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ON LONG RANGE PERCOLATION WITH HEAVY TAILS

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Abstract
Consider independent long range percolation on $\mathbf{Z}^d$, where edges of length n are open with probability p_n. We show that if $\limsup_{n\to\infty} p_n > 0$, then there exists an integer N such that $P_N(0 \leftrightarrow \infty) > 0$, where P_N is the truncated measure obtained by taking $p_{N,n} = p_n$ for $n \leq N$ and $p_{N,n} = 0$ for all $n > N$.

We consider independent long range percolation on the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{G} = \mathbf{Z}^d$, $d \geq 2$, and $\mathcal{E} = \{\langle x, y\rangle \subset \mathcal{V} \times \mathcal{V} : x \neq y\}$. For a given sequence $(p_n)_{n\in\mathbf{N}}$, $p_n \in [0,1]$, we consider the long range percolation process $(\Omega, \mathcal{F}, P)$, where $\Omega = \{0,1\}^{\mathcal{E}}$, $P = \prod_{\langle x,y\rangle\in\mathcal{E}} \mu_{\langle x,y\rangle}$, and $\mu_{\langle x,y\rangle}\{\omega_{\langle x,y\rangle} = 1\} = p_{\|x-y\|}$ is a Bernoulli measure, independent of the state of other edges. Here we use the distance $\|x - y\| = \max_{i=1,\dots,d} |x_i - y_i|$. Given an integer $N \in \mathbf{N}$, we define a truncated sequence $(p_{N,n})_{n\in\mathbf{N}}$ by

$$p_{N,n} = \begin{cases} p_n & \text{if } n \leq N\,, \\ 0 & \text{if } n > N, \end{cases} \tag{1}$$

and a truncated percolation process $(\Omega, \mathcal{F}, P_N)$ by taking $P_N = \prod_{\langle x,y\rangle\in\mathcal{E}} \mu_{N,\langle x,y\rangle}$, where $\mu_{N,\langle x,y\rangle}\{\omega_{\langle x,y\rangle} = 1\} = p_{N,\|x-y\|}$.

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In this note we address the following question: given a sequence $(p_n)_{n\in\mathbf{N}}$ for which $P(0 \leftrightarrow \infty) > 0$, does there always exists some large enough N such that $P_N(0 \leftrightarrow \infty) > 0$? In other words, given a system with infinite range translation invariant interactions which exhibits a phase transition, we ask if the infiniteness of the range is really crucial for this transition to occur. It is known, for instance, that infinite range is essential in one dimensional systems (cf. [FS], [NS]), but it is believed that in dimensions $d \geq 2$, occurrence (if so) of phase transitions for translation invariant interactions is *always* determined by a bounded part of the interaction (excluding cases when interactions are of intrinsically one dimensional structure). Returning to the percolation case, rapid (say, exponential) decay or summability of the p_n's indicates that long range connections may not be necessary for the existence of an infinite cluster. This is the setup of [MS] and partially [B]. On the other hand, heavy tail interactions are still poorly understood, and the only existing studies, [SSV] and [B], rely heavily on asymptotic monotonicity assumptions which are a key ingredient for the use of rather laborious coarse-graining techniques. Although the approach we present here relies on deep and highly nontrivial facts ([GM], [K]), it leads to a much shorter (not to say elementary) proof, which is less sensitive to the geometry of the interactions, and allows to consider a rather general class of systems with connections of irregular, in particular lacunary structure.

Theorem. *If* $\limsup_{n\to\infty} p_n > 0$, *then*

$$P_N(0 \leftrightarrow \infty) > 0 \tag{2}$$

for some large enough N.

Proof. It is sufficient to consider the case $d = 2$. Define $\epsilon > 0$ by $2\epsilon = \limsup_{n\to\infty} p_n$. By [K], Theorem 1 p. 220, there exists some dimension $d_\epsilon \geq 3$ such that

$$p_c(\mathbf{Z}^{d_\epsilon}) < \epsilon/2\,,$$

where $p_c(\mathbf{Z}^{d_\epsilon})$ denotes the critical threshold of independent Bernoulli percolation on the nearest-neighbour lattice $\mathbf{Z}^{d_\epsilon}$. By [GM], Theorem A p. 447, there exists some integer K_ϵ such that

$$p_c\big(\{0,1,\ldots,K_\epsilon-1\}^{d_\epsilon-2}\times\mathbf{Z}^2\big) < p_c(\mathbf{Z}^{d_\epsilon}) + \epsilon/2 < \epsilon\,. \tag{3}$$

Let $n_0 = 0$. For $j \in \{1,2,\ldots,d_\epsilon-1\}$, define recursively

$$n_j = \min\{\ell > (K_\epsilon+1)n_{j-1} :\ p_\ell \geq \epsilon\}\,.$$

For $x \in \mathbf{Z}^2$ and $B \subseteq \mathbf{Z}^2$, define $\mathbf{T}_x B = \{z + x,\ z \in B\}$. Set $B_0 = \{(0,0)\}$. For $j \in \{1,2,\ldots,d_\epsilon-2\}$, define recursively $B_j = \cup_{m=0}^{K_\epsilon-1}\mathbf{T}_{m(n_j,0)}B_{j-1}$. Then, let

$$\mathcal{V}_{d_\epsilon-1} = \bigcup_{(k,m)\in\mathbf{Z}^2}\mathbf{T}_{(kn_{d_\epsilon-1},mn_1)}B_{d_\epsilon-2}\,,$$

and

$$\begin{aligned}\mathcal{E}_{d_\epsilon-1} = \{\langle x,y\rangle \subset \mathcal{V}_{d_\epsilon-1}\times\mathcal{V}_{d_\epsilon-1} : |x_1-y_1| = n_j \text{ for some } 1\leq j\leq d_\epsilon-1\\ \text{and } x_2=y_2, \text{ or } x_1=y_1 \text{ and } |x_2-y_2| = n_1\}\,.\end{aligned}$$

It is straightforward that the subgraph $\mathcal{G}_{d_\epsilon-1} = \big(\mathcal{V}_{d_\epsilon-1},\,\mathcal{E}_{d_\epsilon-1}\big) \subset \mathcal{G}$ is isomorphic to $\{0,1,\ldots,K_\epsilon-1\}^{d_\epsilon-2}\times\mathbf{Z}^2$. An isomorphism $\varphi : \{0,1,\ldots,K_\epsilon-1\}^{d_\epsilon-2}\times\mathbf{Z}^2 \to \mathcal{V}_{d_\epsilon-1}$ is given by $\varphi(x_1,\ldots,x_{d_\epsilon}) := \big(\sum_{i=1}^{d_\epsilon-1} x_i n_i, x_{d_\epsilon}n_1\big)$.

Moreover, by our choice of n_j, $1 \leq j \leq d_\epsilon - 1$, we have that each edge of $\mathcal{G}_{d_\epsilon - 1}$ is open with probability at least ϵ, and using (3) we get (2) with $N = n_{d_\epsilon - 1}$. $\square$

Remark. For further applications of this method to percolation and interacting spin systems see [FL].

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