Elect. Comm. in Probab. **14** (2009), 31–35

ELECTRONIC COMMUNICATIONS in PROBABILITY

UNIQUENESS OF THE MIXING MEASURE FOR A RANDOM WALK IN A RANDOM ENVIRONMENT ON THE POSITIVE INTEGERS

MAREN ECKHOFF
Zentrum Mathematik, Technische Universität München, D-85747 Garching bei München, Germany.
email: `maren.eckhoff@mytum.de`

SILKE W.W. ROLLES
Zentrum Mathematik, Bereich M5, Technische Universität München, D-85747 Garching bei München, Germany.
email: `srolles@ma.tum.de`

Submitted 14 July, 2008, *accepted in final form* November 18, 2008

AMS 2000 Subject classification: Primary 60K37, secondary 60K35
Keywords: random walk in a random environment, mixing measure

Abstract
Consider a random walk in an irreducible random environment on the positive integers. We prove that the annealed law of the random walk determines uniquely the law of the random environment. An application to linearly edge-reinforced random walk is given.

1 Introduction and results

Random walk in a random environment. Let $G=(\mathbb{N}_0,E)$ be the graph with vertex set $\mathbb{N}_0$ and set of undirected edges $E=\{\{n,n+1\},\,n\in\mathbb{N}_0\}$. We consider random walk in a random environment on G defined as follows: Let $\Omega_0\subseteq\mathbb{N}_0^{\mathbb{N}_0}$ denote the set of all nearest-neighbor paths in G starting in 0. We endow Ω_0 with the sigma-field $\mathscr{F}$ generated by the canonical projections $X_t:\Omega_0\to\mathbb{N}_0$ to the t-th coordinate, $t\in\mathbb{N}_0$. For every $p=(p_n)_{n\in\mathbb{N}}\in[0,1]^{\mathbb{N}}$, let Q_p be the probability measure on $(\Omega_0,\mathscr{F})$ such for all $t\in\mathbb{N}_0$ and $n\in\mathbb{N}$, one has

$$Q_p(X_{t+1}=n-1|X_t=n)=1-Q_p(X_{t+1}=n+1|X_t=n)=p_n, \tag{1}$$

$$Q_p(X_{t+1}=1|X_t=0)=1. \tag{2}$$

Thus, Q_p is the distribution of the Markovian nearest-neighbor random walk on G starting in 0 which jumps from n to $n-1$ with probability p_n, $n\in\mathbb{N}$, and from 0 to 1 with probability 1.
We say that $(X_t)_{t\in\mathbb{N}_0}$ is a *random walk in a random environment* on $\mathbb{N}_0$ if there exists a probability measure $\mathbb{P}$ on $[0,1]^{\mathbb{N}}$ such that

$$P((X_t)_{t\in\mathbb{N}_0}\in A)=\int_{[0,1]^{\mathbb{N}}} Q_p((X_t)_{t\in\mathbb{N}_0}\in A)\,\mathbb{P}(dp) \tag{3}$$

holds for all $A \in \mathscr{F}$. We call $\mathbb{P}$ a *mixing measure*.

Uniqueness of the mixing measure. For a recurrent random walk in a random environment on a general locally finite connected graph, the mixing measure is unique if there exists a mixing measure $\mathbb{Q}$ which is supported on transition probabilities of irreducible Markov chains. In this case, the number of transitions from vertex u to vertex v divided by the number of visits to u up to time t converges as $t \to \infty$ to a random variable with law given by the distribution of the corresponding transition probability $Q_p(X_{t+1} = v | X_t = u)$ under $\mathbb{Q}$. Similarly, the $\mathbb{Q}$-distribution of finitely many transition probabilities and, consequently, $\mathbb{Q}$ itself are determined. For this argumentation recurrence is essential. In case the underlying graph is $\mathbb{N}_0$, we show that recurrence is not necessary for the uniqueness of the mixing measure:

Theorem 1.1. *Let $(X_t)_{t\in\mathbb{N}_0}$ be a random walk in a random environment on $\mathbb{N}_0$ with starting vertex 0 and a mixing measure $\mathbb{P}$ supported on $(0,1)^{\mathbb{N}}$. Then the mixing measure is unique.*

Linearly edge-reinforced random walk. Theorem 1.1 can be applied to a representation of *linearly edge-reinforced random walk* on $\mathbb{N}_0$ as a random walk in a random environment. Linearly edge-reinforced random walk on $\mathbb{N}_0$ is a nearest-neighbor random walk with memory on $\mathbb{N}_0$. The walk starts in 0. Initially every edge $e \in E$ has an arbitrary weight $a_e > 0$. The random walker traverses an edge with probability proportional to its weight. After crossing edge e, the weight of e is increased by one. Thus, the weight of edge e at time t is given by

$$w_e(t) := a_e + \sum_{s=1}^{t} 1_e(\{X_{s-1}, X_s\}) \qquad (e \in E,\, t \in \mathbb{N}_0). \tag{4}$$

The random weights $w_e(t)$ represent the memory of the random walker. The more familiar he is with the edge, the more he prefers to use it. We define the transition probability of the random walker, for every $n \in \mathbb{N}_0$, by

$$P_a(X_{t+1} = n | X_0, \dots, X_t) = \frac{w_{\{X_t,n\}}(t)}{w_{\{X_t-1,X_t\}}(t) + w_{\{X_t,X_t+1\}}(t)}\, 1_{\{n-1,n+1\}}(X_t), \tag{5}$$

where we set $w_{\{-1,0\}}(t) = 0$ for all $t \in \mathbb{N}_0$ to simplify the notation. Especially the probability for the random walker jumping to vertex one, given he is in zero, equals one, since this is the only neighbouring vertex to zero.

It was observed by Pemantle [Pem88] that the edge-reinforced random walk on a tree has the same distribution as a certain random walk in a random environment. For $\mathbb{N}_0$, his result states the following:

Theorem 1.2 ([Pem88])**.** *Let $(X_t)_{t\in\mathbb{N}_0}$ be the edge-reinforced random walk on $\mathbb{N}_0$ with starting vertex 0 and initial edge weights $a=(a_e)_{e\in E} \in (0,\infty)^E$. Let $\mathbb{P}_a$ denote the following product of beta distributions:*

$$\mathbb{P}_a := \bigotimes_{n\in\mathbb{N}} beta\left(\frac{a_{\{n-1,n\}}+1}{2}, \frac{a_{\{n,n+1\}}}{2}\right). \tag{6}$$

Then the following holds for all $A \in \mathscr{F}$:

$$P_a((X_t)_{t\in\mathbb{N}_0} \in A) = \int\limits_{[0,1]^{\mathbb{N}}} Q_p((X_t)_{t\in\mathbb{N}_0} \in A)\, \mathbb{P}_a(dp) \tag{7}$$

where Q_p is the measure defined by (1) and (2).

Using this theorem, our uniqueness result Theorem 1.1 implies:

Corollary 1.3. *The mixing measure $\mathbb{P}_a$ in (7) is unique.*

It was shown in [MR07] that the edge-reinforced random walk on a general locally finite graph is a mixture of irreducible Markov chains. In that case, uniqueness of the mixing measure is open.

2 Proofs

We prepare the proof of Theorem 1.1 with three lemmas.

Lemma 2.1. *Let $(X_t)_{t\in\mathbb{N}_0}$ be a random walk in a random environment on $\mathbb{N}_0$ with starting vertex 0 and a mixing measure $\mathbb{P}$ supported on $(0,1)^{\mathbb{N}}$. Then, for every mixing measure $\mathbb{P}^*$, one has*

$$Q_p(\limsup_{t\to\infty} X_t = +\infty) = 1 \quad \text{for } \mathbb{P}^*\text{-almost all } p. \tag{8}$$

Proof. Since $(X_t)_{t\in\mathbb{N}_0}$ is a random walk in a random environment with mixing measure $\mathbb{P}$, one has

$$P(\limsup_{t\to\infty} X_t = +\infty) = \int_{[0,1]^{\mathbb{N}}} Q_p(\limsup_{t\to\infty} X_t = +\infty)\,\mathbb{P}(dp). \tag{9}$$

The measure $\mathbb{P}$ is supported on $(0,1)^{\mathbb{N}}$, so for $\mathbb{P}$-almost all p the Markov chain with law Q_p is irreducible. Every irreducible Markov chain visits all points in the state space, and since the underlying graph is a line this implies $Q_p(\limsup_{t\to\infty} X_t = +\infty) = 1$ for $\mathbb{P}$-almost all p. Inserting this into (9) yields $P(\limsup_{t\to\infty} X_t = +\infty) = 1$.
If $\mathbb{P}^*$ is an arbitrary mixing measure, equation (9) holds for $\mathbb{P}^*$ instead of $\mathbb{P}$ and we already know that the left-hand side equals 1. Thus, the claim (8) follows. ∎
The key idea in the proof of Theorem 1.1 consists in studying the following stopping times T_n, $n \in \mathbb{N}$:

Lemma 2.2. *Let $(X_t)_{t\in\mathbb{N}_0}$ be a random walk in a random environment on $\mathbb{N}_0$ with starting vertex 0 and a mixing measure $\mathbb{P}$ supported on $(0,1)^{\mathbb{N}}$. For $n \in \mathbb{N}$, denote by T_n the number of visits in n before $n+1$ is visited for the first time, and let $\mathbb{P}^*$ be an arbitrary mixing measure. Then for $\mathbb{P}^*$-almost all p, $(T_n)_{n\in\mathbb{N}}$ is a family of independent random variables under the measure Q_p with $T_n \sim$ geometric$(1-p_n)$ for all $n \in \mathbb{N}$.*

Proof. We know from Lemma 2.1 that $Q_p(\limsup_{t\to\infty} X_t = +\infty) = 1$ for $\mathbb{P}^*$-almost all p. Consequently, T_n is finite for every $n \in \mathbb{N}$ Q_p-a.s. for $\mathbb{P}^*$-almost all p. For these p, the following holds: If the random walker is in n, he decides with probability p_n to go back and not to visit $n+1$ right now. In this case, he will return Q_p-almost surely, since $\limsup_{t\to\infty} X_t = +\infty$ Q_p-a.s. With probability $1-p_n$ the walker decides to visit $n+1$, when he is in n and under Q_p his decision does not depend on his decisions at his last visits. Hence, $Q_p(T_n = k) = p_n^{k-1}(1-p_n)$ for all $k \in \mathbb{N}$. The decisions at each vertex are made irrespective of the decisions at the other vertices and under Q_p the transition probabilities do not depend on the past. The independence of $(T_n)_{n\in\mathbb{N}}$ follows. ∎
To prove the uniquenss of the mixing measure for the random walk in a random environment on $\mathbb{N}_0$, we show first that the moments of any mixing measure are uniquely determined.

Lemma 2.3. *Let $(X_t)_{t\in\mathbb{N}_0}$ be a random walk in a random environment on $\mathbb{N}_0$ with starting vertex* 0 *and a mixing measure $\mathbb{P}$ supported on $(0,1)^{\mathbb{N}}$. For every mixing measure $\mathbb{P}^*$, for every $n \in \mathbb{N}$, and for all $k_1,\dots,k_n \in \mathbb{N}_0$, the following holds*

$$\int\limits_{[0,1]^{\mathbb{N}}} \prod_{i=1}^{n} p_i^{k_i}\, \mathbb{P}^*(dp) = \int\limits_{[0,1]^{\mathbb{N}}} \prod_{i=1}^{n} p_i^{k_i}\, \mathbb{P}(dp). \tag{10}$$

Proof. Since $\mathbb{P}^*$ is a mixing measure, Lemma 2.2 implies for all $m \in \mathbb{N}$, $i_1,\dots,i_m \in \mathbb{N}$, with $i_j \neq i_l$ for all $j \neq l$, and for all $k_1,\dots,k_m \in \mathbb{N}$,

$$\begin{aligned} P(T_{i_1} = k_1,\dots,T_{i_m} = k_m) &= \int\limits_{[0,1]^{\mathbb{N}}} Q_p(T_{i_1} = k_1,\dots,T_{i_m} = k_m)\, \mathbb{P}^*(dp) \\ &= \int\limits_{[0,1]^{\mathbb{N}}} \prod_{j=1}^{m} Q_p(T_{i_j} = k_j)\, \mathbb{P}^*(dp) = \int\limits_{[0,1]^{\mathbb{N}}} \prod_{j=1}^{m} p_{i_j}^{k_j-1}(1-p_{i_j})\, \mathbb{P}^*(dp). \end{aligned} \tag{11}$$

By the same argument, this equation holds for the mixing measure $\mathbb{P}$, and hence, we conclude that

$$\int\limits_{[0,1]^{\mathbb{N}}} \prod_{j=1}^{m} p_{i_j}^{k_j-1}(1-p_{i_j})\, \mathbb{P}^*(dp) = \int\limits_{[0,1]^{\mathbb{N}}} \prod_{j=1}^{m} p_{i_j}^{k_j-1}(1-p_{i_j})\, \mathbb{P}(dp). \tag{12}$$

It is sufficient to show for all $m \in \mathbb{N}$, $i_1,\dots,i_m \in \mathbb{N}$ with $i_j \neq i_l$ for all $j \neq l$, and for all $k_1,\dots,k_m \in \mathbb{N}$

$$\int\limits_{[0,1]^{\mathbb{N}}} \prod_{j=1}^{m} p_{i_j}^{k_j}\, \mathbb{P}^*(dp) = \int\limits_{[0,1]^{\mathbb{N}}} \prod_{j=1}^{m} p_{i_j}^{k_j}\, \mathbb{P}(dp). \tag{13}$$

Then choose $i_1,\dots,i_m$ the indices with exponent non-zero in equation (10).
We prove equation (13) by induction over m.
Case $m=1$: For every $i \in \mathbb{N}$ and $k \in \mathbb{N}$, equation (12) yields

$$\int\limits_{[0,1]^{\mathbb{N}}} (p_i^{k-1} - p_i^k)\, \mathbb{P}^*(dp) = \int\limits_{[0,1]^{\mathbb{N}}} (p_i^{k-1} - p_i^k)\, \mathbb{P}(dp). \tag{14}$$

Choose $K \in \mathbb{N}$ arbitrarily. Summing both sides of the last equation over $k \in \{1,\dots,K\}$, we obtain

$$\int\limits_{[0,1]^{\mathbb{N}}} (1 - p_i^K)\, \mathbb{P}^*(dp) = \int\limits_{[0,1]^{\mathbb{N}}} (1 - p_i^K)\, \mathbb{P}(dp). \tag{15}$$

Since $\mathbb{P}$ and $\mathbb{P}^*$ are probability measures, equation (13) holds for $m=1$.
Induction step: Now assume equation (13) holds for $1,\dots,m-1$. Choose $K_1,\dots,K_m \in \mathbb{N}$ arbitrarily. Summing the left-hand side of (12) over $k_j \in \{1,\dots,K_j\}$ for all $j \in \{1,\dots,m\}$ yields

$$\begin{aligned} &\sum_{k_1=1}^{K_1} \cdots \sum_{k_m=1}^{K_m} \int\limits_{[0,1]^{\mathbb{N}}} \prod_{j=1}^{m} (p_{i_j}^{k_j-1} - p_{i_j}^{k_j})\, \mathbb{P}^*(dp) \\ &= \int\limits_{[0,1]^{\mathbb{N}}} \prod_{j=1}^{m} \sum_{k_j=1}^{K_j} \left(p_{i_j}^{k_j-1} - p_{i_j}^{k_j} \right) \mathbb{P}^*(dp) = \int\limits_{[0,1]^{\mathbb{N}}} \prod_{j=1}^{m} (1 - p_{i_j}^{K_j})\, \mathbb{P}^*(dp). \end{aligned} \tag{16}$$

Summing the right-hand side of (12) over $k_j \in \{1, \dots, K_j\}$, $j \in \{1, \dots, m\}$, we obtain the same identity with $\mathbb{P}$ instead of $\mathbb{P}^*$. Hence, we conclude for all $K_1, \dots, K_m \in \mathbb{N}$:

$$\int_{[0,1]^{\mathbb{N}}} \prod_{j=1}^{m} (1 - p_{i_j}^{K_j})\, \mathbb{P}^*(dp) = \int_{[0,1]^{\mathbb{N}}} \prod_{j=1}^{m} (1 - p_{i_j}^{K_j})\, \mathbb{P}(dp). \tag{17}$$

After expanding both products, the same linear combination of integrals remains on both sides. Since only one integral with m different indices occurs, the claim follows from the induction hypothesis. ■

Now we collected everything to prove the main result.

Proof of Theorem 1.1. Let $\mathbb{P}$ be a mixing measure supported on $(0,1)^{\mathbb{N}}$ and let $\mathbb{P}^*$ be an arbitrary mixing measure. The n-dimensional marginals of both measures are distributions on $[0,1]^n$. Therefore, they are determined by their Laplace transforms, which are determined by the joint moments. By Lemma 2.3, these moments agree. Consequently, all finite-dimensional marginals of $\mathbb{P}$ and $\mathbb{P}^*$ agree, and it follows from Kolmogorov's consistency theorem that $\mathbb{P} = \mathbb{P}^*$. ■

Acknowledgement: M.E. would like to thank Steffen Weil for his helpful comments. S.R. would like to thank Franz Merkl for useful discussions.

References

[MR07] F. Merkl and S.W.W. Rolles. A random environment for linearly edge-reinforced random walks on infinite graphs. *Probab. Theory Related Fields*, 138(1-2):157–176, 2007. MR2288067

[Pem88] R. Pemantle. Phase transition in reinforced random walk and RWRE on trees. *Ann. Probab.*, 16(3):1229–1241, 1988. MR0942765