The next lemma is an auxiliary result, which was obtained in [7], section 3.2 and Theorem 5.2. (The automorphism $\mathcal{Z}^\alpha$ on the coordinate space of N^α here corresponds to the (ergodic) automorphism $\mathcal{T}^{(1)}$ on the coordinate space of the martingale M with $M := N^\alpha$ in [7].)

Lemma 2.7. *Let $\alpha > \frac{-1}{2}$ and $T > 0$.*
1. It holds that

$$\Gamma_T\left(\mathcal{Z}^\alpha\left(N^\alpha\right)\right) \;=\; \Gamma_T\left(N^{\alpha,T}\right),$$

where $N_t^{\alpha,T} := N_t^\alpha - \left(\frac{t}{T}\right)^{2\alpha+1} N_T^\alpha$, $t \in [0,T]$, is a bridge of N^α, i.e. a process satisfying $\mathrm{Law}_{\mathbb{P}}\left(N^{\alpha,T}\right) = \mathrm{Law}_{\mathbb{P}}\left(N^\alpha \mid N_T^\alpha = 0\right)$*, and* $\Gamma_T\left(N^{\alpha,T}\right) := \overline{\mathrm{span}\{N_t^{\alpha,T} \mid t \in [0,T]\}}$.
2. We have that

$$\Gamma_T\left(N^\alpha\right) \;=\; \overline{\perp_{n\in\mathbb{N}_0} \mathrm{span}\{\mathcal{Z}_T^{\alpha,n}\left(N^\alpha\right)\}} \tag{2.4}$$

and

$$\Gamma_\infty\left(N^\alpha\right) \;=\; \overline{\perp_{n\in\mathbb{Z}} \mathrm{span}\{\mathcal{Z}_T^{\alpha,n}\left(N^\alpha\right)\}}. \tag{2.5}$$

Here, $\perp$ denotes the orthogonal direct sum.

By combining (2.3) and part 1 of Lemma 2.7, we obtain the following:

Lemma 2.8. *Let $\alpha > \frac{-1}{2}$ and $T > 0$. Then*

$$\Gamma_T\left(\mathcal{Z}^\alpha\left(X\right)\right) \;=\; \Gamma_T\left(N^{\alpha,T}\right).$$

Remark 2.9. Lemma 2.8 is a generalization of identity (1.5). Indeed, we have that $Y_t^H = \sqrt{2-2H}\int_0^t s^{1-2H} dN_s^{H-\frac{1}{2},T}$, a.s., $t \in [0,T]$, where Y^H is the process defined in (1.6). Y^H is a bridge (of some process) if and only if $H = \frac{1}{2}$.

The following generalizes part 2 of Lemma 2.7:

Lemma 2.10. *Let $\alpha > \frac{-1}{2}$ and $T > 0$. Then we have that*

$$\Gamma_T\left(X\right) \;=\; \overline{\oplus_{n\in\mathbb{N}_0} \mathrm{span}\left\{\mathcal{Z}_T^{\alpha,n}(X)\right\}}$$

and

$$\Gamma_\infty\left(X\right) \;=\; \overline{\oplus_{n\in\mathbb{Z}} \mathrm{span}\left\{\mathcal{Z}_T^{\alpha,n}(X)\right\}}.$$

Proof. We assume that $X \neq N^\alpha$. By iterating (2.3) and (2.4), we obtain that

$$\mathcal{Z}_T^{\alpha,n}(X) \;\in\; \Gamma_T\left(\mathcal{Z}^{\alpha,n}\left(X\right)\right) = \;\Gamma_T\left(\mathcal{Z}^{\alpha,n}\left(N^\alpha\right)\right) \;=\; \overline{\perp_{i\geq n} \mathrm{span}\left\{\mathcal{Z}_T^{\alpha,i}\left(N^\alpha\right)\right\}},\; n \in \mathbb{Z}.$$

Moreover, $X_T \not\perp N_T^\alpha$, hence $\mathcal{Z}_T^{\alpha,n}(X) \not\perp \mathcal{Z}_T^{\alpha,n}\left(N^\alpha\right)$, $n \in \mathbb{Z}$, and therefore,

$$\mathcal{Z}_T^{\alpha,n}(X) \;\notin\; \overline{\perp_{i\geq n+1} \mathrm{span}\left\{\mathcal{Z}_T^{\alpha,i}\left(N^\alpha\right)\right\}},\; n \in \mathbb{Z}.$$

From (2.4) and (2.5), it follows that the systems $\left\{\mathcal{Z}_T^{\alpha,n}(X)\right\}_{n\in\mathbb{N}_0}$ and $\left\{\mathcal{Z}_T^{\alpha,n}(X)\right\}_{n\in\mathbb{Z}}$ are free and complete in $\Gamma_T(X)$ and $\Gamma_\infty(X)$, respectively. ☐

Remark 2.11. The process X is an $\mathbb{F}^X$-Markov process if and only if there exists $\alpha > \frac{-1}{2}$ and a constant $c(X)$, such that

$$X_t \;=\; c(X)\cdot t^{\beta-\frac{1}{2}-\alpha}\int_0^t s^\alpha dW_s,\; a.s.,\; t \in (0,\infty). \tag{2.6}$$

The free complete system $\left\{\mathcal{Z}_T^{\alpha,n}(X)\right\}_{n\in\mathbb{Z}}$ is orthogonal if and only if (2.6) is satisfied.

From Lemma 2.10, we obtain the following:

Corollary 2.12. *Let* $\alpha > \frac{-1}{2}$ *and* $T > 0$. *Then*

$$\mathcal{F}_T^X = \bigvee_{n\in\mathbb{N}_0} \sigma\big(\mathcal{Z}_T^{\alpha,n}(X)\big).$$

Furthermore,

$$\mathcal{F} = \mathcal{F}_\infty^X = \bigvee_{n\in\mathbb{Z}} \sigma\big(\mathcal{Z}_T^{\alpha,n}(X)\big).$$

Recall that an automorphism $\mathcal{Z}$ is a *Kolmogorov automorphism*, if there exists a σ-algebra $\mathcal{A} \subseteq \mathcal{F}$, such that $\mathcal{Z}^{-1}\mathcal{A} \subseteq \mathcal{A}$, $\vee_{m\in\mathbb{Z}}\mathcal{Z}^m\mathcal{A} = \mathcal{F}$ and $\cap_{m\in\mathbb{N}_0}\mathcal{Z}^{-m}\mathcal{A} = \{\Omega, \emptyset\}$. A Kolmogorov automorphism is strongly mixing and hence ergodic (see [11], Propositions 5.11 and 5.9 on p. 63 and p. 62). The ergodicity of $\mathcal{Z}^\alpha$ is hence a consequence of the following:

Theorem 2.13. *Let* $\alpha > \frac{-1}{2}$ *and* $T > 0$. *The automorphisms* $\mathcal{Z}^\alpha$ *and* $\mathcal{Z}^{\alpha,-1}$ *are Kolmogorov automorphisms with* $\mathcal{A} = \vee_{n\in-\mathbb{N}}\sigma\big(\mathcal{Z}_T^{\alpha,n}(X)\big)$ *and* $\mathcal{A} = \mathcal{F}_T^X$, *respectively.*

Proof. $\mathcal{Z}^\alpha$ is a Kolmogorov automorphism with $\mathcal{A} = \vee_{n\in-\mathbb{N}}\sigma\big(\mathcal{Z}_T^{\alpha,n}(X)\big)$:
First, $\mathcal{Z}^{\alpha,-1}\mathcal{A} = \vee_{n\in-\mathbb{N}}\sigma\big(\mathcal{Z}_T^{\alpha,n-1}(X)\big) \subseteq \mathcal{A}$.
Second, $\vee_{m\in\mathbb{Z}}\mathcal{Z}^{\alpha,m}\mathcal{A} = \vee_{m\in\mathbb{Z}}\vee_{n\in-\mathbb{N}}\sigma\big(\mathcal{Z}_T^{\alpha,m+n}(X)\big) = \mathcal{F}$.
Third, let $\{Y_n\}_{n\in-\mathbb{N}}$ denote the Hilbert basis of $\overline{\oplus_{n\in-\mathbb{N}}\operatorname{span}\{\mathcal{Z}_T^{\alpha,n}(X)\}}$ which is obtained from $\{\mathcal{Z}_T^{\alpha,n}(X)\}_{n\in-\mathbb{N}}$ via Gram-Schmidt orthonormalization. By using Kolmogorov's zero-one law (see [12], p. 381), we obtain that $\cap_{m\in\mathbb{N}_0}\mathcal{Z}^{\alpha,-m}\mathcal{A} = \cap_{m\in\mathbb{N}_0}\big(\vee_{n\le-m-1}\sigma\big(\mathcal{Z}_T^{\alpha,n}(X)\big)\big) = \cap_{m\in\mathbb{N}_0}\big(\vee_{n\le-m-1}\sigma\big(Y_n\big)\big) = \{\Omega, \emptyset\}$.
Similarly, one shows that $\mathcal{Z}^{\alpha,-1}$ is a Kolmogorov automorphism with $\mathcal{A} = \mathcal{F}_T^X$. $\square$

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References

[1] Deheuvels, P., Invariance of Wiener processes and of Brownian bridges by integral transforms and applications. Stochastic Processes and their Applications 13, 311-318, 1982. MR0671040

[2] Embrechts, P., Maejima, M., Selfsimilar Processes. Princeton University Press, 2002. MR1920153

[3] Erdélyi, A., Magnus, W., Oberhettinger, F., Tricomi, F.G., Tables of integral transforms, Volume 1. McGraw-Hill Book Company, Inc., 1954.

[4] Hida, T., Hitsuda, M., Gaussian Processes. Translations of Mathematical Monographs 120, American Mathematical Society, 1993. MR1216518

[5] Huang, S.T., Cambanis, S., Stochastic and Multiple Wiener Integrals for Gaussian Processes. Annals of Probability 6(4), 585-614, 1978. MR0496408

[6] Jeulin, T., Yor, M., Filtration des ponts browniens et équations différentielles stochastiques linéaires. Séminaire de probabilités de Strasbourg 24, 227-265, 1990. MR1071543

[7] Jost, C., Measure-preserving transformations of Volterra Gaussian processes and related bridges. Available from http://www.arxiv.org/pdf/math.PR/0701888.

[8] Molchan, G.M., Linear problems for a fractional Brownian motion: group approach. Theory of Probability and its Applications, 47(1), 69-78, 2003. MR1978695

[9] Norros, I., Valkeila, E., Virtamo, J., An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions. Bernoulli 5(4), 571-587, 1999. MR1704556

[10] Peccati, G., Explicit formulae for time-space Brownian chaos. Bernoulli 9(1), 25-48, 2003. MR1963671

[11] Petersen, K., Ergodic theory. Cambridge University Press, 1983. MR0833286

[12] Shiryaev, A.N., Probability, Second Edition. Springer, 1989. MR1024077

[13] Yaglom, A.M., Correlation Theory of Stationary and Related Random Functions, Volume 1: Basic Results. Springer, 1987. MR0893393