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THE ZERO-ONE LAW FOR PLANAR RANDOM WALKS IN I.I.D. RANDOM ENVIRONMENTS REVISITED

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Abstract
In this note we present a simplified proof of the zero-one law by Merkl and Zerner (2001) for directional transience of random walks in i.i.d. random environments (RWRE) on $\mathbb{Z}^2$. Also, we indicate how to construct a two-dimensional counterexample in a non-uniformly elliptic and stationary environment which has better ergodic properties than the example given by Merkl and Zerner.

1 Introduction

Let us first recall the model of random walks in random environments (RWRE), see also [Zei04] for a survey. For $d \geq 1$, we denote by $\mathcal{P}$ the set of $2d$-dimensional probability vectors, and set $\Omega = \mathcal{P}^{\mathbb{Z}^d}$. Any $\omega \in \Omega$, written as $\omega = ((\omega(x, x+e))_{|e|=1})_{x\in\mathbb{Z}^d}$, will be called an *environment*. It is called *elliptic* if $\omega(x, x+e) > 0$ for all $x, e \in \mathbb{Z}^d$ with $|e| = 1$ and *uniformly elliptic* if there exists a so-called *ellipticity constant* $\kappa > 0$, such that $\omega(x, x+e) > \kappa$ for all $x, e \in \mathbb{Z}^d$ with $|e| = 1$. Endowing Ω with the canonical product σ-algebra and a probability measure $\mathbb{P}$ turns ω into a collection of random $2d$-vectors, i.e. a *random environment*. The expectation corresponding to $\mathbb{P}$ is denoted by $\mathbb{E}$.
Given an environment $\omega \in \Omega$, the values $\omega(x, x+e)$ serve as transition probabilities for the $\mathbb{Z}^d$-valued Markov chain $(X_n)_{n\geq 0}$, called *random walk in random environment* (RWRE). This process can be defined as the sequence of canonical projections on the sample space $(\mathbb{Z}^d)^{\mathbb{N}}$ endowed with the so-called *quenched measure* $P_{z,\omega}$, which is defined for any starting point $z \in \mathbb{Z}^d$ and any environment $\omega \in \Omega$ and characterized by

$$\begin{aligned} P_{z,\omega}[X_0 = z] &= 1 \quad \text{and} \\ P_{z,\omega}[X_{n+1} = X_n + e \mid X_0, X_1, \ldots, X_n] &= \omega(X_n, X_n + e) \quad P_{z,\omega} - a.s. \end{aligned}$$

for all $e \in \mathbb{Z}^d$ with $|e| = 1$ and all $n \geq 0$. The so-called *annealed measures* P_z, $z \in \mathbb{Z}^d$, are then defined as the semi-direct products $P_z := \mathbb{P} \times P_{z,\omega}$ on $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}$ by $P_z[\cdot] := \mathbb{E}[P_{z,\omega}[\cdot]]$. The

expectations corresponding to $P_{z,\omega}$ and P_z are denoted by $E_{z,\omega}$ and E_z, respectively.
One of the major open questions in the study of RWRE concerns the so-called *0-1 law*, which we shall describe now. For $\ell \in \mathbb{R}^d$, $\ell \neq 0$, define the event

$$A_\ell := \left\{ \lim_{n\to\infty} X_n \cdot \ell = \infty \right\}$$

that the walker tends in a rough sense into direction ℓ, which we call *to the right*. It has been known since the work of Kalikow [Ka81], that if the random vectors $\omega(x,\cdot)$, $x \in \mathbb{Z}^d$, are i.i.d. under $\mathbb{P}$ and $\mathbb{P}$-a.s. uniformly elliptic, then

$$P_0 \left[A_\ell \cup A_{-\ell} \right] \in \{0,1\} . \tag{1}$$

This was extended in [ZerMe01, Proposition 3] to the elliptic i.i.d. case. We shall call (1) Kalikow's zero-one law.
The zero-one law for directional transience is the stronger statement that even $P_0[A_\ell] \in \{0,1\}$. Except for $d = 1$, see e.g. [Zei04, Theorem 2.1.2], it is only partially known under which conditions this statement holds. For $d = 2$ and $\omega(x,\cdot)$, $x \in \mathbb{Z}^d$, being i.i.d. under $\mathbb{P}$, Kalikow [Ka81] presented it as an open question; that case was settled in the affirmative in [ZerMe01], while the case $d \geq 3$ is still wide open.

Theorem 1. (see [ZerMe01, Theorem 1]) *Let $d = 2$, $\ell \in \mathbb{R}^2 \backslash \{0\}$ and let $(\omega(x,\cdot))$, $x \in \mathbb{Z}^2$, be i.i.d. and elliptic under $\mathbb{P}$. Then $P_0[A_\ell] \in \{0,1\}$.*

Let us briefly sketch the proof of Theorem 1 for $\ell = e_1$ as given in [ZerMe01], where e_1 is the first coordinate direction. Assuming that the zero-one law does not hold, one considers two independent random walks in the same environment. The first one starts at the origin, the second one at a point (L, z_L) for some $L > 0$ large. The slab $\{(x_1,x_2) \in \mathbb{Z}^2 \mid 0 \leq x_1 \leq L\}$ is then subdivided into three slabs of equal size. By adjusting z_L and using $d = 2$ one can then force the paths of the two walkers to intersect at some point x in the middle slab with a positive probability, which is bounded away from 0 uniformly in L. In this step some technical result [ZerMe01, Lemma 7] about sums of four independent random variables is used. Now consider a third random walker starting at x. By Kalikow's zero-one law (1) it eventually needs to go either to the left or to the right. Since x has been visited by the first walker which has already traveled a long distance $> L$ to the right and thus most probably will continue to go to the right, the third walker is also likely to go right. However, by the same argument the third walker should also follow the second walker to the left. This leads to the desired contradiction.
The main goal of the present paper is to give a simplified proof of Theorem 1. In Section 2 we are going to present a proof in which the slab between 0 and (L, z_L) is divided into two slabs only. This way the technical lemma [ZerMe01, Lemma 7] is not needed anymore and general directions $\ell \notin \{e_1, e_2\}$ can be handled more easily.
The same paper [ZerMe01] also provides a counterexample of an elliptic, stationary and ergodic environment with $P_0[A_\ell] \notin \{0,1\}$. However, the environment in this example has bad mixing properties. In fact, it is not even totally ergodic since it is not ergodic w.r.t. any spatial shift by $z \in 2\mathbb{Z}^2$. Here we call a measure $\mathbb{P}$ *totally ergodic* if for all $z \in \mathbb{Z}^2 \backslash \{0\}$, $\mathbb{P}$ is ergodic w.r.t. the shift by z. In Section 3 we shall sketch a construction of an elliptic and stationary counterexample which is also totally ergodic.

2 A shorter proof of Theorem 1

Without loss of generality we assume $\|\ell\|_2 = 1$. For $u \in \mathbb{R}$ and $\diamond \in \{<, \leq, >, \geq\}$ we consider the stopping times

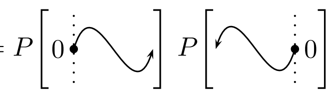
$$T_{\diamond u} := \inf\{n \geq 0 \mid (X_n \cdot \ell) \diamond u\}.$$

By Kalikow's 0-1 law (1), $P_0[A_\ell \cup A_{-\ell}] \in \{0, 1\}$. The case $P_0[A_\ell \cup A_{-\ell}] = 0$ is trivial. So assume

$$P_0[A_\ell \cup A_{-\ell}] = 1. \tag{2}$$

For the proof of the theorem it suffices to show that[1]

$$0 = P_0[T_{<0} = \infty]\ P_0[T_{>0} = \infty] =$$



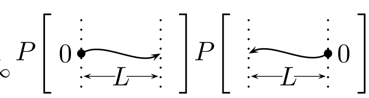
(3)

Indeed, that (3) is sufficient follows from [SzZer99, Proposition 1.2 (1.16)]. For completeness, we repeat the argument here. If (3) holds then either P_0-a.s. $T_{<0} < \infty$ or P_0-a.s. $T_{>0} < \infty$. In the first case, due to translation invariance, $T_{<x\cdot\ell} < \infty$ would hold P_x-a.s. for all $x \in \mathbb{Z}^2$. Hence, $\mathbb{P}$-a.s. $P_{x,\omega}[T_{<x\cdot\ell} < \infty] = 1$ for all $x \in \mathbb{Z}^2$. By the strong Markov property this implies $P_0[A_\ell] = 0$. Similarly, we get in the second case that $P_0[A_{-\ell}] = 0$, which yields due to (2) $P_0[A_\ell] = 1 - P_0[A_{-\ell}] = 1$.
For the proof of (3) observe that

$$T_{\geq L}, T_{\leq -L} \geq L \quad P_0\text{-a.s. for all } L \geq 0, \tag{4}$$

since the walk is moving between nearest neighbors. Therefore, the right hand side of (3) is equal to

$$\lim_{L\to\infty} P_0[T_{\geq L} < T_{<0}]\ P_0[T_{\leq -L} < T_{>0}] = \lim_{L\to\infty}$$



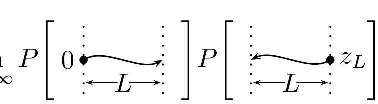
(5)

Now fix a unit vector $\ell^\perp \in \mathbb{R}^2$ which is perpendicular to ℓ. Then for $L \in \mathbb{N}$ we choose $z_L \in \mathbb{Z}^2$ such that

$$\begin{aligned}
&x_L := z_L \cdot \ell \geq 2L,\\
&z_L \text{ has a nearest neighbor } w_L \in \mathbb{Z}^2 \text{ with } w_L \cdot \ell < 2L,\\
&y_L := z_L \cdot \ell^\perp \text{ is a median of the distribution of } X_{T_{\geq 2L}} \cdot \ell^\perp \text{ under}\\
&\quad P_0[\,\cdot \mid T_{\geq 2L} < T_{<0}], \quad \text{i.e.} \quad P_0[X_{T_{\geq 2L}} \cdot \ell^\perp \diamond y_L \mid T_{\geq 2L} < T_{<0}] \leq 1/2 \quad \text{for } \diamond \in \{<, >\}.
\end{aligned} \tag{6}$$

(If $\ell = e_1$ then we can choose $\ell^\perp = e_2$, $x_L = 2L$ and $z_L = (2L, y_L)$.) In order to make the events in the two probabilities in (5) depend on disjoint and therefore independent parts of the environment we shift the starting point in the second factor in (5) from 0 to z_L. Thus, by translation invariance, we can rewrite (5) as

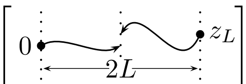
$$\lim_{L\to\infty} P_0[T_{\geq L} < T_{<0}]\ P_{z_L}[T_{\leq L} < T_{>x_L}] = \lim_{L\to\infty}$$



(7)

[1] Here and in the following the sole purpose of the figures is to illustrate the term immediately preceeding the figure. The proof is complete without the figures.

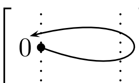
To write the product of probabilities in (7) as a single probability, we introduce two independent random walks moving in the same environment, one starting at 0, the other starting at z_L. So for any $\omega \in \Omega$ and $L \in \mathbb{N}$ let $P_{0,z_L,\omega}$ be a probability measure on $(\mathbb{Z}^2)^{\mathbb{N}} \times (\mathbb{Z}^2)^{\mathbb{N}}$ such that the two canonical processes of projections $(X_n^1)_n$ and $(X_n^2)_n$ on this space are independent of each other and have distributions $P_{0,\omega}$ and $P_{z_L,\omega}$, respectively, and denote by P_{0,z_L} the corresponding annealed measure. Stopping times referring to the walk $(X_n^i)_n$ will be marked with an upper index i $(i = 1, 2)$. Then, by independence, (7) is equal to

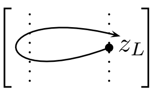
$$\lim_{L\to\infty} P_{0,z_L}[T^1_{\geq L} < T^1_{<0}, T^2_{\leq L} < T^2_{>x_L}] = \lim_{L\to\infty} P\Big[$$


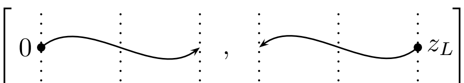
$$\Big]. \qquad (8)$$

After crossing the line $\{x \mid x \cdot \ell = L\}$ any walk must due to (2) a.s. cross the line $\{x \mid x \cdot \ell = 0\}$ or the line $\{x \mid x \cdot \ell = 2L\}$. Consequently, (8) is less than or equal to

$$\liminf_{L\to\infty} P_0[T_{\geq L} < T_{<0} < \infty] + P_{z_L}[T_{\leq L} < T_{>x_L} < \infty] + P_{0,z_L}[T^1_{\geq 2L} < T^1_{<0}, T^2_{\leq 0} < T^2_{>x_L}] \qquad (9)$$

$$= \liminf_{L\to\infty} P\Big[$$


$$\Big] + P\Big[$$


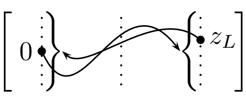
$$\Big] + P\Big[$$


$$\Big].$$

Due to (4) the first term in (9) is $\leq P_0[\exists n \geq L : \ |X_n \cdot \ell| \leq 1]$. The same holds for the second term, which is $\leq P_0[T_{\leq -L} < T_{>0} < \infty]$ due to translation invariance. Therefore, both terms vanish as $L \to \infty$ due to (2). Summarizing, we have shown

$$P_0[T_{<0} = \infty] \ P_0[T_{>0} = \infty] \leq \liminf_{L\to\infty} P_{0,z_L}[T^1_{\geq 2L} < T^1_{<0}, T^2_{\leq 0} < T^2_{>x_L}]. \qquad (10)$$

Now consider the event on the right-hand side of (10). There are two possibilities: Either the paths of the two random walks cross each other (with probability, say, C_L) before $T^1_{\geq 2L}$ and $T^2_{\leq 0}$, respectively, or they avoid each other (with probability, say, N_L). Therefore, defining the hitting time of $x \in \mathbb{Z}^2$ by $H(x) := \inf\{n \geq 0 \mid X_n = x\}$, we can rewrite the probability on the right-hand side of (10) as $C_L + N_L$, where

$$C_L := P_{0,z_L}[\exists x : \ H^1(x) \leq T^1_{\geq 2L} < T^1_{<0}, \ H^2(x) \leq T^2_{\leq 0} < T^2_{>x_L}] = P\Big[$$


$$\Big]$$

and

$$N_L := P_{0,z_L}[T^1_{\geq 2L} < T^1_{<0}, \ T^2_{\leq 0} < T^2_{>x_L}, \{X_n^1 \mid \ n \leq T^1_{\geq 2L}\} \cap \{X_n^2 \mid \ n \leq T^2_{\leq 0}\} = \emptyset].$$

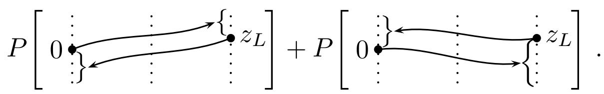
Hence we get from (10)

$$P_0[T_{<0} = \infty] \ P_0[T_{>0} = \infty] \leq \limsup_{L\to\infty} C_L + \limsup_{L\to\infty} N_L. \qquad (11)$$

To estimate N_L, observe that on the event in the definition of N_L we have that $y_L - X^1_{T^1_{\geq 2L}} \cdot \ell^\perp$ and $X^2_{T^2_{\leq 0}} \cdot \ell^\perp$ are either both strictly positive or both strictly negative since the dimension is equal to two. Indeed, otherwise the two paths $\{X_n^1 \mid \ n \leq T^1_{\geq 2L}\}$ and $\{X_n^2 \mid \ n \leq T^2_{\leq 0}\}$ would intersect each other, since the two diagonals of any planar quadrangle intersect each other.

Therefore,

$$\begin{aligned} N_L &= \sum_{s=\pm 1} P_{0,z_L}\Big[T^1_{\geq 2L} < T^1_{<0},\ T^2_{\leq 0} < T^2_{>x_L}, \{X^1_n \mid n \leq T^1_{\geq 2L}\} \cap \{X^2_n \mid n \leq T^2_{\leq 0}\} = \emptyset, \\ &\qquad s = \operatorname{sign}\Big(y_L - X^1_{T^1_{\geq 2L}} \cdot \ell^\perp\Big) = \operatorname{sign}\Big(X^2_{T^2_{\leq 0}} \cdot \ell^\perp\Big)\Big] \end{aligned}$$

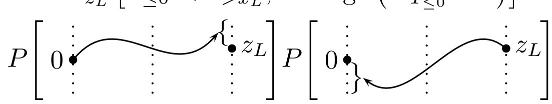
$=$ 

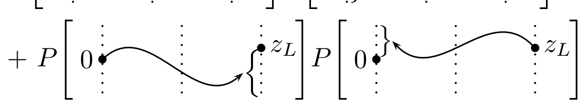
Denoting by $\Pi_{L,s}$ ($L \in \mathbb{N}, s \in \{-1,0,+1\}$) the set of all finite nearest-neighbor paths that start at z_L and leave the strip $\{x \mid 0 \leq x \cdot \ell \leq x_L\}$ on the opposite side through a vertex x with $\operatorname{sign}(x \cdot \ell^\perp) = s$, we rewrite N_L as

$$\begin{aligned} N_L &= \sum_{s=\pm 1} \sum_{\pi \in \Pi_{L,s}} P_{0,z_L}\Big[T^1_{\geq 2L} < T^1_{<0}, \{X^1_n \mid n \leq T^1_{\geq 2L}\} \cap \pi = \emptyset,\ (X^2_n)_n \text{ follows } \pi, \\ &\qquad s = \operatorname{sign}\Big(y_L - X^1_{T^1_{\geq 2L}} \cdot \ell^\perp\Big)\Big]. \end{aligned}$$

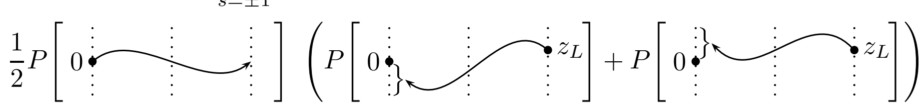
Using the disjointness of the paths we get by independence in the environment,

$$\begin{aligned} N_L &= \sum_{s=\pm 1} \sum_{\pi \in \Pi_{L,s}} P_0\left[T_{\geq 2L} < T_{<0}, \{X_n \mid n \leq T_{\geq 2L}\} \cap \pi = \emptyset, s = \operatorname{sign}\left(y_L - X_{T_{\geq 2L}} \cdot \ell^\perp\right)\right] \\ &\qquad P_{z_L}\left[(X_n)_n \text{ follows } \pi\right] \\ &\leq \sum_{s=\pm 1} \sum_{\pi \in \Pi_{L,s}} P_0\left[T_{\geq 2L} < T_{<0}, s = \operatorname{sign}\left(y_L - X_{T_{\geq 2L}} \cdot \ell^\perp\right)\right]\ P_{z_L}\left[(X_n)_n \text{ follows } \pi\right] \\ &= \sum_{s=\pm 1} P_0\left[T_{\geq 2L} < T_{<0}, s = \operatorname{sign}\left(y_L - X_{T_{\geq 2L}} \cdot \ell^\perp\right)\right] \\ &\qquad P_{z_L}\left[T_{\leq 0} < T_{>x_L}, s = \operatorname{sign}\left(X_{T_{\leq 0}} \cdot \ell^\perp\right)\right] \end{aligned}$$

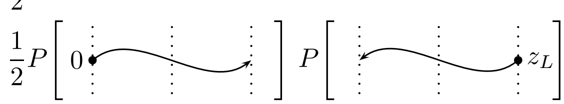
$=$ 



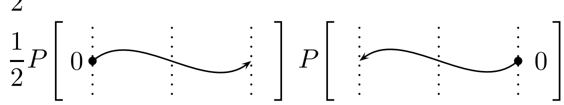
$$\overset{(6)}{\leq} \frac{1}{2} P_0\left[T_{\geq 2L} < T_{<0}\right] \sum_{s=\pm 1} P_{z_L}\left[T_{\leq 0} < T_{>x_L}, s = \operatorname{sign}\left(X_{T_{\leq 0}} \cdot \ell^\perp\right)\right]$$

$=$ 

$$\leq \frac{1}{2} P_0\left[T_{\geq 2L} < T_{<0}\right]\ P_{z_L}\left[T_{\leq 0} < T_{>x_L}\right]$$

$=$ 

$$\leq \frac{1}{2} P_0\left[T_{\geq 2L} < T_{<0}\right]\ P_0\left[T_{\leq -2L} < T_{>0}\right]$$

$=$ 

$$\overset{(4)}{\leq} \quad \frac{1}{2}P_0[2L < T_{<0}]\ P_0[2L < T_{>0}]$$
$$\underset{L\to\infty}{\longrightarrow} \quad \frac{1}{2}P_0[T_{<0} = \infty]\ P_0[T_{>0} = \infty] \ = \ \frac{1}{2}P\Big[\,0\,\Big]\ P\Big[\,0\,\Big].$$

Hence, due to (11),

$$\frac{1}{2}P_0[T_{<0} = \infty]\ P_0[T_{>0} = \infty] \leq \limsup_{L\to\infty} C_L.$$

For the proof of (3) it therefore suffices to show

$$\lim_{L\to\infty} C_L = 0. \tag{12}$$

By considering the possible locations of the intersections of the two paths, we estimate C_L by

$$C_L \quad \leq \quad C_0^L + C_L^{x_L} \ = \ P\Big[\,0 \quad x \quad z_L\,\Big] + P\Big[\,0 \quad x \quad z_L\,\Big],$$

where

$$C_a^b := P_{0,z_L}[\exists x:\ a \leq x\cdot\ell \leq b,\ H^1(x) \leq T^1_{\geq 2L} < T^1_{<0},\ H^2(x) \leq T^2_{\leq 0} < T^2_{>x_L}].$$

Due to symmetry and translation invariance it suffices to show for the proof of (12) that $C_0^L \to 0$. To this end let $\varepsilon > 0$ and set $r(x,\omega) := P_{x,\omega}[A_\ell]$. Then

$$C_0^L \quad \leq \quad C_{0,1}^L + C_{0,2}^L, \qquad \text{where} \tag{13}$$
$$C_{0,1}^L \quad := \quad P_0[\exists x:\ r(x,\omega) \leq \varepsilon,\ H(x) \leq T_{\geq L} < \infty] = P\Big[\,0 \quad x \quad , r(x,\omega) \leq \varepsilon\,\Big] \quad \text{and}$$
$$C_{0,2}^L \quad := \quad P_{z_L}[\exists x:\ x\cdot\ell \leq L,\ r(x,\omega) \geq \varepsilon,\ H(x) < \infty] \ = \ P\Big[\,x \quad z_L, r(x,\omega) \geq \varepsilon\,\Big].$$

In order to bound $C_{0,1}^L$, consider $\sigma := \inf\{n \geq 0 \mid r(X_n,\omega) \leq \varepsilon\}$. Note that σ is a stopping time w.r.t. the filtration $(\mathcal{F}_n)_{n\geq 0}$, where $\mathcal{F}_n$ is the σ-field generated by $X_0,\ldots,X_n$ and ω. Therefore, by the strong Markov property,

$$C_{0,1}^L = P_0[\sigma \leq T_{\geq L} < \infty] = E_0\left[P_{X_\sigma,\omega}[T_{\geq L} < \infty], \sigma \leq T_{\geq L}, \sigma < \infty\right].$$

Since for all $x \in \mathbb{Z}^2$ and almost all ω, $P_{x,\omega}$-a.s. $\{T_{\geq L} < \infty\} \searrow A_\ell$ as $L \to \infty$ due to (2), we get by dominated convergence

$$\lim_{L\to\infty} C_{0,1}^L = E_0\left[P_{X_\sigma,\omega}[A_\ell], \sigma < \infty\right] \ = \ E_0\left[r\left(X_\sigma,\omega\right),\ \sigma < \infty\right] \ \leq \ \varepsilon \tag{14}$$

by definition of σ. Now consider $C_{0,2}^L$. By translation invariance

$$C_{0,2}^L \quad \leq \quad P_0[\exists x:\ x\cdot\ell \leq -L,\ r(x,\omega) \geq \varepsilon, H(x) < \infty]$$
$$\overset{(2),(4)}{\leq} \quad P_0[T_{\leq -L} < \infty, A_\ell] + P_0[\exists n \geq L: r(X_n,\omega) \geq \varepsilon, A_{-\ell}]. \tag{15}$$

Obviously, the first term in (15) goes to 0 as $L \to \infty$. The same holds for the second term since due to the martingale convergence theorem, P_0-a.s. $r(X_n,\omega) = P_0[A_\ell \mid \mathcal{F}_n] \to \mathbf{1}_{A_\ell}$ as $n \to \infty$, cf. [ZerMe01, Lemma 5]. Together with (13) and (14) this yields $\limsup_L C_0^L \leq \varepsilon$. Letting $\varepsilon \searrow 0$ gives $\lim_L C_0^L = 0$. This finishes the proof of (12) and concludes the proof of Theorem 1.

3 A stationary and totally ergodic counterexample

The stationary and ergodic environment constructed in [ZerMe01, Section 3] is based on two disjoint trees which together span $\mathbb{Z}^2$. The branches of these trees are paths of coalescing random walks which for one tree go either up or right and for the other tree go either left or down. In order to allocate enough space for both trees some periodicity was introduced which destroyed total ergodicity of the environment.

In this section we shall sketch an alternative construction which gives a totally ergodic environment. It has been inspired by the Poisson tree considered in [FeLaTh04, Section 3]. The main difference to the tree in [FeLaTh04] is that the nodes and leaves of our tree do not form a Poisson point process but are obtained from the discrete equivalent of a Poisson point process by a local thinning procedure as follows: We first color the vertices of $\mathbb{Z}^2$ independently black with some fixed probability $0 < p < 1$ (in Figure 1, $p = 1/7$) and white otherwise. Then all the black points $x \in \mathbb{Z}^2$ for which the set $x + \{\pm e_2, \pm 2e_2, -e_1 \pm 2e_2\}$ contains another black point are removed simultaneously, i.e. painted white again. The remaining set of black points is called $B \subseteq \mathbb{Z}^2$. Obviously, the random variables $\mathbf{1}_{x\in B},\ x \in \mathbb{Z}^2$, are only finite range dependent.

Now each black point grows a gray line to the right until the line's tip reaches an ℓ^1-neighbor of another black point, see also Figure 1 for a realization on a discrete torus. The set of the gray points obtained this way is called $G \subseteq \mathbb{Z}^2$. More formally, for $x \in \mathbb{Z}^2$ let

$$\begin{aligned} g(x) &:= \inf\{n \geq 0 \mid x + ne_1 \in (B + \{e_2, -e_2, -e_1\})\} \quad \text{and set} \\ G &:= \{x + ke_1 \mid x \in B, 1 \leq k \leq g(x)\}. \end{aligned}$$

Note that almost surely all $g(x),\ x \in \mathbb{Z}^2$, are finite. Now consider the set $T := B \cup G$.

Lemma 2. *If $x \in T$ then $x + e_1 \in T$ or $\{x + e_2, x + e_1 + e_2\} \subseteq T$ or $\{x - e_2, x + e_1 - e_2\} \subseteq T$. Similarly, if $x \in T^c$ then $x - e_1 \in T^c$ or $\{x + e_2, x - e_1 + e_2\} \subseteq T^c$ or $\{x - e_2, x - e_1 - e_2\} \subseteq T^c$. Moreover, $T^c \neq \emptyset$.*

Proof. First note that

$$x \in B \Rightarrow x + e_1 \in T \tag{16}$$

since either $x + e_1$ is black or, by construction of B, $g(x) \geq 1$ and hence $x + e_1$ is gray as the right neighbor of a black point.

For the first statement of the lemma let $x \in T$ and assume $x + e_1 \in T^c$. Then by (16), $x \in T\backslash B = G$. Hence it suffices to show that

$$x \in G, x + e_1 \in T^c \Rightarrow ((x + e_2 \in B \wedge x + e_1 + e_2 \in T) \vee (x - e_2 \in B \wedge x + e_1 - e_2 \in T)), \tag{17}$$

where $\wedge$ and $\vee$ denote "and" and "or", respectively. Since x is gray but $x + e_1$ is white, the gray line to which x belongs must have stopped growing in x. This means that one of the neighbors $x + e_1, x + e_2$ or $x - e_2$ must be black. Hence, since $x + e_1$ is white $x + e_2$ or $x - e_2$ must be black. By construction of B, only one of them can be black. By symmetry we may assume $x + e_2 \in B$. Then, due to (16), $x + e_2 + e_1 \in T$. Thus (17) has been shown and the first statement follows.

For the second statement of the lemma let $x \in T^c$ and assume $x - e_1 \in T$. Then applying (16) to $x - e_1$ instead of x yields $x - e_1 \in T\backslash B = G$. Consequently, an application of (17) to $x - e_1$ instead of x yields without loss of generality (due to symmetry) that $x - e_1 + e_2 \in B$ and $x + e_2 \in T$. Now it suffices to show that $x - e_2, x - e_1 - e_2 \in T^c$. Neither of these points

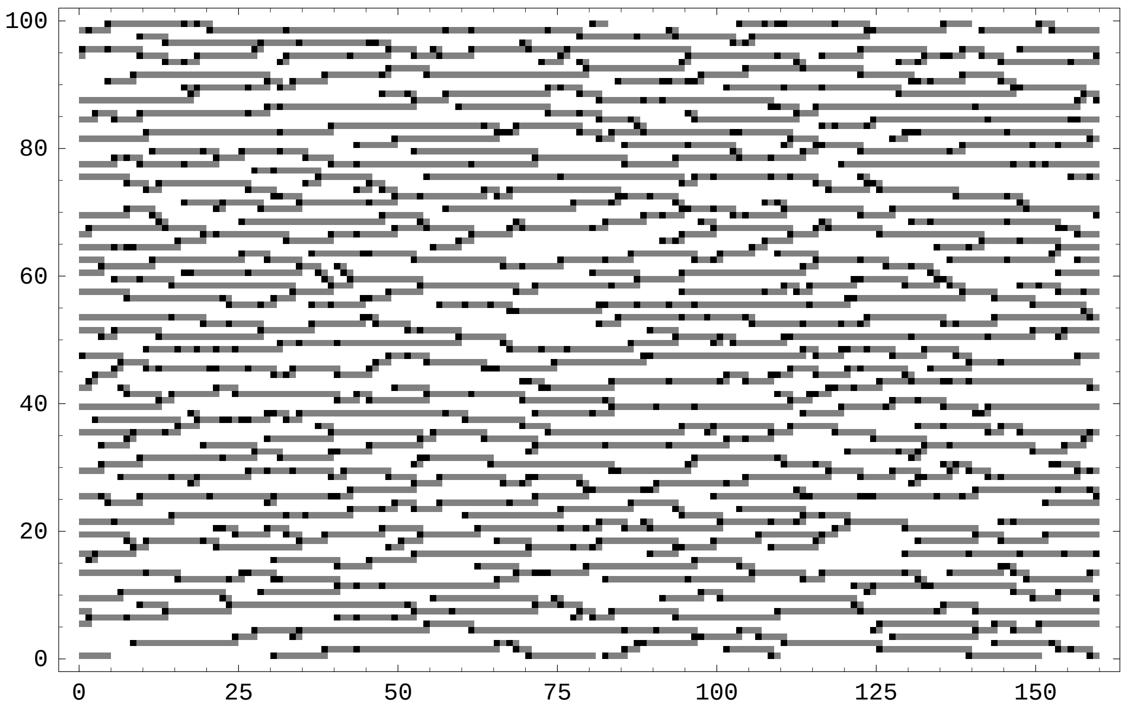


Figure 1: The black points constitute some in a certain way thinned lattice point process on a torus. From each black point a gray line grew to the right until its tip became an ℓ^1-neighbor of another black point.

can be black by construction of B since $x - e_1 + e_2$ is already black. So it suffices to show that neither of them is gray. This is done by contradiction. Assume that one of them is gray. If $x - e_2$ were gray then $x - e_1 - e_2$ would have to be black or gray. By construction of B, $x - e_1 - e_2$ cannot be black since $x - e_1 + e_2$ is already black. Hence $x - e_1 - e_2$ would have to be gray, too. So we may assume that $x - e_1 - e_2$ is gray. By construction of the gray lines, there is some $k \geq 2$ such that $x - ke_1 - e_2$ is black and all the points $x - ie_1 - e_2$ $(1 \leq i < k)$ in between are gray. Now recall that $x - e_1$ is gray, too. Hence by the same argument, there is some $m \geq 2$ such that $x - me_1$ is black and all the points $x - ie_1$ $(1 \leq i < m)$ in between are gray. By construction of B, k and m cannot be equal, since this would give two black points, $x - ke_1 - e_2$ and $x - ke_1$, on top of each other. So assume $2 \leq k < m$. The case $2 \leq m < k$ is treated similarly. Then the gray line starting at the black point $x - me_2$ passes through the neighbor $x - ke_2$ of the black vertex $x - ke_1 - e_2$. Hence, it has to stop there, i.e. the next point $x - (k-1)e_1 - e_2$ cannot be gray, which it is. This gives the desired contradiction and proves the second statement.

For the final statement of the lemma, $T^c \neq \emptyset$, we show that $x \in B$ implies $x + e_1 + e_2 \in T^c$ or $x + e_1 - e_2 \in T^c$. Firstly, by construction of B, not both $x + e_1 + e_2$ and $x + e_1 - e_2$ can be black. Secondly, none of them can be gray. Indeed, assume that for example $x + e_1 + e_2$ were gray. As above, this would imply that its left neighbor $x + e_2$ would be black or gray, too. However, by construction of B, $x + e_2$ cannot be black since x is already black. Hence $x + e_2$

and $x + e_1 + e_2$ would belong to the same gray line starting at some black point $x - ke_1 + e_2$ with $k \geq 1$. However, this line would have to stop in $x + e_2$ and not extend to $x + e_1 + e_2$ since $x + e_2$ is a neighbor of the black point x, which would give a contradiction. $\square$

Now we define for any vertex $x \in \mathbb{Z}^2$ its ancestor $a(x)$ as follows: For $x \in T$ we set

$$a(x) := \begin{cases} x + e_1 & \text{if } x + e_1 \in T, \\ x + e_2 & \text{else if } x + e_2, x + e_1 + e_2 \in T, \\ x - e_2 & \text{else if } x - e_2, x + e_1 - e_2 \in T \end{cases}$$

and for $x \in T^c$ we define

$$a(x) := \begin{cases} x - e_1 & \text{if } x - e_1 \in T^c, \\ x + e_2 & \text{else if } x + e_2, x - e_1 + e_2 \in T^c, \\ x - e_2 & \text{else if } x - e_2, x - e_1 - e_2 \in T^c. \end{cases}$$

Due to Lemma 2 the function $a : \mathbb{Z}^2 \to \mathbb{Z}^2$ is well defined and determines two disjoint infinite directed trees, i.e. loop-free graphs, with sets of vertices T and T^c, respectively. (The thinning at the beginning of the initial point process was necessary to prevent the black and gray tree to disconnect the white complement into finite pieces, possibly leaving it without an infinite component.) Moreover, if we define recursively $a^0(x) := x$ and $a^{n+1}(x) := a^n(a(x))$ for $n \geq 0$, then we have that

$$\begin{aligned} &(a^n(x) \cdot e_1)_{n \geq 0} \text{ is monotone increasing for } x \in T \text{ and decreasing for } x \in T^c \text{ with} \\ &a^2(x) \cdot e_1 \geq x \cdot e_1 + 1 \text{ for } x \in T \text{ and } a^2(x) \cdot e_1 \leq x \cdot e_1 - 1 \text{ for } x \in T^c. \end{aligned} \tag{18}$$

It can be shown, cf. [FeLaTh04, Theorem 3.1(d)], that all the branches in the tree on T are a.s. finite, i.e. the length $h(x) := \sup\{n \geq 0 \mid \exists y \ a^n(y) = x\}$ of the longest line of descendants of x in T is a.s. finite for all $x \in T$. Moreover, it can be shown, cf. [FeLaTh04, Theorem 3.1(b)], that the tree on T is a.s. connected, i.e. it is not a forest consisting of several disjoint connected trees. This implies that a.s. all the branches of the white tree on T^c have finite height $h(x)$ as well. The rest of the construction is the same as in [ZerMe01, pp. 1730, 1732]: We define the environment in terms of the ancestor function a for $x, y \in \mathbb{Z}^2$ with $|x - y| = 1$ by

$$\omega(x, y) = \begin{cases} 1 - 3/(h^2(x) + 4) & \text{if } y = a(x) \\ 1/(h^2(x) + 4) & \text{else.} \end{cases}$$

By Borel Cantelli there is a constant $c > 0$ such that $P_{x,\omega}[\forall n \ X_n = a^n(x)] > c$ for all $x \in \mathbb{Z}^2$ and almost all ω. Consequently, due to (18), and since neither tree is empty,

$$P_0 \left[\liminf_{n \to \infty} \frac{X_n \cdot e_1}{n} \geq \frac{1}{2} \right] > 0 \quad \text{and} \quad P_0 \left[\liminf_{n \to \infty} \frac{X_n \cdot (-e_1)}{n} \geq \frac{1}{2} \right] > 0$$

and in particular $P_0[A_{e_1}] \notin \{0, 1\}$. Since $(\omega(x, \cdot))_{x \in \mathbb{Z}^2}$ has been obtained from B by the application of a deterministic function which commutes with all spatial shifts in $\mathbb{Z}^2$ and since B itself is stationary and totally ergodic, $(\omega(x, \cdot))_{x \in \mathbb{Z}^2}$ is stationary and totally ergodic as well. We refrained from investigating the mixing properties of this environment, which has been done for a similar counterexample for $d \geq 3$ in [BrZeiZer06].

Open problems: The gap between positive and negative results concerning the directional zero-one law in $d = 2$ could be narrowed by answering one of the following questions: (1)

Is there a stationary and ergodic counterexample to the directional zero-one law for $d = 2$, which is uniformly elliptic? For $d \geq 3$ there is such a counterexample, which is even polynomially mixing, see [BrZeiZer06]. (2) Can the directional zero-one law for $d = 2$ be extended to stationary, ergodic and uniformly elliptic environments which have weaker independence properties than finite range dependence?

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