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RECURRENCE FOR BRANCHING MARKOV CHAINS

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Abstract
The question of recurrence and transience of branching Markov chains is more subtle than for ordinary Markov chains; they can be classified in transience, weak recurrence, and strong recurrence. We review criteria for transience and weak recurrence and give several new conditions for weak recurrence and strong recurrence. These conditions make a unified treatment of known and new examples possible and provide enough information to distinguish between weak and strong recurrence. This represents a step towards a general classification of branching Markov chains. In particular, we show that in *homogeneous* cases weak recurrence and strong recurrence coincide. Furthermore, we discuss the generalization of positive and null recurrence to branching Markov chains and show that branching random walks on $\mathbb{Z}$ are either transient or positive recurrent.

1 Introduction

A branching Markov chain (BMC) (X,P,μ) is a system of particles in discrete time on a discrete state space X. The process starts with one particle in some starting position $x \in X$. At each time particles split up in offspring particles independently according to some probability distributions $(\mu(x))_{x\in X}$. The new particles then move independently according to some irreducible Markov chain (X,P) with transition probabilities P. Processes of this type are studied in various articles with different notations and variations of the model.
Ordinary Markov chains are either transient or recurrent, i.e., the starting position is either visited a finite or an infinite number of times. This $0-1$-law does not hold for branching Markov chains in general, compare with [2], [10], and [20]. Let $\alpha(x)$ be the probability that starting the BMC in x the state x is visited an infinite number of times by some particles. We can classify the BMC in transient ($\alpha(x)=0$ for all x), weakly recurrent ($0<\alpha(x)<1$ for all x), and strongly recurrent ($\alpha(x)=1$ for all x). In cases where we do not want to distinguish between weak and strong recurrence we just say that the process is recurrent. Effects of this type occur also in a varied

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model of BMC in which the branching is governed by a fixed tree, compare with [2] for more details on Markov chains indexed by trees.

Let $m(x)$ be the mean of the offspring distribution $\mu(x)$. If $m(x) = m$ for all x there is a nice and general criterion for transience and recurrence: the BMC (X, P, μ) is transient if and only if $m \leq 1/\rho(P)$, where $\rho(P)$ is the spectral radius of the underlying Markov chain (X, P). Observe that the type of the process depends only on two characteristics, m and $\rho(P)$, of the process. We give several description of the spectral radius $\rho(P)$. These description are useful to give several different proofs for the above criterion and offer various possibilities to decide whether a process is transient or recurrent.

Another purpose of this paper is to review and to continue the work of [2], [8], [10], [12], [16], [17], [18], [21], [20], [23], [25], [26], and [28]. We present a unifying description and give new conditions for weak and strong recurrence. Our results suggest that the type of the process only depends on the mean number of offspring and some variation of the spectral radius of the underlying Markov chain, too.

A useful observation concerning the different types is that transience / recurrence is sensitive to local changes of the underlying process but criteria for strong recurrence have to contain global information of the process. This fact is exploited by the idea of *seeds,* finite regions of the process that may produce an infinite number of particles independently of the rest of the process. Eventually, infinitely many particles leave this finite regions and return to the starting position. Hence, the BMC is recurrent. While the existence of one such seed already implies recurrence, we need *sufficiently many good* seeds to guarantee strong recurrence, compare with Section 3. In *homogeneous* cases, where local properties become global, recurrence and strong recurrence coincide, compare with Subsection 3.2.

In Section 2 we recall some facts about Markov chains, Green functions, and the corresponding spectral radius that are crucial for our further development. In particular, we give several description of the spectral radius of a Markov chain, e.g. in terms of power series, superharmonic functions, Perron-Frobenius eigenvalues, and rate function of large deviations. In Subsection 2.2 we define the model of BMC formally, recall the classification results for transience and recurrence of [10] and [20], and give concrete examples in Subsection 2.3.

In Section 3 we first give general conditions for strong recurrence, see Subsection 3.1, and show that in *homogeneous* cases recurrence and strong recurrence coincide, see Subsection 3.2 where we prove known (Theorems 3.7 and 3.9) and new results (Theorems 3.8 and 3.11 as well as Subsection 3.2.5) with a unified method. We then discuss, see Subsections 3.3, several conditions for strong recurrence that, while not leading to a complete classification in general, work well in concrete situations and offer a toolbox to handle further concrete problems. In particular, there is a second critical value that divides the weakly and strongly recurrent phase. We want to point out that at the second critical value the process may be either strongly or weakly recurrent, compare with Theorem 3.14. In order to develop a complete classification we study BMC on graphs in Subsection 3.3.2 and give conjectures in Subsection 3.4.

In Section 4 we generalize the concept of positive and null recurrence to BMC and give conditions for positive recurrence in terms of a functional equation, Theorem 4.4, that fits in the context of Theorems 2.11 and 3.2 and can be seen as the natural generalization of the criterion of Foster for positive recurrence of Markov chains. Eventually, we conclude in showing that homogeneous branching Markov chains on $\mathbb{Z}$ with bounded jumps are either transient or positive recurrent, compare with Theorem 4.8.

2 Preliminaries

2.1 Markov chains

A Markov chain (X,P) is defined by a countable state space X and transition probabilities $P = \big(p(x,y)\big)_{x,y\in X}$. The elements $p(x,y)$ of P define the probability of moving from x to y in one step. As usual, let X_n denote the position of the Markov chain at time n. The transition operator P can be interpreted as a (countable) stochastic matrix, so that, on the one hand, $p^{(n)}(x,y)$ is the (x,y)-entry of the matrix power P^n and, on the other hand, we have that $p^{(n)}(x,y) = \mathbb{P}_x(X_n = y)$ is the probability to get from x to y in n steps. We set $P^0 = I$, the identity matrix over X. Throughout this paper we assume that the state space is infinite and that the transition operator P is irreducible, i.e., for every pair $x,y \in X$ there exists some $k \in \mathbb{N}$ such that $p^{(k)}(x,y) > 0$.
Markov chains are related to random walks on graphs. So let us recall some basic standard notations of graphs. A (directed) graph $G = (V,E)$ consists of a finite or countable set of vertices V and an adjacency relation $\sim$ that defines the set of edges $E \subset V \times V$. A path from a vertex x to some vertex y is a sequence $x = x_0, x_1, \dots, x_n = y$ with $x_i \sim x_{i+1}$ for all $0 \le i < n$. The number n is the length of the path. A graph is (strongly) connected if every ordered pair of vertices is joined by a path. The usual graph distance $d(x,y)$ is the minimum among the length of all paths from x to y. A vertex y is called a neighbor of x if $x \sim y$. The degree $deg(x)$ of a vertex x is the numbers of its neighbors. A graph is called locally finite if $deg(x)$ is finite for all vertices x. We say a graph G has bounded geometry if $deg(\cdot)$ is bounded and is M-regular if all vertices have degree M.
Every Markov chain (X,P) defines a graph $G = (V,E)$, with a set of vertices $V = X$ and a set of edges $E := \{(x,y) : p(x,y) > 0 \ x,y \in X\}$. It is clear that a Markov chain is irreducible if and only if its corresponding graph is connected. If the transition probabilities of the Markov chain are in some kind adapted to the structure of G, we shall speak of a random walk on G with transition probabilities P. We shall call a Markov chain on a graph with symmetric adjacency relation a simple random walk (SRW) if the *walker* chooses every neighbor with the same probability, i.e., $p(x,y) = 1/deg(x)$ for $x \sim y$ and 0 otherwise.
We recall the Green function and the spectral radius of an irreducible Markov chain. These two characteristics will be crucial for our further development, compare with §1 in [29] for proofs and more.

Definition 2.1. The Green function of (X,P) is the power series

$$G(x,y|z) := \sum_{n=0}^{\infty} p^{(n)}(x,y) z^n, \ x,y \in X, \ z \in \mathbb{C}.$$

We write $G(x,x)$ for $G(x,x|1)$.

Observe, that, due to the irreducibility, $G(x,y|z)$ either converges for all $x,y \in X$ or diverges for all $x,y \in X$. Therefore, we can define R as the finite convergence radius of the series $G(x,y|z)$ and call $1/R$ the spectral radius of the Markov chain.

Definition 2.2. The spectral radius of (X,P) is defined as

$$\rho(P) := \limsup_{n\to\infty} \left(p^{(n)}(x,y)\right)^{1/n} \in (0,1]. \tag{2.1}$$

We denote $\rho(G)$ the spectral radius of the SRW on the graph G.

Lemma 2.3. *We have*

$$p^{(n)}(x,x) \le \rho(P)^n, \text{ and } \lim_{n\to\infty}\left(p^{(nd)}(x,x)\right)^{1/nd} = \rho(P),$$

where $d := d(P) := gcd\{n:\ p^{(n)}(x,x) > 0\ \forall x\}$ *is the period of P.*

If X and P are finite, the spectral radius of $\rho(P)$ becomes the largest, in absolute value, eigenvalue of the matrix P and equals 1. The spectral radius of a Markov chain with infinite state space can be approximated with spectral radii of finite sub-stochastic matrices. To this end we consider (general) finite nonnegative matrices Q. A matrix $Q = (Q_{i,j})_{i,j\in\mathbb{R}^{N\times N}}$ with nonnegative entries is called irreducible if for any pair of indices i,j we have $Q^m(i,j) > 0$ for some $m \in \mathbb{N}$. The well-known Perron-Frobenius Theorem states (e.g. Theorem 3.1.1 in [7]), among other things, that Q possesses a largest, in absolute value, real eigenvalue $\rho(Q)$. Furthermore, we have (e.g. with part *(e)* of Theorem 3.1.1 in [7])

$$\rho(Q) = \limsup_{n\to\infty}\left(Q^n(i,i)\right)^{1/n} \quad \forall 1 \le i \le N.$$

Remark 2.4. If Q is a symmetric matrix with $\rho(Q) < \infty$ then Q acts on $l^2(X)$ as a bounded linear operator with norm $\|Q\| = \rho(Q)$. The same holds true for reversible Markov chains (X,P) with P acting on some appropriate Hilbert space.

Now, let us consider an infinite irreducible Markov chain (X,P). A subset $Y \subset X$ is called irreducible if the sub-stochastic operator

$$P_Y = (p_Y(x,y))_{x,y\in Y}$$

defined by $p_Y(x,y) := p(x,y)$ for all $x,y \in Y$ is irreducible.
It is straightforward to show the next characterization.

Lemma 2.5. *Let* (X,P) *be an irreducible Markov chain. Then,*

$$\rho(P) = \sup_Y \rho(P_Y), \tag{2.2}$$

where the supremum is over finite and irreducible subsets $Y \subset X$. *Furthermore,* $\rho(P_F) < \rho(P_G)$ *if* $F \subset G$.

For finite irreducible matrices Q the spectral radius as defined in (2.1) equals the largest eigenvalue of Q. This does not longer hold true for general infinite irreducible transition kernels P, compare with [27], since the existence of a Perron-Frobenius eigenvalue can not be guaranteed. Nevertheless, the transition operator P acts on functions $f : X \to \mathbb{R}$ by

$$Pf(x) := \sum_y p(x,y)f(y), \tag{2.3}$$

where we assume that $P|f| < \infty$. It turns out that the spectral radius $\rho(P)$ can be characterized in terms of t-superharmonic functions.

Definition 2.6. Fix $t > 0$. A t-superharmonic function is a function $f : X \to \mathbb{R}$ satisfying

$$Pf(x) \le t\cdot f(x) \quad \forall x \in X.$$

We obtain the following (well-)known characterization of the spectral radius in terms of t-superharmonic functions, e.g. compare with §7 in [29].

Lemma 2.7.

$$\rho(P) = \min\{t > 0 : \exists f(\cdot) > 0 \textit{ such that } Pf \le tf\}$$

We can express the spectral radius in terms of the rate function $I(\cdot)$ of a large deviation principle (LDP). Let us assume that a LDP holds for the distance, i.e.,

$$\begin{aligned}
\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_o\left(\frac{d(X_n,o)}{n} \in O\right) &\ge -\inf_{a\in O} I(a),\\
\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_o\left(\frac{d(X_n,o)}{n} \in C\right) &\le -\inf_{a\in C} I(a),
\end{aligned}$$

for all open sets $O \subset \mathbb{R}$ and all closed sets $C \subset \mathbb{R}$. Furthermore, we assume ellipticity, i.e., that there is some constant $c > 0$ such that $p(x,y) \ge c \cdot \mathbf{1}\{p(x,y) > 0\}$ for all $x, y \in X$.
We do not claim to be the first who make the following observation that is quite intuitive since LDP's for Markov chains are closely linked with Perron-Frobenius theory for irreducible matrices.

Lemma 2.8. *Let (X,P) be an irreducible Markov chain with $p(x,y) \ge c \cdot \mathbf{1}\{p(x,y) > 0\}$ for some constant $c > 0$. Assume furthermore that a LDP holds for the distance with rate function $I(\cdot)$, then*

$$-\log \rho(P) = I(0). \tag{2.4}$$

Proof. We have

$$\begin{aligned}
-I(0) &= \lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_o\left(\frac{d(X_n,o)}{n} \le \varepsilon\right)\\
&\ge \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_o\left(X_n = o\right) = \log \rho(P).
\end{aligned}$$

For the converse we use our assumption: $p(x,y) \ge c \cdot \mathbf{1}\{p(x,y) > 0\}$ for some $c > 0$. With $0 < \varepsilon < 1$ we obtain

$$p^{(n)}(o,o) \ge \mathbb{P}_o\left(d(X_{\lceil n(1-\varepsilon)\rceil}, o) \le \lfloor n\varepsilon\rfloor\right) c^{\lfloor n\varepsilon\rfloor}.$$

Therefore,

$$\begin{aligned}
\frac{1}{n} \log \mathbb{P}_o\left(\frac{d(X_{\lceil n(1-\varepsilon)\rceil}, o)}{\lceil n(1-\varepsilon)\rceil} \le \frac{\lfloor n\varepsilon\rfloor}{\lceil n(1-\varepsilon)\rceil}\right) &= \frac{1}{n} \log \mathbb{P}_o\left(d(X_{\lceil n(1-\varepsilon)\rceil}, o) \le \lfloor n\varepsilon\rfloor\right)\\
&\le \frac{1}{n} \log\left(p^{(n)}(o,o) c^{-\lfloor n\varepsilon\rfloor}\right)\\
&= \log\left(p^{(n)}(o,o)\right)^{1/n} - \frac{\lfloor n\varepsilon\rfloor}{n} \log c.
\end{aligned}$$

Hence, for all $0 < \varepsilon < 1$

$$-I(0) \le \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_o\left(\frac{d(X_n,o)}{n} \le \frac{\lfloor n\varepsilon\rfloor}{\lceil n(1-\varepsilon)\rceil}\right) \le \log \rho - \varepsilon \log c.$$

Now, letting $\varepsilon \to 0$ finishes the proof. □

2.2 Branching Markov chains

A branching Markov chain (BMC) (X, P, μ) consists of two dynamics on the state space X: branching, $(\mu(x))_{x\in X}$, and movement, P. Here

$$\mu(x) = \big(\mu_k(x)\big)_{k\geq 1}$$

is a sequence of nonnegative numbers satisfying

$$\sum_{k=1}^{\infty} \mu_k(x) = 1 \text{ and } m(x) := \sum_{k=1}^{\infty} k\mu_k(x) < \infty,$$

and $\mu_k(x)$ is the probability that a particle in x splits up in k particles. The movement of the particles is governed through an irreducible and infinite Markov chain (X, P). Note that we always assume that each particle has at least one offspring, i.e., $\mu_0 = 0$, and hence the process survives forever. Similar results can be obtained by conditioning on the survival of the branching process.
The BMC is defined as follows. At time 0 we start with one particle in an arbitrary starting position $x \in X$. At time 1 this particle splits up in k offspring particles with probability $\mu_k(x)$. Still at time $n = 1$, these k offspring particles then move independently according to the Markov chain (X, P). The process is defined inductively: At each time each particle in position x splits up according to $\mu(x)$ and the offspring particles move according to (X, P). At any time, all particles move and branch independently of the other particles and the previous history of the process. If the underlying Markov chain is a random walk on a graph G and the branching distributions are adapted to G we shall also speak of a branching random walk (BRW). In the case where the underlying random walk is a simple random walk on a graph G we denote the process (G, μ).
We introduce the following notations in order to describe the process. Let $\eta(n)$ be the total number of particles at time n and let $x_i(n)$ denote the position of the ith particle at time n. Denote $\mathbb{P}_x(\cdot) = \mathbb{P}(\cdot|x_1(0) = x)$ the probability measure for a BMC started with one particle in x. A BMC can be viewed as a Markov chain on the *big* state space

$$\mathscr{X} := X \cup X^2 \cup X^3 \cup \dots \,.$$

Clearly, this Markov chain on $\mathscr{X}$ is transient if $m(x) > 1$ for some $x \in X$. A priori it is not clear in which sense transience and recurrence of Markov chains can be generalized to BMC. One possibility is to say a BMC is recurrent if with probability 1 at least one particle returns to the starting position. This approach was followed for example in [18]. We choose a different one, e.g. compare with [2], [10], and [20], since this approach offers a finer partition in recurrence and transience, and gives interesting generalizations of equivalent conditions for transience of (nonbranching) Markov chains, compare with Theorem 2.11.

Definition 2.9. Let

$$\alpha(x) := \mathbb{P}_x\left(\sum_{n=1}^{\infty}\sum_{i=1}^{\eta(n)} \mathbf{1}\{x_i(n) = x\} = \infty\right). \tag{2.5}$$

be the probability that x is visited an infinite number of times, given the process starts in x with one particle. A BMC is called transient if $\alpha(x) = 0$ for some ($\Longleftrightarrow$ all) $x \in X$, weakly recurrent if $0 < \alpha(x) < 1$ for some ($\Longleftrightarrow$ all) $x \in X$, and strongly recurrent if $\alpha(x) = 1$ for some ($\Longleftrightarrow$ all) $x \in X$. We write $\alpha = 0$ if $\alpha(x) = 0$ for all $x \in X$ and $\alpha > 0$ and $\alpha \equiv 1$, respectively. We call a BMC recurrent if it is not transient.

Definition 2.9 is justified through the next result.

Lemma 2.10. *We have that* $\alpha(x)=0,\ \alpha(x)>0,$ *and* $\alpha(x)=1$ *either hold for all or none* $x\in X$.

Proof. Let $x,y\in X$. Due to irreducibility and the independence of branching and movement we have that

$$\mathbb{P}_x\left(\sum_{n=1}^{\infty}\sum_{i=1}^{\eta(n)}\mathbf{1}\{x_i(n)=y\}=\infty\right)\begin{array}{l}=0\\>0\\=1\end{array}$$

is equivalent to

$$\mathbb{P}_y\left(\sum_{n=1}^{\infty}\sum_{i=1}^{\eta(n)}\mathbf{1}\{x_i(n)=y\}=\infty\right)\begin{array}{l}=0\\>0\\=1\end{array}.$$

Hence, it suffices to show that

$$\mathbb{P}_x\left(\sum_{n=1}^{\infty}\sum_{i=1}^{\eta(n)}\mathbf{1}\{x_i(n)=x\}=\infty \text{ and } \sum_{n=1}^{\infty}\sum_{i=1}^{\eta(n)}\mathbf{1}\{x_i(n)=y\}<\infty\right)=0. \tag{2.6}$$

We follow the line of the proof of Lemma 3.3 in [2]. Since (X,P) is irreducible we have $p^{(l)}(x,y)=\delta>0$ for some $l\in\mathbb{N}$. Let $N,M\in\mathbb{N}$. The probability that there are times $M<n_1,\dots,n_N$ with $n_{j-1}+l<n_j$ for all $1\le j\le N$ such that $x_i(n_j)=x$ for some $1\le i\le\eta(n_j)$ for all j but $x_i(n)\neq y$ for all $n>M$ and all $1\le i\le\eta(n)$ is at most $(1-\delta)^N$. Letting $N\to\infty$, this yields

$$\mathbb{P}_x\left(\sum_{n=1}^{\infty}\sum_{i=1}^{\eta(n)}\mathbf{1}\{x_i(n)=x\}=\infty \text{ and } \sum_{n=M}^{\infty}\sum_{i=1}^{\eta(n)}\mathbf{1}\{x_i(n)=y\}=0\right)=0.$$

Let A_M be the event in the last formula. Notice that

$$\bigcup_{M\ge1}A_M=\left\{\sum_{n=1}^{\infty}\sum_{i=1}^{\eta(n)}\mathbf{1}\{x_i(n)=x\}=\infty \text{ and } \sum_{n=1}^{\infty}\sum_{i=1}^{\eta(n)}\mathbf{1}\{x_i(n)=y\}<\infty\right\}.$$

This proves the claim. □

In analogy to [18], see also the superposition principle in [12], we introduce the following modified version of BMC. We fix some position $o\in X$, which we denote the origin of X. The new process is like the original BMC at time $n=1$, but is different for $n>1$. After the first time step we conceive the origin as *freezing*: if a particle reaches the origin, it stays there forever and stops splitting up. We denote this new process with **BMC***. The process BMC* is analogous to the original process BMC except that $p(o,o)=1,\ p(o,x)=0\ \forall x\neq o$ and $\mu_1(o)=1$ from the second time step on. Let $\eta(n,x)$ be the number of particles at position x at time n. We define the random variable $\nu(o)$ as

$$\nu(o):=\lim_{n\to\infty}\eta(n,o)\in\{0,1,\dots\}\cup\{\infty\}.$$

We write $\mathbb{E}_x\nu(o)$ for the expectation of $\nu(o)$ given that $x_1(0)=x$. Note that our notation of $\eta(n,o)$ and $\nu(o)$ is different from the one in [10]. Since the choice of the origin may affect the behavior of the BMC*, we keep track of the dependence of the variables η and ν on the choice of the origin

and write $\eta(n,o)$ and $\nu(o)$. Furthermore, our definition of the process BMC* differs from the one given in [18]: in our definition, the origin is not absorbing at time $n=1$.
The Green function $G(x,x|m)$ at $z=m$ gives the expected number of particles that visits x of a BMC with constant mean offspring $m=m(x)$ started in x. Due to this interpretation, $G(x,y|m)<\infty$ implies transience of a BMC with constant mean offspring m. While in some cases, e.g. random walk on homogeneous trees [12], the converse is true, it does not hold in general, compare with Remark 2.13. It turns out that another generating function is decisive for transience. Let

$$T_y := \min_{n\geq 1}\{X_n = y\}$$

be the time of the first return to y and

$$U(x,y|z) := \sum_{n=1}^{\infty} \mathbb{P}_x(T_y = n) z^n$$

its corresponding generating function. Due to the definition of the process BMC* we have the useful identity

$$\mathbb{E}_x \nu(y) = U(x,y|m)$$

for a BMC with constant mean offspring m. Furthermore, the equality offers a probabilistic interpretation for the generating function $U(x,y|z)$ with $z\geq 1$. The generating functions G and U are naturally connected through

$$G(x,x|z) = \frac{1}{1-U(x,x|z)}$$

and hence one can show that

$$\rho(P) = \max\{z>0:\ U(x,x|z)\leq 1\}. \tag{2.7}$$

The next Theorem, due to [20], gives several sufficient and necessary conditions for transience. Notice that the expected number of particles visiting x the first time in their ancestry line, i.e., $\mathbb{E}_x\nu(x)$, takes the role of the Green function in the theory of Markov chains and that the criterion of transience in terms of the existence of nonconstant superharmonic functions becomes (iii).

Theorem 2.11. *A BMC (X,P,μ) with $m(y)>1$ for some y is transient if and only if the three equivalent conditions hold:*

(i) $\mathbb{E}_o\nu(o)\leq 1$ *for some (⟺ all)* $o\in X$.

(ii) $\mathbb{E}_x\nu(o)<\infty$ *for all* $x,o\in X$.

(iii) There exists a strictly positive function $f(\cdot)$ *such that*

$$Pf(x) \leq \frac{f(x)}{m(x)} \quad \forall x\in X.$$

In particular if the mean offspring is constant, i.e., $m(x)=m\ \forall x\in X$, we have, due to Lemma 2.7, the following result of [10]. Observe that in this case we can speak about a *critical behaviour.*

Theorem 2.12. *The BMC (X,P,μ) with constant mean offspring $m>1$ is transient if and only if $m\leq 1/\rho(P)$.*

Theorem 2.12 follows directly from Lemma 2.7 and part *(iii)* of Theorem 2.11. Another way, compare with [30], to see Theorem 2.12 is combining Theorem 2.11 *(i)* and the fact that $\rho(P) = \max\{z > 0: \ U(x,x|z) \leq 1\}$ (see Equation (2.7)) and conclude with $\mathbb{E}_x \nu(y) = U(x,y|m)$. We give a direct proof, without using the abstract arguments of Lemma 2.7 or Equation (2.7), since the arguments used are helpful to understand the reasonings in the remainder of the paper.

Proof. The case $m < 1/\rho(P)$ is clear, since $G(x,x|m) < \infty$ implies transience. To show that $m > 1/\rho(P)$ implies recurrence we compare the original BMC with an embedded process and prove that this process with fewer particles is recurrent. We start the BMC in $o \in X$. We know from the hypothesis and the definition of $\rho(P)$ that there exists a $k = k(o)$ such that

$$p^{(k)}(o,o) > m^{-k}.$$

We construct the embedded process $(\xi_i)_{i\geq 0}$ by observing the BMC only at times $k, 2k, 3k, \ldots$ and by neglecting all the particles not being in position o at these times. Let ξ_i be the number of particles of the new process in o at time ik. The process $(\xi_i)_{i\geq 0}$ is a Galton-Watson process with mean $p^{(k)}(o,o) \cdot m^k > 1$, thus survives with positive probability. Eventually, the origin is hit infinitely often with positive probability.
In order to prove transience at the critical value $m = 1/\rho(P)$, we use a continuity argument to show that the subset $\{m : \ (X,P,\mu) \text{ is recurrent}\} \subset \mathbb{R}$ is open. In other words for any recurrent BMC with mean offspring m there exists some $\varepsilon > 0$ such that the BMC with mean offspring $m - \varepsilon$ is still recurrent and hence the critical BMC must be transient. So assume the BMC to be recurrent. Due to Theorem 2.11 *(i)* there exists some k such that $\mathbb{E}_o \eta(k,o) > 1$. We define an embedded Galton-Watson process $(\zeta_i)_{i\geq 0}$. We start a BMC* with origin o with one particle in o. Let Ψ_1 be the particles that are the first particles in their ancestry line to return to o before time k. We define Ψ_i inductively as the number of particles that have an ancestor in Ψ_{i-1} and are the first in the ancestry line of this ancestor to return to o in at most k time steps. Clearly $\zeta_0 := 1$ and $\zeta_i := |\Psi_i|, \ i \geq 1$, defines a supercritical Galton-Watson process since $E\zeta_1 = \mathbb{E}_o \eta(k,o) > 1$. Furthermore, $E\zeta_1 = \mathbb{E}_o \eta(k,o)$ and

$$\mathbb{E}_x \eta(k,o) = m \cdot \sum_{y \neq o} p(x,y) \mathbb{E}_y \eta(k-1,o) + m \cdot p(x,o).$$

Now, it is easy to see that $E\zeta_1$ is continuous in m. Eventually, for $\varepsilon > 0$ sufficiently small the BMC with mean offspring $m - \varepsilon$ is still recurrent.

□

Remark 2.13. Theorem 2.12 implies that $G(x,x|m) < \infty$ is equivalent to transience of the process if and only if $G(x,x|1/\rho(P)) < \infty$, i.e., the underlying Markov chain is ρ-transient.

Remark 2.14. The fact that $m\rho(P) > 1$ implies the recurrence of the BMC can be also seen by dint of the interpretation as a general branching process and Lemma 2.5. Due to the latter there exists a finite and irreducible Y such that $m\rho(P_Y) > 1$. Now, let us consider only particles in Y and neglect the particles leaving Y. This defines a supercritical multi-type Galton-Watson process with first moments $m \cdot P_Y$ that survives with positive probability, compare with Chapter V in [1], and hence $\alpha(x) > 0$. Finite regions as Y, that may produce an infinite number of particles without help from outside, are referred to as *seeds.* Note that in [6] regions of these kind are called *recurrent seeds.*

2.3 Examples

Example 2.15. Consider the random walk on $\mathbb{Z}^d$, $d \in \mathbb{N}$. Let $e_i \in \mathbb{Z}^d$ with $(e_i)_j = \delta_{ij}$ for $i, j \in \{1, \dots, d\}$, $d \geq 1$, and define transition probabilities P by

$$p(x, x+e_i) = p_i^+,\ p(x, x-e_i) = p_i^- \text{ such that}$$

$$\sum_{i=1}^d p_i^+ + \sum_{i=1}^d p_i^- = 1, \quad \forall x \in \mathbb{Z}^d$$

and such that P is irreducible. Take branching distributions with constant mean offspring m. We calculate, using for example large deviation estimates (compare with Lemma 2.2):

$$\rho(P) = 2\sum_{i=1}^d \sqrt{p_i^+ p_i^-}.$$

Hence, the corresponding BMC is transient if and only if

$$m \leq 1 \Big/ \left(2\sum_{i=1}^d \sqrt{p_i^+ p_i^-}\right).$$

In particular, the BRW $(\mathbb{Z}^d, \mu)$ with constant mean offspring is recurrent if $m > 1$.

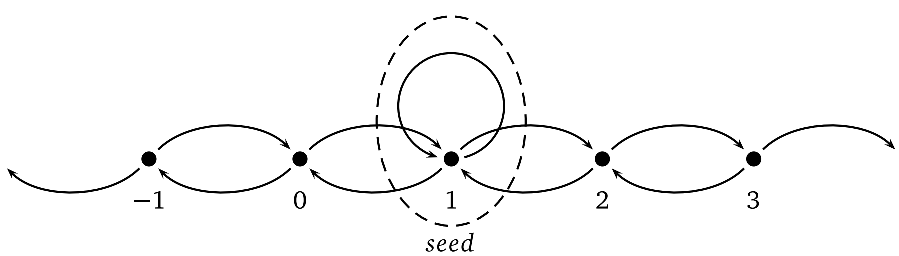
Example 2.16. We consider an irreducible symmetric random walk on a finitely generated group and constant mean offspring m. We can classify groups in amenable or nonamenable using branching random walks: a finitely generated group G is amenable if and only if every BRW on G with constant mean offspring $m > 1$ is recurrent. This statement is a variation of Proposition 1.5 in [3] where tree-indexed Markov chains are considered. To proof it, we merely need to combine Theorem 2.12 with the well-known result of Kesten stating that every irreducible and symmetric Random Walk on a finitely generated group G has spectral radius 1 if and only if G is amenable, e.g. compare with Corollary 12.5 in [29].

Example 2.17. Let (X, P) be the SRW on the regular tree $\mathbb{T}_M$. We have $\rho(P) = \rho(\mathbb{T}_M) = \frac{2\sqrt{M-1}}{M}$ (compare with Lemma 1.24 in [29]). The BMC $(\mathbb{T}_M, \mu)$ with constant offspring distribution μ is transient if and only if

$$m \leq \frac{M}{2\sqrt{M-1}}.$$

Example 2.18. We consider the example of Section 5 in [5] on $\mathbb{Z}$ with binary branching, i.e., $\mu_2(x) = 1$ for all $x \in \mathbb{Z}$. The transition probabilities are $p(1,0) = p(1,2) = 1/8$, $p(1,1) = 3/4$ and

$$p(x, x+1) = 1 - p(x, x-1) = \frac{2+\sqrt{3}}{4} \quad x \neq 1.$$



The BMC (X, P, μ) is not strongly recurrent since the spatially homogeneous BMC with $m(x) = 2$ and $p(x, x+1) = 1 - p(x, x-1) = \frac{2+\sqrt{3}}{4}$ for all $x \in \mathbb{Z}$ is transient, see Example 2.15. Let us first take 0 as the origin o of the corresponding BMC*. We show that $E_1 \nu(0) = \infty$. The number of particles which never leave state 1 is given by a Galton-Watson process with mean number $2 \cdot 3/4 > 1$. And so, with positive probability, an infinite number of particles visits state 1. This clearly implies $\mathbb{E}_1 \nu(0) = \infty$. Eventually, the BMC (X, P, μ) is recurrent, but not strongly recurrent. Notice, if $o = 1$ then we have for the corresponding BMC* that $\mathbb{E}_x \nu(1) < \infty$ for all x.

Remark 2.19. Example 2.18 illustrates very well the idea of *seeds* that make BMCs recurrent: state 1 can be seen as a seed that may create an infinite number of particles without help from outside. Since in this case the seed is just a *local inhomogeneity,* the process may *escape* the seed and is not strongly recurrent.

3 Criteria for strong recurrence

In this section we discuss criteria for strong recurrence of BMC. In Subsection 3.1 we present known and new conditions for general Markov chains and offspring distributions. While Theorem 3.2 of [18] is more of theoretical interest the Lemma 3.4 and Proposition 3.6 are new useful tools to prove strong recurrence. In particular, if the underlying Markov chain and the offspring distributions are *homogeneous,* recurrence and strong recurrence coincide, compare with Subsection 3.2.

In Subsection 3.3 we present several approaches in order to develop sufficient and necessary conditions for strong recurrence for general BMC, see Lemma 3.12, Theorem 3.14, and Lemma 3.17. An interesting observation is that transience / recurrence depend on local properties and recurrence / strong recurrence on global properties of the process. Therefore a classification result would demand a suitable description of infinite structures and would deliver a measure for inhomogeneity of the process. The conditions for strong recurrence are given in terms of appropriate spectral radii. While a general and applicable criterion for strong recurrence remains challenging, our condition work well in concrete situations, e.g. compare with Theorem 3.15 and Example 3.26. The section ends with a short discussion including conjectures in Subsection 3.4.

3.1 General Criteria

The criteria for transience and recurrence, see Theorems 2.11 and 2.12, do not depend on the precise structure of the branching mechanism but only on the mean offspring m. This can no longer hold true for criteria for strong recurrence since we can choose the branching distribution such that with positive probability no branching occurs at all, see the following Example 3.1.

Example 3.1. Consider the random walk on $\mathbb{Z}$ with drift to the right, i.e., $p(x, x+1) = 1 - p(x, x-1) = p > 1/2$ $\forall x \in \mathbb{Z}$. In order to construct a nontrivial BMC where with positive probability no branching occurs, i.e., $\mathbb{P}(\eta(n) = 1 \ \forall n \geq 1) > 0$, we first investigate the underlying random walk. We know, Law of Large Numbers, that $S_n/n \to s := 2p - 1$ as $n \to \infty$. Hence for each realization $S_n(\omega)$ of the random walk there exists some $T(\omega)$ such that $S_n(\omega) > (s - \varepsilon)n$ for all $n > T(\omega)$ for some small $\varepsilon > 0$. Define

$$C_T := \{\omega : S_n(\omega) > (s - \varepsilon)n \ \forall n > T\} \text{ and } C_\infty := \bigcup_{T=1}^{\infty} C_T.$$

Due to the Law of Large Numbers and since the C_T are increasing, we have $1 = \mathbb{P}(C_\infty) = \lim_{T\to\infty} \mathbb{P}(C_T)$. Hence, there exists some $T > 0$ such that $\mathbb{P}(A) > 0$, with $A := \{\omega : S_n(\omega) > (s-\varepsilon)n \ \forall n > T\}$. Now we choose the branching distributions such that on the event A with positive probability no branching occurs. We define $\big(\mu(x)\big)_{x\in\mathbb{Z}}$ such that $m(x) = m > 1/\rho(P)$ and $\mu_1(x) = 1 - e^{-bx}$ for $x > 0$, and $\mu_1(x) = (1 - e^{-b})$ for $x \le 0$, where b is some positive constant. Eventually,

$$\mathbb{P}(\eta(n) = 1 \ \forall n \ge 1|A) \ge \left(1 - e^{-b}\right)^T \prod_{n=T}^{\infty} \left(1 - e^{-b(s-\varepsilon)n}\right) > 0$$

and the BMC (X, P, μ) is not strongly recurrent but recurrent since $m > 1/\rho(P)$. On the other hand if $\tilde\mu(x) = \tilde\mu$ with $m > 1/\rho(P)$ and hence is homogeneous, then the BMC $(X, P, \tilde\mu)$ is strongly recurrent, compare with Subsection 3.2.

Despite the above discussion, there exists a sufficient and necessary condition for strong recurrence where the offspring distribution may depend on the states. Let

$$\Psi(x,z) := \sum_{k=1}^{\infty} z^k \mu_k(x)$$

be the generating function of $\mu(x)$. We have the following necessary and sufficient condition for strong recurrence of [18].

Theorem 3.2. *The BMC (X, P, μ) is not strongly recurrent if and only if there exists a finite subset M of X and a function $0 < g \le 1$, such that*

$$\Psi\left(x, Pg(x)\right) \ge g(x) \quad \forall x \notin M \tag{3.1}$$

and

$$\exists y \notin M : \ g(y) > \max_{x\in M} g(x). \tag{3.2}$$

Proof. We give a sketch of the proof in [18]. We start the BMC in y and define

$$\widetilde{Q}(n) := \prod_{i=1}^{\eta(n)} g(x_i(n)).$$

Furthermore, let

$$\tau := \min_{n\ge 0}\{\exists i \in \{1,\dots,\eta(n)\} : \ x_i(n) \in M\}$$

be the entrance time in M. It turns out that

$$Q(n) := \widetilde{Q}(n \wedge \tau)$$

is a submartingal for $n \ge 0$. Since $Q(n)$ is bounded, it converges a.s. and in L^1. Hence there exists some random variable Q_∞ such that

$$Q_\infty = \lim_{n\to\infty} Q(n)$$

and

$$\mathbb{E}_y Q_\infty \quad = \quad \lim_{n\to\infty} \mathbb{E}_y Q(n) \ge \mathbb{E}_y Q(0) = g(y). \tag{3.3}$$

Assuming that the BMC is strongly recurrent, we obtain that $\tau < \infty$ a.s. and therefore $Q_\infty \le \max_{x\in M} g(x)$. This contradicts inequality (3.3) since $g(y) > \max_{x\in M} g(x)$. The converse is more constructive. Assume the BMC not to be strongly recurrent and consider the probability that starting the BMC in x no particles hits some $o \in X$:

$$g(x) := \mathbb{P}_x\left(\forall n \geq 0\ \forall i = 1, \ldots, \eta(n):\ x_i(n) \neq o\right) \quad \text{for } x \neq o$$

and $g(o) := 0$. One easily checks that g verifies the requested conditions for $M := \{o\}$. □

The conditions in Theorem 3.2 are difficult to check. We did not find a more explicit formulation. Furthermore, it is not clear if strong recurrence depends on the whole structure of the branching distributions, since the above conditions are written in term of the generating function of μ. Nevertheless, we see in Subsection 3.2 that in homogenous cases the necessary and sufficient condition for strong recurrence does only depend on the mean offspring $m(x)$ and conjecture that this fact holds in general, see Conjecture 3.29.

Remark 3.3. Theorem 3.2 implies in particular that strong recurrence depends on global properties of the BMC since local properties can be excluded by the choice of the finite set M.

A useful tool are induced Markov chains that couple a Markov chain to the branching Markov chain. The induced Markov chain X_n is defined inductively. We can think of it as the position of a label. The BMC starts, at time $n = 0$, with one particle that is labelled. At time n the labelled particle picks at random one of its offspring and hand over the label. It is easy to check that the position of the label defines a Markov chain with transition probabilities P. Another way to interpret the induced Markov chain is to modify the original process in a way that particles do not die but produce offspring with distribution $\tilde{\mu}_{i-1} = \mu_i,\ i \geq 1$. In this case we can speak of the trace of a particle which has the same distribution as the trace of a Markov chain with transition kernel P.
The next Lemma is our main tool to show strong recurrence.

Lemma 3.4. *Let $c > 0$ and define $C := \{\alpha(x) \geq c\}$. If the set C is recurrent with respect to the Markov chain (X, P), i.e., is a.s. hit infinitely often by the trace of the Markov chain, the BMC is strongly recurrent.*

Proof. The idea is to define a sequence of embedded supercritical Galton-Watson processes and show that at least one of them survives. We start the process with $x = x_1 \in C$. Let us define the first Galton-Watson process $(\zeta_i^{(1)})_{i\geq 0}$. To this end, let Ψ_1 be the particles that are the first particles in their ancestry line to return to x before time k (to be chosen later) and define Ψ_i inductively as the number of particles that have an ancestor in Ψ_{i-1} and are the first in the ancestry line of this ancestor to return to x in at most k time steps. Clearly $\zeta_0^{(1)} := 1$ and $\zeta_i^{(1)} := |\Psi_i|,\ i \geq 1$, defines a Galton-Watson process. Due to the definition of the process BMC* we have that $E\zeta_1^{(1)} = \mathbb{E}_x \eta(k, x)$.
Claim: There is some k such that $E\zeta_1^{(1)} > 1$ and that the probability of survival of $(\zeta_i^{(1)})_{i\geq 0}$ is larger than $c/2$.
We choose k such that the probability of survival of $(\zeta_i^{(1)})_{i\geq 0}$ is larger than $c/2$. If this first Galton-Watson process dies out we wait until the induced Markov chain hits a point $x_2 \in C$; this happens with probability one since C is recurrent w.r.t the induced Markov chain. Then we start a second process $(\zeta_i^{(2)})_{i\geq 0}$, defined in the same way as the first but started in position x_2. If the second process dies out, we construct a third one, and so on. We obtain a sequence of independent Galton-Watson processes $\left((\zeta_i^{(j)})_{i\geq 0}\right)_{j\geq 1}$. The probability that all these processes die out is less than $\prod_{j=1}^{\infty}(1 - c/2) = 0$. Eventually, at least one process survives and we have $\alpha(x) = 1$ for all $x \in X$.

It remains to prove the claim. Consider the Galton-Watson process $(Z_i)_{i\geq 0}$ constructed as $(\zeta_i^{(1)})_{i\geq 0}$ but with $k=\infty$. Hence $EZ_1 = \mathbb{E}_x \nu(x) \in (0,\infty]$. Let $f(s)=\sum_{j=1}^{\infty}\mu_j s^j,\ |s|\leq 1$ be the generation function of $(Z_i)_{i\geq 0}$. From the definition of f as a power series with nonnegative coefficients, we have that it is strictly convex and increasing in $[0,1)$. Furthermore, the extinction probability q of $(Z_i)_{i\geq 0}$ is the smallest nonnegative root of the equation $t=f(t)$. For every k we define a process $(Z_i^k)_{i\geq 0}$ with corresponding mean offspring $\eta(k,x)$, distribution $\mu^k=(\mu_1^k,\mu_2^k,\ldots)$, and generating function f^k. The probabilities μ^k converge pointwise to μ and so do the generating functions. Using the fact that $f(q)=q$ and $1-q\geq c$ we find a k such that $f^k(1-c/2)\leq 1-c/2$, thus $(Z_i^k)_{i\geq 0}$ survives with probability at least $c/2$. □

Remark 3.5. In Lemma 3.4 we can replace the condition that C is recurrent w.r.t. the Markov chain by the condition that C is recurrent w.r.t. the BMC, i.e., C is visited infinitely often by some particles of the BMC.

Let $F(x)$ denote the return probability of the Markov chain (X,P), i.e., the probability that the Markov chain started in x returns to x. If we assume the branching distributions to be constant, i.e., $\mu(x)=\mu$ for all $x\in X$, we have the following sufficient condition for strong recurrence in terms of the mean offspring and the return probability of the underlying Markov chain.

Proposition 3.6. *The BMC (X,P,μ) with constant offspring distribution is strongly recurrent if*

$$m > \sup_{x\in X}\frac{1}{F(x)}.$$

Proof. Due to Lemma 3.4 we have to show that $\alpha(x)\geq c$ for all $x\in X$ and some $c>0$. Consider the Galton-Watson process $(\tilde{\xi}_i)_{i\geq 0}$ with offspring distribution μ and mean m. Furthermore, let p such that

$$\frac{1}{m} < p < \inf_{x\in X} F(x)$$

and percolate the process $(\tilde{\xi}_i)_{i\geq 0}$ with survival parameter p. This leads to a Galton-Watson process $(\xi_i)_{i\geq 0}$ with mean $mp>1$ and some survival probability $c>0$, compare with [14]. Back on BMC, we start the process (X,P,μ) with one particle in some arbitrary position, say o, and compare the original process with the BMC $(X,P,\tilde{\mu})$ with fewer particles: $\tilde{\mu}(o):=\mu(o)$ and $\tilde{\mu}_1(x):=1$ for all $x\neq o$. In other words, $(X,P,\tilde{\mu})$ does only branch in o. Observe that the number of particles returning to o in this process can be described by dint of a percolated Galton-Watson process $(\zeta_i)_{i\geq 0}$ with offspring distribution μ and survival parameter $F(o)$. Since $F(o)<p$ we can use a standard coupling of Bernoulli percolation, compare with Chapter 4 in [15], to prove that the survival probability of $(\zeta_i)_{i\geq 0}$ is at least the one of $(\xi_i)_{i\geq 0}$. If $(\zeta_i)_{i\geq 0}$ survives, an infinite number of particles visits o in $(X,P,\tilde{\mu})$ and hence in (X,P,μ) as well. We can conclude that $\alpha(o)\geq c$ for the original BMC (X,P,μ). □

3.2 Homogeneous BMC

Lemma 3.4 offers a general argument to show strong recurrence. In particular, it is used to prove that homogeneous BMC are strong recurrent if and only if they are recurrent. This fact is also plausible from the viewpoint of seeds. An infinite number of seeds are visited and each of these gives birth to a supercritical multi-type Galton-Watson process with extinction probability bounded from below. We give several known (3.2.1, 3.2.3) and new (3.2.2, 3.2.4, and 3.2.5) examples of homogeneous processes. They are all consequences of Theorem 2.12 and Lemma 3.4.

3.2.1 Quasi-transitive BMC

Let X be a locally finite, connected graph with discrete metric d. An automorphism of X is a self-isometry of X with respect to d, and $AUT(X)$ is the group of all automorphisms of X. Recall that when a group Γ acts on a set X, this process is called a group action: it permutes the elements of X. The group orbit of an element x is defined as $\Gamma x := \{\gamma x : \gamma \in \Gamma\}$. A group Γ acts transitivly on X if it possesses only a single group orbit, i.e., for every pair of elements x and y of X, there is a group element $\gamma \in \Gamma$ such that $\gamma x = y$. The graph X is called transitive if $AUT(X)$ acts transitively on X, and quasi-transitive if $AUT(X)$ acts with a finite number of orbits. Let P be the transition matrix of an irreducible random walk on X and $AUT(X,P)$ be the group of all $\gamma \in AUT(X)$ which satisfy $p(\gamma x, \gamma y) = p(x,y)$ for all $x, y \in X$. We say the Markov chain (X,P) is transitive if the group $AUT(X,P)$ acts transitively on X and quasi-transitive if $AUT(X,P)$ acts with a finite number of orbits on X.
The definition of quasi-transitivity can be extended to BMC. We say a BMC is quasi-transitive if the group $AUT(X,P,\mu)$ of all $\gamma \in AUT(X,P)$ which satisfy $\mu_k(x) = \mu_k(\gamma x)$ $\forall k \geq 1$ for all $x \in X$ acts with a finite number of orbits on X. Observing that $\alpha(x)$ attains only a finite number of values and hence $\alpha(x) \geq c$ for some $c > 0$ we obtain due to Theorem 2.12 and Lemma 3.4 the following result. It is due to [10] and also generalizes some results of [26].

Theorem 3.7. *Let (X,P,μ) be a quasi-transitive BMC with constant mean offspring $m(x) = m > 1$. It holds that*

- *the BMC is transient ($\alpha = 0$) if $m \leq 1/\rho(P)$.*
- *the BMC is strongly recurrent ($\alpha = 1$) if $m > 1/\rho(P)$.*

3.2.2 Branching random walk on trees with finitely many cone types

An important class of *homogeneous* trees are periodic trees that are also known as trees with finitely many cone types, compare with [15] and [22] . These trees arise as the directed cover (based on r) of some finite connected directed graph G : the tree T has as vertices the finite paths in G, i.e., $\langle r, i_1, \ldots, i_n \rangle$. We join two vertices in T by an edge if one path is an extension by one vertex of the other. The cone T^x at $x \in T$ is the subtree of T rooted at x and spannend by all vertices y such that x lies on the geodesic from r to y. We say that T^x and T^y have the same cone type if they are isomorphic as rooted trees and every cone type corresponds in a natural way to some vertex in G. Let $\tau(x)$ be the function that maps a vertex $x \in T$ to its cone type in G. If G is strongly connected, i.e., for every pair x, y there is a directed path in G from x to y, we call the cone types irreducible. In this case every cone *contains* every other cone type as a subtree.
We consider the nearest neighbour random walk on T according to [22]. Suppose we are given transition probabilities $q(i,j)_{i,j \in G}$ on G. We may hereby assume, w.l.o.g., that $q(i,j) > 0$ if and only if there is an edge from i to j in G. Furthermore, suppose we are given *backward probabilities* $p(-i) \in (0,1)$ for each $i \in G$. Then the random walk on the tree T is defined through the following transition probabilities $p(x,y)$, where $x, y \in T$:

$$p(o,y) := q(r, \tau(y)), \text{ if } x = y^-,$$

and for $x \neq o$ with $\tau(x) = i$

$$p(x,y) := \begin{cases} (1-p(-i))q\big(\tau(x), \tau(y)\big), & \text{if } x = y^- \\ p(-i), & \text{if } y = x^-, \end{cases}$$

where x^- is the *ancestor* of x. It's worth to mention, that if the cone type are irreducible, then $\rho(P) < 1$ if and only if the random walk is transient, compare with Theorem B in [22].
Furthermore, we assign branching distributions μ to the vertices of G and define the BRW (T, P, μ) with $\mu(x) = \mu(\tau(x))$. We speak of a BRW of finitely many (irreducible) cone types and have the following classification:

Theorem 3.8. *Let (T, P, μ) be BRW with finitely many irreducible cone types and constant mean offspring $m(x) = m > 1$. It holds that the BMC*

- *is transient $(\alpha = 0)$ if $m \leq 1/\rho(P)$,*
- *is strongly recurrent $(\alpha = 1)$ if $m > 1/\rho(P)$.*

Proof. First observe that the process is not quasi-transitive and $\alpha(x) \neq \alpha(y)$ for $\tau(x) = \tau(y)$. We prove that for every cone type $\alpha(x) \geq c(\tau(x)) > 0$ and conclude with Theorem 2.12 and Lemma 3.4. Due to Lemma 2.5 there exists a finite subset F such that $m \cdot \rho(P_F) > 1$. Now, since the cone types are irreducible every cone T^x contains F as a subset (hereby we mean a subset that is isomorphic to F). We fix a cone type say $i \in G$ and and let x such that $\tau(x) = i$. There exists some $n \in \mathbb{N}$ such that the ball $B_n(x)$ of radius n around x contains F. Lemma 2.5 yields that $\rho(T^x \cap B_n(x)) \cdot m > 1$. Recalling Remark 2.14 we notice that the embedded multi-type Galton-Watson process living on $T^x \cap B_n(x)$ is supercritical and survives mit positive probability, say $c(i)$. We can conclude with a standard coupling arguments that $\alpha(x) \geq c(i)$ for all $\tau(x) = i$. □

If the cone types are not irreducible all three phases may occur, compare with Theorem 3.15.

3.2.3 Branching random walk in random environment on Cayley graphs

Let G be a finitely generated group. Unless G is abelian, we write the group operation multiplicatively. Let S be a finite symmetric generating set of G, i.e., every element of G can be expressed as the product of finitely many elements of S and $S = S^{-1}$. The Cayley graph $X(G, S)$ with respect to S has vertex set G, and two vertices $x, y \in G$ are joined by an edge if and only if $x^{-1}y \in S$. Now, let q be some probability measure on S. The random walk on $X(G, S)$ with transition probabilities q is the Markov chain with state space $X = G$ and transition probabilities

$$p(x, y) = q(x^{-1}y) \quad \text{for } x^{-1}y \in S$$

and 0 otherwise. The discrete convolution is defined as $q * q(x) = \sum_y q(y)q(y^{-1}x)$. The n-step transition probabilities are

$$p^{(n)}(x, y) = q^n(x^{-1}y),$$

where q^n is the n-fold discrete convolution of q with itself. We start the random walk at time 0 in some position $o \in X$.
We introduce the random environment. Let $\mathscr{M}$ be the collection of all probability measures on S and let $(\omega_x)_{x \in X}$ be a collection of iid random variables with values in $\mathscr{M}$ which serve as an environment. For each realization $\omega := (\omega_x)_{x \in X}$ of this environment, we define a Markov chain $(X_n)_{n \in \mathbb{N}}$ on $X = G$ with starting position o and

$$\mathbb{P}_{\omega,o}(X_{n+1} = y | X_n = x) := p_\omega(x, y) := \omega_x(x^{-1}y) \quad \forall n \geq 1.$$

We denote by P_ω the transition kernel of the Markov chain on the state space X.
Let η be the distribution of this environment. We assume that η is a product measure with one-dimensional marginal Q. The support of Q is denoted by $\mathscr{K}$ and its convex hull by $\hat{\mathscr{K}}$. We always

assume the following condition on Q that ensures the irreducibility of a random walk with transition probabilities $q \in \mathscr{K}$:

$$Q\{\omega : \omega(s) > \gamma \ \forall s \in S'\} = 1 \text{ for some } \gamma > 0, \tag{3.4}$$

where $S' \subseteq S$ is a minimal set of generators, i.e., every proper subset $T \subsetneq S'$ is not a generating set.

In addition to the environment which determines the random walk we introduce a random environment determining the branching mechanism. Let $\mathscr{B}$ be the set of all infinite positive sequences $\mu = (\mu_k)_{k\geq 1}$ satisfying $\sum_{k=1}^{\infty} \mu_k = 1$ and $m(\mu) := \sum_{k=1}^{\infty} k\mu_k < \infty$. Let $\widetilde{Q}$ be a probability distribution on $\mathscr{B}$ and set

$$m^* := \sup\{m(\mu) : \mu \in \mathrm{supp}(\widetilde{Q})\} \tag{3.5}$$

which may take values in $\mathbb{R} \cup \{\infty\}$. Let $(\omega_x)_{x\in X}$ be a collection of iid random variables with values in $\mathscr{M}$ and $(\mu_x)_{x\in X}$ be a collection of iid random variables with values in $\mathscr{B}$ such that $(\omega_x)_{x\in X}$ and $(\mu_x)_{x\in X}$ are independent, too. Let Θ be the corresponding product measure with one-dimensional marginal $Q \times \widetilde{Q}$. For each realization $(\omega, \mu) := (\omega_x, \mu_x)_{x\in X}$ let P_ω be the transition kernel of the underlying Markov chain and branching distribution $\mu(x) = \mu_x$. Thus, each realization (ω, μ) defines a BMC (X, P_ω, μ). We denote by $\mathbb{P}_{\omega,\mu}$ the corresponding probability measure.

We assume that $m^* > 1$, excluding the case where the BMC is reduced to a Markov chain without branching.

The classification is due to [20] where it is proved for branching random walk in random environment (BRWRE) on Cayley graphs. Furthermore, compare to [21] where it is shown for a model where branching and movement may be dependent. The interesting fact is that the type only depends on some extremal points of the support of the random environment, namely the highest mean offspring and the *less transient* homogeneous random walk.

We obtain due to Lemma 3.4 and Theorem 2.12 that the spectral radius is deterministic, i.e., $\rho(P_\omega) = \rho$ for Θ-a.a. realizations. This can also be seen directly from the observation that $\rho(P_\omega) = \limsup \left(p^{(n)}(x,x)\right)^{1/n}$ does not depend on x and hence by ergodicity of the environment is constant a.s. .

Theorem 3.9. *If $m^* \leq 1/\rho$ then the BRWRE is transient for Θ-a.a. realizations (ω, μ), otherwise it is strongly recurrent for Θ-a.a. realizations (ω, μ).*

In the special case of the lattice the spectral radius ρ can be calculated explicitly.

Corollary 3.10. *The BRWRE on $\mathbb{Z}^d$ is strongly recurrent for Θ-a.a. realizations if*

$$(m^*)^{-1} < \sup_{p\in\mathscr{K}} \inf_{\theta\in\mathbb{R}^d} \left(\sum_s e^{\langle\theta,s\rangle} p(s) \right).$$

Otherwise it is transient for Θ-a.a. realizations.

3.2.4 BRW on percolation clusters

Let in this subsection G be a graph with bounded geometry and origin o. We consider Bernoulli(p) percolation on G, i.e., for fixed $p \in [0,1]$, each edge is kept with probability p and removed otherwise, independently of the other edges. Denote the random subgraph of G that remains by $C(\omega)$ and $C(\omega, x)$ the connected component containing x. We refer to Chapter 6 in [15] for more information and references on percolation models.

Theorem 3.11. *The BRW with constant offspring distribution and $m > 1$ and underlying SRW on a connected component of $C(\omega)$ is a.s. strongly recurrent.*

Proof. We start the BRW in o and consider $C(\omega, o)$. Clearly if the component is finite then the BRW is strongly recurrent. Due to Lemma 2.5 and $\rho(\mathbb{Z}) = 1$ there exists a subset Y of $\mathbb{Z}$ such that $\rho(P_Y) \cdot m > 1$. W.l.o.g. we can assume Y to be a line segment of length k. Now, let us imagine that the percolation is constructed during the evolution of the BRW. For $n \geq 1$ we denote $B_n = B_n(o)$ the ball of radius n around the origin o. We percolate the edges in the ball B_k. The percolation cluster is now defined inductively. If one vertex of the border, say x_i, of the ball B_{ik} is hit by some particle we percolate the edges in $B_{(i+1)k} \setminus B_{ik}$. With positive probability $\beta(x_i)$ we have that $C(\omega, x_i) \cap (B_{(i+1)k} \setminus B_{ik})$ equals the line segment of length k. Since G is of bounded geometry we have that $\beta(x_i) \geq \delta > 0$ for all x_i. Observe that $\alpha(x_i)$ is at least the survival probability, say c, of the multi-type Galton-Watson process restricted on the line segment of length k, compare with Remark 2.14. Eventually, either $C(\omega, o)$ is finite or the set $\{\alpha(x) \geq c \cdot \delta\}$ is recurrent w.r.t. the BRW and we can conclude with Remark 3.5. $\square$

3.2.5 Uniform BMC

Let us assume that $(p^{(l)}(x,x))^{1/l}$ converges uniformly in x, i.e., $\forall\, \varepsilon > 0\ \exists\, l$ such that $(p^{(l)}(x,x))^{1/l} > \rho(P) - \varepsilon\ \forall x \in X$, and that there is a $k \in \mathbb{N}$ such that $\inf_x \sum_{i=1}^{k} i\mu_i(x) \geq 1/\rho(P)$. Now, consider a modified BMC with branching distributions

$$\tilde{\mu}_0(x) = \sum_{i=k+1}^{\infty} \mu_i(x) \text{ and } \tilde{\mu}_i(x) = \mu_i(x) \text{ for } i = 1, \ldots, k \text{ and } x \in X.$$

For this new process we obtain a sequence of supercritical Galton-Watson processes $((\zeta_i^{(j)})_i)_j$ with bounded variances and means bounded away from 1, since l and k do not depend on the starting position s_j. Observe that we have for the generating function f of a Galton Watson process with mean m and variance σ^2 that $f'(1) = m$ and $f''(1) = \sigma^2/(m - m^2)$. The extinction probability q of a Galton-Watson process is the unique nonnegative solution less than 1 of the equation $s = f(s)$. Using Taylor's Theorem and the convexity of f' we can conclude that the extinction probabilities q_j of $(\zeta_i^{(j)})_{i \geq 1}$ are bounded away from 1.

3.3 Inhomogeneous BMC

In this section we give conditions for strong recurrence, $\alpha \equiv 1$, and recurrence, $\alpha < 1$, that, although failing to produce a complete classification, work well in concrete examples. We assume throughout this section that $\mu(x) = \mu$ for all $x \in X$.

3.3.1 Connecting Markov chains at a common root

We present a method to glue different Markov chains following Chapter 9 in [29]. Let $(X_i, P_i), i \in I$, be a family of irreducible Markov chains. We choose a root r_i in each X_i and connect the X_i by identifying all these roots. The rest of the X_i remains disjoint. This gives a set $X = \bigcup_i X_i$ with root r, $\{r\} = \bigcap_i X_i$. In order to define the transition matrix P on X, we choose constants $\alpha_i > 0$ such that $\sum_i \alpha_i = 1$ and set

$$p(x,y) = \begin{cases} p_i(x,y) & x,y \in X_i,\ x \neq r, \\ \alpha_i p_i(r,y) & x = r,\ y \in X_i \setminus \{r\}, \\ \sum_i \alpha_i p_i(r,r) & x = y = r, \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

When each X_i is a graph and P_i is the SRW on X_i, then X is the graph obtained by connecting the X_i at a common root r. Choosing $\alpha_i = deg_{X_i}(r)/deg_X(r)$, we obtain the SRW on X. Due to this construction the calculation of the spectral radius of (X,P) can be done with the help of generating functions of (X_i,P_i), compare with Chapter 9 in [29].
For these types of Markov chains we obtain a condition for strong recurrence in terms of

$$\varrho(P) := \inf_{i\in I} \rho(P_i) \in [0,1]. \tag{3.7}$$

Lemma 3.12. *Let $(X_i,P_i), i \in I$, be a family of irreducible Markov chains and (X,P) as defined in (3.6). The BMC (X,P,μ) with constant branching distribution is not strongly recurrent, i.e., $\alpha < 1$, if*

$$m < 1/\varrho(P).$$

If the inf *is attained then $m = 1/\varrho(P)$ implies $\alpha < 1$, too.*

Proof. There exists $i \in I$ such that $m \leq 1/\rho(P_i)$. Due to Theorem 2.12 we know that the BMC (X_i,P_i,μ) is transient. Hence, there exists some $x \in X_i$ such that the BMC (X_i,P_i,μ) started in x never hits $r_i = r$ with positive probability. Therefore, with positive probability the BMC (X,P,μ) started in x never hits r. □

Remark 3.13. If $m = 1/\varrho(P)$ and the $\inf_{i\in I} \rho(P_i)$ is not attained then both cases can occur. The BMC is strongly recurrent if all (X_i,P_i) are quasi-transitive, compare with Theorem 3.14. In order to construct an example that is not strongly recurrent, let (X_1,P_1) be as in Example 2.18. For $i \geq 2$, let (X_i,P_i) be the random walk on $\mathbb{Z}$ with drift defined by $p_i(x,x+1) = \frac{2+\sqrt{3}}{4} - \frac{1}{i+1}$. We glue the Markov chains in $r = r_i = 0$ and obtain $\varrho(P) = \frac{1}{2}$. Since the BMC (X_1,P_1,μ) with $m = 2$ is not strongly recurrent, this follows for the BMC (X,P,μ) as well.

For certain Markov chains, constructed as above, we can give a complete classification in transience, recurrence and strong recurrence. Observe that we can replace *quasi-transitive* by any other homogeneous process of Subsection 3.2. Interesting is the subtle behavior in the second critical value; the BMC may be strongly recurrent or weakly recurrent.

Theorem 3.14. *Let $(X_i,P_i), i \in I$, be a family of quasi-transitive irreducible Markov chains and (X,P) as defined in (3.6). We have the following classification for the BMC (X,P,μ) with constant mean offspring m :*

$$\begin{array}{lrcll} (i) & & m & \leq 1/\rho(P) & \Longleftrightarrow \quad \alpha \equiv 0, \\ (ii) & 1/\rho(P) < & m & < 1/\varrho(P) & \Longleftrightarrow \quad 0 < \alpha(x) < 1, \\ (iii) & 1/\varrho(P) < & m & & \Longleftrightarrow \quad \alpha \equiv 1. \end{array} \tag{3.8}$$

If the inf *in the definition of $\varrho(P)$ is attained, then $m = 1/\varrho(P)$ implies $\alpha < 1$, and if the* inf *is not attained, then $m = 1/\varrho(P)$ implies that $\alpha \equiv 1$.*

Proof. The part (i) is Theorem 2.12, (ii) is Lemma 3.12 and (iii) follows from Theorem 3.7 by observing that each BMC (X_i, P_i, μ) is strongly recurrent. The same argumentation holds if the inf is not attained. The case when the inf is attained follows with Lemma 3.12. $\square$

Analogous arguments yield the classification for trees with finitely many cone types that are not necessarily irreducible. For this purpose let G_i be the irreducible classes of G, T_i the directed cover of G_i, and $\widetilde{\rho}(T) := \min_i \rho(T_i)$.

Theorem 3.15. *Let (T, P, μ) be a BRW with finitely many cone types and constant mean offspring $m(x) = m > 1$. We have*

(i) $\alpha = 0$ *if* $m \leq 1/\rho(P)$,

(ii) $0 < \alpha < 1$ *if* $1/\rho(P) < m \leq 1/\widetilde{\rho}(P)$,

(iii) $\alpha = 1$ *if* $m > 1/\widetilde{\rho}(P)$.

Remark 3.16. The example in Theorem 3.15 illustrates very well the two *exponential* effects that compete. The first is the exponential decay of the return probabilities represented by $\rho(P_i)$ and the other the exponential growth of the particles represented by m. If m is smaller that $1/\rho(P_i)$ for all i the decay of the return probabilities always wins and the process is transient. In the middle regime where $1/\rho(P_i) < m < 1/\rho(P_j)$ for some i, j the exponential growth may win but if $m > 1/\rho(P_i)$ for all i the exponential growth always wins and the process is strongly recurrent.

3.3.2 Simple random walks on graphs

In order to find conditions for strong recurrence we inverse the action of connecting graphs at a common root and split up some given graph in appropriate subgraphs. In the remaining part of this section we assume for sake of simplicity the Markov chain to be a simple random walk on a graph $G = (V, E)$, where $V = X$ is the vertex set and E is the set of edges. Keeping in mind that $\rho(P) = \sup_{|F|<\infty} \rho(P_F)$, compare with equation (2.2), we define

$$\widetilde{\rho}(P) := \inf_{|\partial F|<\infty} \rho(P_F),$$

where the inf is over all infinite irreducible $F \subset X$ such that the (inner) boundary of F, $\partial F := \{x \in F : x \sim F^c\}$ is a finite set. We associate a subset $F \subset X$ with the induced subgraph $F \subset G$ that has vertex set F and contains all the edges $xy \in E$ with $x, y \in F$. We can express $\widetilde{\rho}(P)$ in terms of transient SRWs on induced subgraphs with $|\partial F| < \infty$. For such a graph F we obtain

$$\begin{aligned} p_F^{(n)}(x,y) &= \mathbb{P}_x(X_n = y, X_i \in F \ \forall i \leq n) \\ &= \mathbb{P}_x(X_n = y | X_i \in F \ \forall i \leq n) \cdot \mathbb{P}_x(X_i \in F \ \forall i \leq n) \\ &= q^{(n)}(x,y) \cdot \mathbb{P}_x(X_i \in F \ \forall i \leq n), \end{aligned}$$

where $q^{(n)}(x,y)$ are the nth step probabilities of the SRW on F with transition kernel Q. Since F is transient and ∂F is finite, we have for $x \in F \setminus \partial F$ that

$$\mathbb{P}_x(X_i \in F \ \forall i \leq n) \geq \mathbb{P}_x(X_i \in F \ \forall i) > 0$$

and hence

$$\limsup_{n\to\infty} \left(p_F^{(n)}(x,y)\right)^{1/n} = \limsup_{n\to\infty} \left(q^{(n)}(x,y)\right)^{1/n}, \quad \forall x, y \in F.$$

Eventually, we can write

$$\widetilde{\rho}(P) = \widetilde{\rho}(G) = \inf_{|\partial F|<\infty} \rho(F),$$

where the inf is over all induced connected infinite subgraphs $F \subset G$ with finite boundaries. In analogy to the proof of Lemma 3.12 we obtain a necessary condition for strong recurrence that we conjecture to be sufficient for graphs with bounded degrees.

Lemma 3.17. *The BMC (G, μ) is not strongly recurrent, $\alpha < 1$, if*

$$m < \frac{1}{\widetilde{\rho}(G)}.$$

If the inf *is attained then $m = 1/\widetilde{\rho}(G)$ implies $\alpha < 1$.*

Remark 3.18. Lemma 3.17 holds true for any locally finite graph. However, $m > 1/\widetilde{\rho}(P)$ does not imply strong recurrence in general, see the following Example 3.19 and Subsection 3.4 for a more detailed discussion.

Example 3.19. Consider the following tree T with exploding degrees bearing copies of $\mathbb{Z}^+$ on each vertex. Let r be the root with degree 7. First, define inductively the *skeleton* of our tree: $deg(x) = 2^{2n+3} - 1$ for vertices x with $d(r,x) = n$. Now, glue on each vertex a copy of $\mathbb{Z}^+$, such that in the final tree a vertex with distance n from the root has degree 2^{2n+3} or 2. Due to this construction we have $\rho(T) = \widetilde{\rho}(T) = 1$. Consider the BRW (T, μ) with $\mu_2(x) = 1$ for all $x \in T$ and start the process with one particle in r. The probability that no copy of $\mathbb{Z}^+$ is visited is at least the probability that the process lives only on the skeleton and moves always away from the root:

$$\left(1 - \frac{1}{8}\right)^2 \cdot \prod_{n=1}^{\infty} \left(1 - \frac{1}{2^{2n+2}}\right)^{2^{n+1}} > 0.$$

Hence the BRW is not strongly recurrent.

In order to give a sufficient condition for strong recurrence we define

$$\check{\rho}(P) := \limsup_{n\to\infty} \inf_{x\in X} \rho(P_{B_n(x)}) \tag{3.9}$$

and write $\check{\rho}(G)$ for the SRW on G. Here, $B_n(x)$ is the ball of radius n around x.
Notice that this can be seen as a variation of of the spectral radius since

$$\rho(P) = \limsup_{n\to\infty} \sup_{x\in X} \rho(P_{B_n(x)}).$$

Proposition 3.20. *Let G be a graph with bounded geometry. The BRW (G, μ) with constant offspring distribution is strongly recurrent if*

$$m > \frac{1}{\check{\rho}(G)}.$$

Proof. There exists some $n \in \mathbb{N}$ such that for all $x \in X$ we have $m > 1/\rho(P_{B_n(x)})$. We follow the lines of the proof of Theorem 3.7 and construct and infinite number of supercritical Galton-Watson processes. Observe that since the maximal degree of G is bounded, there are only a finite number of different possibilities for the graphs $B_{x,n}$. Therefore, the extinction probabilities of the Galton-Watson processes are bounded away from 1 and and we can conclude with Lemma 3.4. □

Remark 3.21. The sufficient condition in Proposition 3.20 is not necessary for strong recurrence in general. Consider the following tree T that is a combination of $\mathbb{T}_3$ and $\mathbb{Z}$: Let r be the root with degree 2. The tree T is defined such that $deg(x)=3$ for all x such that $d(r,x)\in\left[2^{2k},2^{2k+1}-1\right]$ and $deg(x)=2$ for all x such that $d(r,x)\in\left[2^{2k+1},2^{2k+2}-1\right]$ for $k\geq 0$. We have $\rho(T)=1$, $\check{\rho}(T)=\rho(\mathbb{T}_3)=2\sqrt{2}/3<1$ and that the BRW (T,μ) is strongly recurrent for all $m>1$. To see the latter observe that for all $m>1$ there exists some k such that $p_{\mathbb{Z}}^{(k)}(0,0)\cdot m^k>1$, where $P_{\mathbb{Z}}$ is the transition kernel of the SRW on $\mathbb{Z}$. Thus each part of $\mathbb{Z}$ of length k constitutes a seed and we conclude with Lemma 3.4.

3.3.3 Simple random walks on trees

Let T be a tree of degree bounded by $M\in\mathbb{N}$ and denote P_T for the transition matrix of the SRW on T and $P_{\mathbb{T}_M}$ for the transition matrix for the SRW on $\mathbb{T}_M$, the M-regular tree. We consider T to be an infinite subtree of $\mathbb{T}_M$. One shows by induction on n :

Lemma 3.22.

$$p_T^{(n)}(x,y)\geq p_{\mathbb{T}_M}^{(n)}(x,y)\quad \forall x,y\in T\ \forall n\in\mathbb{N}.$$

Since the spectral radius of the SRW on $\mathbb{T}_M$ is $\rho(\mathbb{T}_M)=\frac{2\sqrt{M-1}}{M}$, compare with Example 2.17, we immediately obtain a lower bound for the spectral radius of SRW on trees with bounded degrees.

Lemma 3.23. *Let T be a tree with degrees bounded by M. Then the simple random walk on T satisfies*

$$\rho(T)\geq\frac{2\sqrt{M-1}}{M}.$$

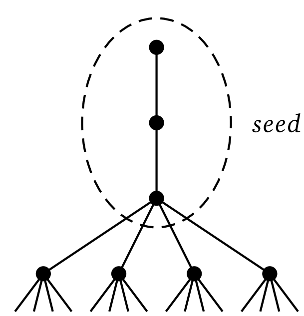
We obtain the following Corollary of Lemma 3.23 and Proposition 3.20.

Corollary 3.24. *Let T be a tree with maximal degree M. The BRW (T,μ) with constant offspring distribution is strongly recurrent if $m>\frac{2\sqrt{M-1}}{M}$.*

Remark 3.25. Observe that Lemma 3.23 does hold for general graphs with degrees bounded by M, compare with Theorem 11.1 in [29]. Therefore, Corollary 3.24 does hold true for graphs with degrees bounded by M as well.

We conclude this chapter with an interesting and illustrative example, gathered from [29] (Chapter 9), where we can give a complete classification in transience, recurrence and strong recurrence.

Example 3.26. We construct a graph that looks like a rooted M-ary tree with a hair of length 2 at the root. Let G_1 be the tree where each vertex has degree $M>1$, with the exception of the root o, which has degree $M-1$. As G_2 we choose the finite path $[0,1,2]$. The graph G is obtained by identifying 0 with o, compare with Subsection 3.3.1. The SRW on G is obtained by setting $\alpha_1=\frac{M-1}{M}$ and $\alpha_2=\frac{1}{M}$, compare with Equation (3.6). Let us first consider the case where $M\geq 5$. One calculates the spectral radius of the SRW on G :



$$\rho(G)=\sqrt{\frac{M-1}{2(M-2)}}.$$

Recall that $\varrho(G) = \min\{\rho(G_1), \rho(G_2)\}$. Due to Lemma 3.23 we have $\varrho(G) \geq \frac{2\sqrt{M-1}}{M}$. Since $\rho(G_1) = \frac{2\sqrt{M-1}}{M}$ we have $\varrho(G) = \frac{2\sqrt{M-1}}{M}$. Notice that $\varrho(G) = \widetilde{\rho}(G) = \check{\rho}(G)$. Now, Theorem 2.12, Lemma 3.12 and the proof of Theorem 3.14 yields

(i) $m \leq 1/\rho(G) \implies (G,\mu)$ is transient,

(ii) $1/\rho(G) < m \leq 1/\varrho(G) \implies (G,\mu)$ is recurrent,

(iii) $m > 1/\varrho(G) \implies (G,\mu)$ is strongly recurrent.

Observe that in this example the graph G_2 can be seen as a seed that makes the BRW recurrent. The first critical value $1/\rho(G)$ is such that the G_2 becomes a seed, where the second critical value $1/\varrho(G)$ is such that the branching compensates the drift induced by the graph G_1. Furthermore, notice that for $M = 3, 4$ the spectral radius of G is $\rho(G) = \frac{2\sqrt{M-1}}{M}$ and recurrence and strong recurrence coincide. Thus in this case, the branching which is necessary to produce a seed in G_2 must be at least as high as the branching that is needed to compensate the drift of the SRW on G_1.

3.4 Outlook

We know that if the offspring distributions depend on the state, any criterion for strong recurrence must incorporate more information on the offspring distributions than the mean. If the offspring distributions do not depend on the state, we conjecture, compare with the results obtained Section 3, that there is a second threshold:

Conjecture 3.27. *Let (X, P, μ) be a BMC with constant offspring distribution. Then there exists some $\widetilde{m}$ such that the BMC is strongly recurrent if $m > \widetilde{m}$ and not strongly recurrent if $m < \widetilde{m}$.*

Let us state the conjecture made in Subsection 3.3.2. Recall

$$\widetilde{\rho}(G) = \inf_{|\partial F| < \infty} \rho(F),$$

where the inf is over all induced connected infinite subgraphs $F \subset G$ with finite boundaries.

Conjecture 3.28. *Let G be a graph with bounded degrees. The BRW (G, μ) with constant offspring distribution is strongly recurrent if*

$$m > \frac{1}{\widetilde{\rho}(G)}.$$

For SRWs on locally finite graphs this is not true, compare with Example 3.19. This example suggests to consider transient subsets. Let

$$\widetilde{\rho}(G, m) := \inf_F \rho(F),$$

where the inf is over all irreducible $F \subset G$ where ∂F is transient with respect to the BRW (F, μ). Observe that $\widetilde{\rho}(G, m)$ does depend on m since transience is w.r.t. the BRW. In analogy to the proof of Lemma 3.17 we can prove that the BRW (G, μ) is not strongly recurrent if $m < 1/\widetilde{\rho}(G, m)$. We conjecture that for BRWs $1/\widetilde{\rho}(G, m)$ is decisive for strong recurrence, compare with Lemma 3.17.

Conjecture 3.29. *Let G be a graph. The BRW (G, μ) with constant offspring distribution is strongly recurrent if*

$$m > \frac{1}{\widetilde{\rho}(G, m)}.$$

4 Positive recurrence

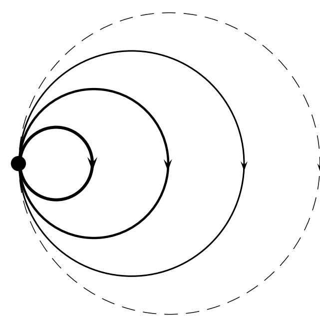
An irreducible Markov chain is called positive recurrent if the expected time to return is finite for all possible starting positions. We generalize this definition to BMC and say the process returns to its starting position if the starting position is hit by at least one particle. If the expected time to return is finite, we call the BMC positive recurrent.

Definition 4.1. A recurrent BMC is positive recurrent if

$$\mathbb{E}_x T_x < \infty \quad \forall x \in X, \tag{4.1}$$

with $T_x := \inf\{n > 0 : \exists i \in \{1, 2, \ldots, \eta(n)\} : x_i(n) = x\}$. Otherwise it is called null recurrent.

In contrast to the question of transience and recurrence of BMC, it is now also interesting to consider underlying null recurrent Markov chains and ask whether the corresponding BMC is null or positive recurrent. For a Markov chain we have that either $\mathbb{E}_x T_x < \infty$ for all or for none $x \in X$. This does no longer hold for BMCs, as we can see in the following Example 4.2. Furthermore, positive recurrence does depend on more information of the offspring distribution then just the mean offspring.



Example 4.2. We consider a random walk on a directed graph with denumerable many directed cycles of *exploding* length emanating from the origin o. Let $C_i = (c_0^{(i)}, \cdots, c_{2^i}^{(i)})$ be cycles of length 2^i with $c_0^{(i)} = c_{2^i}^{(i)} = o$ for $i \geq 1$. The origin o is the only common vertex of these cycles, i.e., $c_k^{(i)} \neq c_l^{(j)}$, for $1 \leq k < 2^i$ and $1 \leq l < 2^j$ $\forall i, j \in \mathbb{N}$. The transition probabilities P on $X := \bigcup_i C_i$ are defined as

$$\begin{aligned} p(o, c_1^{(i)}) &:= \left(\frac{1}{2}\right)^i, \quad i \geq 1, \\ p(c_k^{(i)}, c_{k+1}^{(i)}) &:= 1 \quad \forall 1 \leq k < 2^i,\ i \geq 1. \end{aligned}$$

The Markov chain (X, P) is null recurrent. We consider the BMC (X, P, μ) with $\mu_1(o) = \mu_3(o) = \frac{1}{2}$ and $\mu_2(x) = 1$ $\forall x \neq o$. It is now straightforward to show that $\mathbb{E}_o T_o = \infty$ but $\mathbb{E}_{c_1^{(1)}} T_{c_1^{(i)}} < \infty$. Observe that the BMC with the same constant mean offspring $m = 2$ but $\mu_2(x) = 1$ for all $x \in X$ is positive recurrent, i.e., $\mathbb{E}_x T_x < \infty$ for all $x \in X$.
Despite Example 4.2 we have under some natural assumptions on the branching that $\mathbb{E}_x T_x < \infty$ holds either for all or none $x \in X$.

Lemma 4.3. *Let (X, P, μ) be a BMC and assume that $0 < \mu_1(x) < 1$ for all $x \in X$. If $\mathbb{E}_o T_o < \infty$ for some $o \in X$ we have*

$$\mathbb{E}_x T_y < \infty \quad \forall x, y \in X.$$

Proof. We start the BMC in o. Let $y \in X$ and choose k such that $p^{(k)}(o, y) > 0$. Since $\mu_1(x) > 0$ $\forall x$ we have that with positive probability the total number of particles at time k is 1 and that this particle is in y, i.e., $\eta(k) = \eta(k, y) = 1$. Hence, $\mathbb{E}_y T_o < \infty$. In order to show $\mathbb{E}_o T_y < \infty$ we use that $\mu_1(x) < 1$. Let τ_i be independent random variables distributed like T_o under $\mathbb{P}_o[\cdot | \eta(1) = 1]$. We proceed with a *geometric waiting time argument*: We start the process with one particle in o and wait a random time τ_i until a first particle returns. This particle splits up in at least two particles with positive probability $1-\mu_1(o)$. One of these particles starts a new process that returns to o after τ_2 time steps. The remaining particles, if there exists any, hit y after k time steps with positive probability at least $p^{(k)}(x,y)$. This is repeated until y is hit. Therefore, we obtain with $q := (1-\mu_1(o))p^{(k)}(o,y)$:

$$\mathbb{E}_o T_y \leq k + \sum_{i=1}^{\infty} \left((1-q)^{i-1} q \cdot \sum_{j=1}^{i} E\tau_j \right) < \infty,$$

since $E\tau_i = \mathbb{E}_o[T_o|\eta(1) = 1] < \infty$. Hence, $\mathbb{E}_o T_y < \infty$ and $\mathbb{E}_y T_o < \infty$ for all $y \in X$. Since $\mathbb{E}_x T_y \leq \mathbb{E}_x T_o + \mathbb{E}_o T_y$ $\forall x, y \in X$ we are done. □

There is a *branching* analog to the 2nd criterion of Foster, compare with Theorem 2.2.3. in [9], for positive recurrence of Markov chains.

Theorem 4.4. *Let $o \in X$. If there exists a nonnegative function f with $f(o) > 0$ such that*

$$Pf(x) \leq \frac{f(x)-\varepsilon}{m(x)} \quad \forall x \neq o \text{ for some } \varepsilon > 0, \tag{4.2}$$

then $\mathbb{E}_x T_o < \infty$ for all $x \neq o$.

Proof. Let $x \neq o$. We define

$$Q(n) := \sum_{i=1}^{\eta(n)} f(x_i(n))$$

and

$$Z(n) := Q(n \wedge T_o) + \varepsilon \cdot (n \wedge T_o). \tag{4.3}$$

We write $\omega(n) := \{x_1(n), \dots, x_{\eta(n)}(n)\}$ for the positions of particles at time n and obtain using Equation (4.2):

$$\mathbb{E}_x[Q(n+1)|\omega(n) = \omega] \leq Q(n) - \varepsilon\eta(n)$$

under $\{T_o > n\}$. Hence, under $\{T_o > n\}$ we have

$$\begin{aligned} \mathbb{E}_x[Z(n+1)|\omega(n) = \omega] &= \mathbb{E}_x[Q(n+1) + \varepsilon(n+1)] \\ &\leq Q(n) - \varepsilon\eta(n) + \varepsilon(n+1) \\ &\leq Q(n) + \varepsilon n. \end{aligned}$$

Therefore, $Z(n)$ is a nonnegative supermartingale. We obtain with Equation (4.3)

$$\mathbb{E}_x[n \wedge T_o] \leq \frac{\mathbb{E}_x[Z(n)]}{\varepsilon} \leq \frac{\mathbb{E}_x[Z(0)]}{\varepsilon} = \frac{f(x)}{\varepsilon}.$$

Letting $n \to \infty$ yields

$$\mathbb{E}_x[T_o] \leq \frac{f(x)}{\varepsilon} < \infty \quad \forall x \neq o.$$

□

In general it is not possible to give criteria for the positive recurrence in terms of the mean offspring $m(x)$, compare with Example 4.2. Despite this fact, it turns out that for *homogeneous* BMC, e.g. quasi-transitive BMC, strong recurrence and positive recurrence coincide. We refer to [6] where the asymptotic of the tail of the distributions of the hitting times are studied even for branching random walks in random environment.

In the following subsection,we present another method to show positive recurrence of BRW on $\mathbb{Z}$ using large deviation estimates and the rate of escape of the BRW.

4.1 BRW on $\mathbb{Z}$

Let us consider an irreducible, transient random walk, $S_n = \sum_{i=1}^n X_i$, on $\mathbb{Z}$ with i.i.d. increments X_i. Furthermore, we assume bounded jumps, i.e., $|X_i| \leq d$ for some $d \in \mathbb{N}$. This assumption will be crucial in the proof of Lemmata 4.9 and 4.10 but can be replaced for Theorem 4.5 and Corollary 4.6 by the assumption that S_n/n satisfies a large deviation principle. Without loss of generality we assume that the random walk has drift to the right, i.e., $E[X_i] > 0$. Let $I(\cdot)$ be the strictly monotone rate function defined by

$$-I(a) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(S_n \leq an) \text{ for } a \leq EX_i.$$

We consider the BMC (X,P,μ) with constant offspring distribution μ with mean offspring m and denote M_n the leftmost particle at time n.
We have the well-known result for the minimal position of a BRW, compare with [11], [13], [4], [24], and [19] for multi-type BRWs.

Theorem 4.5.

$$\liminf_{n\to\infty} \frac{M_n}{n} = \sup\{s : I(s) \geq \log m\} \quad \mathbb{P}\text{-}a.s.$$

Since $\rho(P) = e^{-I(0)}$, compare with Lemma 2.8, we immediately obtain the following result on the speed of the leftmost particle.

Corollary 4.6. *Let (X,P) be a random walk with bounded jumps on $X = \mathbb{Z}$ and drift to the right. For a BRW (X,P,μ) with constant offspring distribution the following holds true:*

(i) If $m > 1/\rho(P)$, then

$$\liminf_{n\to\infty} \frac{M_n}{n} < 0 \quad \mathbb{P}\text{-}a.s.$$

(ii) If $m = 1/\rho(P)$, then

$$\liminf_{n\to\infty} \frac{M_n}{n} = 0 \quad \mathbb{P}\text{-}a.s.$$

(iii) If $m < 1/\rho(P)$, then

$$\liminf_{n\to\infty} \frac{M_n}{n} > 0 \quad \mathbb{P}\text{-}a.s..$$

Remark 4.7. In particular, Corollary 4.6 implies transience if $m < 1/\rho(P)$ and strong recurrence if $m > 1/\rho(P)$.

Eventually, we obtain that under the above conditions strong recurrence implies positive recurrence:

Theorem 4.8. *Let (X,P) be a random walk with bounded jumps on $X = \mathbb{Z}$ and drift to the right. The BRW (X,P,μ) with constant offspring distribution is positive recurrent if $m > 1/\rho(P)$.*

The proof follows from the following Lemmata 4.9 and 4.10. Beside T_x we consider the following stopping time

$$\widetilde{T}_x := \inf_{n>0}\{\exists i \in \{1,2,\ldots,\eta(n)\} : x_i(n) \in [x-d,x]\}.$$

Lemma 4.9. *Let (X,P) be a random walk with bounded jumps on $X=\mathbb{Z}$ and drift to the right. For a BRW (X,P,μ) with constant offspring distribution and $m>1/\rho(P)$ we have*

$$\mathbb{E}_x\widetilde{T}_x<\infty \quad \forall x\in X.$$

Proof. We show the claim for $x=0$ and write $\widetilde{T}$ for $\widetilde{T}_0$. Since

$$\mathbb{E}\widetilde{T}=\sum_{n\geq 0}\mathbb{P}(\widetilde{T}>n)$$

it suffices to study the behavior of $\mathbb{P}(\widetilde{T}>n)$ for large n and to show that it is summable. To this end we split the sum into two terms:

$$\begin{aligned}\mathbb{P}(\widetilde{T}>n) &= \sum_{k=1}^{\lambda n}\mathbb{P}\left(\widetilde{T}>n|\eta(\gamma n)=k\right)\mathbb{P}\left(\eta(\gamma n)=k\right)\\ &\quad+\sum_{k=\lambda n+1}^{\infty}\mathbb{P}\left(\widetilde{T}>n|\eta(\gamma n)=k\right)\mathbb{P}\left(\eta(\gamma n)=k\right)\end{aligned}$$

with $\lambda>0$ and $\gamma\in(0,1)$ to be chosen later. Here we assume that λn and γn take values in $\mathbb{N}$. We obtain

$$\mathbb{P}(\widetilde{T}>n)\leq\mathbb{P}\left(\eta(\gamma n)\leq\lambda n\right)+\sum_{k=\lambda n+1}^{\infty}\mathbb{P}\left(\widetilde{T}>n|\eta(\gamma n)=k\right). \tag{4.4}$$

In order to estimate the second summand we observe that at time γn all particles are *at worst* at position $\gamma n d$ and obtain

$$\begin{aligned}\mathbb{P}(\widetilde{T}>t|\eta(\gamma n)=k) &\leq \mathbb{P}\left(M_{(1-\gamma)n}>-\gamma n d\right)^k\\ &= \mathbb{P}\left(\frac{M_{(1-\gamma)n}}{(1-\gamma)n}>-\frac{\gamma d}{1-\gamma}\right)^k.\end{aligned}$$

Due to Corollary 4.6 we have

$$\liminf_n\frac{M_{(1-\gamma)n}}{(1-\gamma)n}=s$$

for some $s<0$. Now, we choose γ such that

$$-\frac{\gamma d}{1-\gamma}>s.$$

Hence, there exists $\theta<1$ with

$$\mathbb{P}\left(\frac{M_{(1-\gamma)n}}{(1-\gamma)n}>-\frac{\gamma d}{1-\gamma}\right)\leq\theta<1,$$

for sufficiently large n. Therefore,

$$\mathbb{P}(\widetilde{T}>n|\eta(\gamma n)=k)\leq\theta^k \tag{4.5}$$

and the second summand in Equation (4.4) is bounded by $\theta^{\lambda(n+1)}/(1-\theta)$.

It remains to bound the first term in Equation (4.4). To do this we do not consider the whole BRW but focus on the induced random walk. Denote Y_n the number of times the labelled particle is not the only offspring of its ancestor. In other words, when we think about the process where particles live forever and produce offspring according to $\tilde{\mu}_{i-1} = \mu_i$ $i \geq 1$, then Y_n is just the number of offspring of the starting particle at time n. Hence, $Y_n \sim Bin(n,p)$, where $p := \sum_{i=2}^{\infty} \mu_i > 0$. Observe that a Large Deviation Principle holds for Y_n, i.e., $P(Y_n \leq an)$ decays exponentially fast for $a < p$. Due to the definition of Y_n we have $\eta(\gamma n) \geq Y_{\gamma n}$ and obtain with $l := \gamma n$ that

$$P(\eta(\gamma n) \leq \lambda n) \leq P(Y_{\gamma n} \leq \lambda n) = P(Y_l \leq \lambda \gamma^{-1} l).$$

The last term decays exponentially fast for $\lambda\gamma^{-1} < p$. Therefore, choosing $\lambda < p\gamma$, we obtain that $\mathbb{P}\left(\eta(\gamma n) \leq \lambda n\right)$ decays exponentially fast. □

Lemma 4.10. *Under the assumptions of Lemma 4.9 we have*

$$\mathbb{E}_x \widetilde{T}_x < \infty \implies \mathbb{E}_x T_x < \infty.$$

Proof. For simplicity let $x = 0$ and write $\widetilde{T}$ and T for $\widetilde{T}_0$ and T_0, respectively. Analogous to the proof of Lemma 4.9, it is proven that $\mathbb{E}_y \widetilde{T} < \infty$ $\forall y \in X$. We start the BRW in 0 and consider the random time τ_1 when $[-d,0]$ is hit for the first time by some particle. We pick one particle of those being in $[-d,0]$ at time τ_1 and consider the BRW originating from this particle. Due to the irreducibility, there exists $k \in \mathbb{N}$ and $q > 0$ such that $\forall y \in [-d,0]$ we have $p^{(l)}(y,0) \geq q > 0$ for some $l \leq k$. Hence, 0 is visited by some particle up to time k with probability at least q. If 0 is not hit after k time steps we consider a BRW starting in some occupied position in $[-d(k+1), d(k+1)]$ and wait the random time τ_2 until $[-d,0]$ is hit by some particle. This is repeated until 0 is hit at the random time W. Since $\mathbb{E}_x \widetilde{T} < \infty$ for all $x \in X$, there exists some $C > 0$ such that $\mathbb{E}_x \widetilde{T} \leq C$ $\forall x \in [-d(k+1), d(k+1)]$. We conclude with

$$\begin{aligned} \mathbb{E}T \leq \mathbb{E}W &\leq \mathbb{E}\left[(\tau_1 + k)q + (1-q)q(\tau_1 + \tau_2 + k) + \cdots\right] \\ &\leq \sum_{i=1}^{\infty} (iC + k)(1-q)^{i-1} q < \infty. \end{aligned}$$

□

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