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A NON-MARKOVIAN PROCESS WITH UNBOUNDED P-VARIATION

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Abstract

A recent theorem in [3] provided a link between a certain function of transition probabilities of a strong Markov process and the boundedness of the p-variation of its trajectories. Here one assumption of that theorem is relaxed and an example is constructed to show that the Markov property cannot be easily dispensed with.

Introduction

Let ξ_t, $t \in [0,T]$, be a strong Markov process defined on some complete probability space $(\Omega, \mathcal{F}, P)$ and with values in a complete separable metric space (X, ρ). Denote the transition probability function of ξ_t by $P_{s,t}(x, dy)$, $0 \le s \le t \le T$, $x \in X$. For any $h \in [0,T]$ and $a > 0$ consider the function

$$\alpha(h,a) = \sup\{P_{s,t}(x, \{y : \rho(x,y) \ge a\}) : x \in X, 0 \le s \le t \le (s+h) \wedge T\}. \tag{1}$$

The behavior of $\alpha(h,a)$ as a function of h gives sufficient conditions for regularity properties of the trajectories of the process ξ_t. As Kinney showed in [1], ξ_t has an almost surely càdlàg version if $\alpha(h,a) \to 0$ as $h \to 0$ for any fixed $a > 0$, and an almost surely continuous version if $\alpha(h,a) = o(h)$ as $h \to 0$ for any fixed $a > 0$. (See also an earlier paper by Dynkin [2]). Recently we have established an interesting connection between the function $\alpha(h,a)$ and the p-variation of the trajectories of the process ξ_t (see Theorem 2 below). The objective of this paper is twofold: first to relax an assumption on the parameter β in Theorem 2 and then to show that if ξ_t is no longer Markov, the claim of Theorem 2 is false. To be more precise, we will construct an example of a càdlàg non-Markov process η_t on $[0,1]$ with unbounded p-variation for any $p > 0$ and such that the inequality (2) below holds up to a logarithmic factor and hence satisfies the relaxed condition on β stated in Theorem 3. Some of the properties of this η_t are given in Theorem 4.
Recall that for a $p \in (0,\infty)$ and a function f defined on the interval $[0,T]$ and taking values

in (X,ρ) its p-variation is

$$v_p(f) := \sup\{\sum_{k=0}^{m-1} \rho(f(t_{k+1}), f(t_k))^p : 0 = t_0 < t_1 < \cdots < t_m = T, m = 1, 2, \dots\}.$$

The range of the parameter β in the following definition is extended from "$\beta \geq 1$" used in [3] to "$\beta > 0$" needed here.

Definition 1 *Let $\beta > 0$ and $\gamma > 0$. We say that a Markov process $\xi_t, t \in [0,T]$, belongs to the class $\mathcal{M}(\beta,\gamma)$ ($\mathcal{M}$ for Markov) if there exist constants $a_0 > 0$ and $K > 0$ such that for all $h \in [0,T]$ and $a \in (0,a_0]$*

$$\alpha(h,a) \leq K\frac{h^\beta}{a^\gamma}. \tag{2}$$

Theorem 2 [3, Theorem 1.3] *Let ξ_t, $t \in [0,T]$, be a strong Markov process with values in a complete separable metric space (X,ρ). Suppose ξ_t belongs to the class $\mathcal{M}(\beta,\gamma)$, for some $\beta \geq 1$. Then for any $p > \gamma/\beta$ the p-variation $v_p(\xi)$ of ξ_t is finite almost surely.*

Preliminaries and Results

As mentioned above, we first relax condition "$\beta \geq 1$" in Theorem 2 as follows:

Theorem 3 *Theorem 2 holds if the condition "$\beta \geq 1$" is replaced by "$\beta > (3-e)/(e-1)$".*

Remark 1 Most of the interesting processes (e.g. Brownian motion, Levy processes on $\mathbb{R}^n$) satisfy (2) with $\beta \geq 1$ and the condition "$p > \gamma/\beta$" in Theorem 2 is sharp. Yet this need not be true in general. To see this consider a symmetric real-valued α-stable Lévy motion X_t, $t \in [0,1]$, with $\alpha \in (0,2)$. Furthermore, let $f(x) = x^\delta$ for some $\delta \in (0,1)$ and define $Y_t = X_{f(t)}$, $t \in [0,1]$. Since the function f is a strictly increasing continuous bijection of $[0,1]$ onto itself, both the strong Markov and the p-variation properties are preserved (for the latter see e.g. Lemma 4.4 of [4]). This implies that Y_t, like X_t, is also a strong Markov process with bounded p-variation for any $p > \alpha$ (see e.g. [3]). On the other hand, X_t has independent increments, so we also get for $h \in (0,1]$, $a > 0$ and some constant $K > 0$

$$\begin{aligned}\alpha(h,a) &= \sup\{P(|Y_t - Y_s| \geq a | Y_s = x) : x \in \mathbb{R},\, 0 \leq s \leq t \leq (s+h)\wedge 1\} \\ &= \sup\left\{P\big(\left|X_{f(t)} - X_{f(s)}\right| \geq a\big) : 0 \leq s \leq t \leq (s+h)\wedge 1\right\} \\ &\leq \sup\left\{\frac{K(f(t)-f(s))}{a^\alpha} : 0 \leq s \leq t \leq (s+h)\wedge 1\right\} = K\frac{h^\delta}{a^\alpha}.\end{aligned} \tag{3}$$

So Theorem 3 (but not Theorem 2) applies if $\delta \in ((3-e)/(e-1), 1)$ and yields the boundedness of p-variation of paths of Y_t only for any $p > \alpha/\delta > \alpha$.

The following is needed for the construction of the example described previously. Let $\{\delta_m\}_{m=1}^\infty$ be a monotone decreasing sequence of positive numbers such that $\sum_{m=1}^\infty \delta_m = 1$. Set $\delta_0 := 0$. In the future we will conveniently choose $\delta_m := 2\cdot 3^{-m}$, for $m \geq 1$, but most of the arguments can be adapted for any sequence $\{\delta_m\}$ which sums to one. Denote the partial sums by Δ_m, i.e. for any $m = 1, 2, \dots$ let $\Delta_m := \sum_{k=0}^m \delta_k$, and set $J_m = [\Delta_{m-1}, \Delta_m)$. This way we get a partition of $[0,1)$ into a union of disjoint subintervals J_m of length δ_m. Next consider an increasing sequence of integers $\{n_m\}_{m=1}^{\infty}$ such that

$$\log_3(n_m) = 3^m - m - 1 + \log_3 2, \tag{4}$$

for any $m \geq 1$, and let

$$a_m = \frac{\delta_m}{3(n_m+3)}, \quad d_m = \Delta_{m-1} + \frac{\delta_m}{3}. \tag{5}$$

Furthermore, let $\{c_m\}_{m=0}^{\infty}$ be an increasing sequence of real numbers, converging to $1/2$ as $m \to \infty$, but rather slowly. To be more specific we take

$$c_m = \frac{1}{2}\left(1 - \frac{1}{1+2\ln(m+1)}\right). \tag{6}$$

This choice is influenced by the desire to have a process with unbounded p-variation for any $p > 0$ and is, of course, not unique. Any sequence which converges to $1/2$ even slower than $\{c_m\}$ will suit as well. Let $X_0, X_1, \ldots$ be a sequence of independent (but not identically distributed) Bernoulli random variables such that

$$P(X_i = 0) = \begin{cases} 1-\delta_i, & \text{if } i \geq 1; \\ 1/2, & \text{if } i = 0, \end{cases} \qquad P(X_i = 1) = 1 - P(X_i = 0).$$

Consider a sequence $\{Z_m\}_{m=1}^{\infty}$ of independent random variables, independent from $\{X_k\}_{k=0}^{\infty}$. Assume that for each $m = 1, 2, \ldots$, a random variable Z_m is distributed uniformly on the interval $[a_m, (1+n_m)a_m]$. Without loss of generality we will assume that all random variables $X_i, i \geq 0$, and $Z_m, m \geq 1$, are defined on the same probability space $(\Omega, \mathcal{F}, P)$.

Furthermore, for $m = 1, 2 \ldots$, set $V_m := d_m + Z_m$ and $V'_m := V_m + a_m$. Now on the interval $[0,1]$ define a random process $\eta_t \equiv \eta(t)$ as follows: set $\eta_0 = X_0$, $\eta_1 = 1/2$ and on a subinterval J_m for any $m = 1, 2, \ldots$ let

$$\eta_t = \begin{cases} \eta(\Delta_{m-1}-), & \text{if } \Delta_{m-1} \leq t < V_m \text{ or} \\ & \text{if } V'_m \leq t < \Delta_m \text{ and } X_m = 0; \\ 1 - \eta(\Delta_{m-1}-), & \text{if } V_m \leq t < V'_m; \\ c_m(1-X_0) + (1-c_m)X_0, & \text{if } V'_m \leq t < \Delta_m \text{ and } X_m = 1; \end{cases} \tag{7}$$

where we set $\eta(0-) \equiv \eta_0$ and $\eta(s-)(\omega) = \lim_{u \uparrow s} \eta_u(\omega)$, i.e. we use the left hand side limit ω-pointwise, for $s \in (0,1)$. The idea behind this definition is simple: we consider a process which with probability $1/2$ starts either at 0 or at 1 and shortly before the end of the first subinterval J_1 returns back to $\eta(0)$, if $X_1 = 0$, or, depending on X_0, jumps to c_1 or $1 - c_1$, in case $X_1 = 1$. Inside J_2, η_t starts where it left J_1, and at time $t = \Delta_2$ remains at $\eta(\Delta_1)$, if $X_2 = 0$, or, depending on X_0, jumps to c_2 or $1 - c_2$, in case $X_2 = 1$, i.e. closer to $1/2$. The same happens in any other interval J_m.

Another way to look at what is happening to η_t at $t = \Delta_m, m = 1, 2, \ldots$ is via Markov chains. Indeed, set $W_0 := X_0$, $W_m := \eta(\Delta_m)$, for $m = 1, 2, \ldots$, and notice that $\{W_k\}_{k=0}^{\infty}$ defines a Markov chain on a state space

$$A_\infty = \{c_k, 1 - c_k : k = 0, 1, \ldots\}. \tag{8}$$

For each $m = 1, 2, \dots$, W_m is either equal to W_{m-1}, c_m or $1 - c_m$. Furthermore,

$$\begin{aligned}
P(W_m = W_{m-1}) &= P(X_m = 0) = 1 - \delta_m, \\
P(W_m = c_m) &= P(X_0 = 0, X_m = 1) = \frac{\delta_m}{2}, \\
P(W_m = 1 - c_m) &= P(X_0 = 1, X_m = 1) = \frac{\delta_m}{2}, \\
P(W_m = c_m | W_{m-1}) &= \delta_m \mathbf{1}_{\{W_{m-1} < 1/2\}} = \delta_m \mathbf{1}_{\{X_0 = 0\}}, \\
P(W_m = 1 - c_m | W_{m-1}) &= \delta_m \mathbf{1}_{\{W_{m-1} > 1/2\}} = \delta_m \mathbf{1}_{\{X_0 = 1\}}, \\
P(W_m = W_{m-1} | W_{m-1}) &= 1 - \delta_m.
\end{aligned}$$

Later we will also use the sets

$$A_m = \{c_k, 1 - c_k \,|\, k = 0, 1, \dots, m\}, \quad m \geq 0. \tag{9}$$

The following theorem lists some of the interesting properties of the process η_t.

Theorem 4 *Let η_t be defined by (7). Then*

(i) η_t is almost surely left-continuous at $t = 1$, and hence the paths of η_t are almost surely càdlàg;

(ii) the p-variation $v_p(\eta)$ of η_t is unbounded for any $p > 0$;

(iii) there exists a constant $K > 0$ such that for any $h \in (0, 1)$ and $a > 0$

$$\alpha(h, a) \leq Kh \left(1 \vee \ln \frac{1}{h}\right). \tag{10}$$

Remark 2 Slightly abusing notation we define $\alpha(h, a)$ in (10) as

$$\alpha(h, a) = \sup\{P(\rho(\eta_t, \eta_s) \geq a | \eta_s = x) : x \in X, 0 \leq s \leq t \leq (s + h) \wedge T\}.$$

For Markov processes this definition agrees with (1).

As an easy corollary we have

Corollary 5 *The conclusion of Theorem 3 no longer holds if $\xi_t, t \in [0, T]$ is not necessarily Markov.*

Proofs

The proof of Theorem 3 requires only minor adjustments to the proof of Theorem 2. To be specific, we first slightly improve Lemma 2.5(iii) of [3] as follows:

Lemma 6 *Let $\gamma(a, x) = \int_0^x u^{a-1} e^{-u} du$ be the incomplete gamma function defined for $a > 0$ and $x \geq 0$. Then for any $q \in (0, 1 - 1/e)$ and $a \geq 2/q - 3$*

$$0 < a\gamma(a, 1) - \frac{1}{e} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k + a + 1)} \leq q.$$

Proof. The proof is identical to that of Lemma 2.5(iii) except for the right-hand inequality. Since the series alternates plus, minus, etc., and the terms decrease in absolute value, three terms provide an upper bound:

$$a\gamma(a,1)-\frac{1}{e}<\frac{1}{(a+1)(a+2)}+\frac{1}{2(a+3)}\leq\frac{1}{a+2}+\frac{1}{2(a+3)}\leq\frac{2}{a+3}\leq q,$$

if $a\geq 2/q-3>(3-e)/(e-1)$. □

Before modifying Corollary 2.6 of [3], recall a few definitions. Let $\mathcal{F}^\xi$ be the natural filtration generated by the process ξ, i.e. $\mathcal{F}^\xi=\{\mathcal{F}^\xi_t, t\in[0,T]\}$, where $\mathcal{F}^\xi_t=\sigma(\xi_u, 0\leq u\leq t)\subset\mathcal{F}$. Recall that a random variable $\tau:\Omega\to[0,\infty]$ is an $\mathcal{F}^\xi$-Markov time iff for all $u\in[0,T]$, $\{\tau<u\}\in\mathcal{F}^\xi_u$. If τ is an $\mathcal{F}^\xi$-Markov time, define $\mathcal{F}_\tau:=\{A: A\cap\{\tau<u\}\in\mathcal{F}^\xi_u, u\in[0,T]\}$. Also set $\mathcal{F}_0:=\{\emptyset,\Omega\}$. Furthermore, for any $0\leq a<b\leq T$ let

$$R(a,b):=\sup_{a\leq s\leq t\leq b}\rho(\xi_s,\xi_t)=\sup_{s,t\in(\mathbb{Q}\cap[a,b])\cup\{b\}}\rho(\xi_s,\xi_t),$$

since ξ_t has càdlàg paths. Hence $R(a,b)$ is $\mathcal{F}^\xi_b$-measurable. Moreover, for any sequence $0<a_n\downarrow a\leq b\leq T$ we have $R(a_n,b)\uparrow R(a,b)$ as $n\to\infty$ since the intervals $[a_n,b]$ are expanding and ξ_t is right continuous.
For any $r=0,\pm1,\pm2,\ldots$, define $M_r:=2^{-r-1}$ and let $\{\tau_{l,r}\}$, $l=0,1,2,\ldots$ be the sequence of random times defined as follows:

$$\tau_{0,r}:=0,\quad \tau_{l,r}:=\begin{cases}\inf\{t\in[\tau_{l-1,r},T]: & R(\tau_{l-1,r},t)>M_r\},\\ T+1, & \text{if the set above is empty.}\end{cases}$$

It is shown in [3] that each $\{\tau_{l,r}\}$ is an $\mathcal{F}^\xi$-Markov time. For all $i=1,2,\ldots$ also define $\zeta_{i,r}:=\tau_{i,r}-\tau_{i-1,r}$. Here is the modified corollary:

Corollary 7 *Assume that a strong Markov process $\xi_t, t\in[0,T]$, belongs to $\mathcal{M}(\beta,\gamma)$ for some $\beta>(3-e)/(e-1)$. Let r be any integer. Then for any $i=1,2,\ldots$ and for any number $q\in\big(2/(\beta+3),1-1/e\big)$ almost surely on $\{\tau_{i-1,r}<T\}$ we have*

$$E\left(e^{-\zeta_{i,r}}\,\middle|\,\mathcal{F}_{\tau_{i-1,r}}\right)\leq\begin{cases}\beta\gamma(\beta,T_r)T_r^{-\beta} & \text{if } T_r<1,\\ e^{-1}+q & \text{if } T_r=1,\end{cases}$$

where $T_r=\min\Big\{\big((M_{r+2}\wedge a_0)^\gamma/(2K)\big)^{1/\beta},T,1\Big\}$.

Proof. One only needs to replace "$e^{-1}+7/24<0.660$" in the display of Case 1 by "$e^{-1}+q<1$" in view of Lemma 6 and the fact that for $q\in(2/(\beta+3),1-1/e)$ we have $\beta>2/q-3$. □

Next change the statement of Lemma 2.7 of [3] as follows:

Lemma 8 *Let r be any integer, $\beta>(3-e)/(e-1)$ and $q\in(2/(\beta+3),1-1/e)$. For any $j=1,2,\ldots$*

$$P\big(\tau_{j,r}\leq T\big)\leq e^T\begin{cases}\big(\beta\gamma(\beta,T_r)T_r^{-\beta}\big)^j & \text{if } T_r<1,\\ \big(e^{-1}+q\big)^j & \text{if } T_r=1.\end{cases}$$

Proof. The proof of this lemma is identical to that of Lemma 2.7 of [3]. Just use Corollary 7 instead of Corollary 2.6 of [3]. $\square$

To present the necessary changes to Lemma 3.1 of [3] we once again need a few definitions from [3]. Let $PP := \{\kappa : \kappa = \{t_i : 0 = t_0 < t_1 < \cdots < t_{m_\kappa} = T\}\}$ be the set of all point partitions of $[0, T]$. For any integer r, recall $M_r = 2^{-r-1}$ and let $\kappa \in PP$, $\kappa = \{t_i\}_{i=0}^{m_\kappa}$ be an arbitrary point partition. Define the random sets

$$K_r(\omega) := K_r(\omega, \kappa) := \{k : 1 \leq k \leq m_\kappa,\ M_r \leq \rho(\xi_{t_k}, \xi_{t_{k-1}}) < M_{r-1}\}.$$

Let r_1 be the largest integer less or equal to $-(\log_2 a_0 + 3)$, so that $M_{r+2} \geq a_0$ for all $r \leq r_1$.

Lemma 9 *Let ξ_t, r, β and T_r be as in Corollary 7 . Suppose that $q \in (2/(\beta+3), 1 - 1/e)$ and $r > r_1$. Then*

$$E \sup_{\kappa \in PP} \sum_{k \in K_r(\omega)} 1 \leq \begin{cases} 4T_r^{-1} e^T & \text{if } T_r < 1, \\ e^T \frac{q+1/e}{1-q-1/e} & \text{if } T_r = 1. \end{cases}$$

Proof. The only difference between the proof of this lemma and Lemma 3.1 of [3] is in the case $T_r = 1$ where the geometric series with general term $(e^{-1} + 7/24)^j$ is replaced by $(e^{-1} + q)^j$ leading to the sum $(q + 1/e)/(1 - q - 1/e)$ in place of $0.66/0.34 < 1.95$. $\square$

And finally replace 1.95 in the bound of P_1 on page 2063 of [3] by $\frac{q+1/e}{1-q-1/e}$ to complete the proof of Theorem 3.

Now switch attention to the proof of Theorem 4. The first two properties of η_t are easy to establish while the third is somewhat more involved.

Proof of property (i): by definition of η_t (see (7)) it is clear that η_t is right continuous for any $t \in [0, 1)$ and has left-limits for any $t \in (0, 1)$. The only uncertainty is the left-limit at $t = 1$. We claim that η_t is left continuous at $t = 1$ almost surely. To get this we first show that η_t converges in probability to 1/2 as $t \uparrow 1$. Let $\varepsilon \in (0, 1/(2(1 + 2\ln 2)))$ be arbitrary and consider

$$m_\varepsilon = \max\{m \geq 1 : m < \exp\{((2\varepsilon)^{-1} - 1)/2\} - 1\},$$

so that $c_m < 1/2 - \varepsilon$, for all $m > m_\varepsilon$. Now set $\delta = 3^{-m_\varepsilon - 1}$ and for all $t \in (1 - \delta, 1)$ get

$$P(|\eta_t - 1/2| > \varepsilon) \leq P(X_m = 0, \forall m > m_\varepsilon) = \prod_{m=m_\varepsilon+1}^{\infty} (1 - \delta_m) = 0 < \varepsilon.$$

Furthermore, by definition of η_t, for each ω, $|\eta_t(\omega) - 1/2|$ is nonincreasing as $t \uparrow 1$. So, in fact, we have a stronger statement:

$$P\left(\sup_{t \leq s < 1} |\eta_s - 1/2| > \varepsilon\right) = 0 < \varepsilon, \quad \text{if} \quad t \in (1 - \delta, 1).$$

Therefore, for almost all ω, $\sup_{t \leq s < 1} |\eta_s(\omega) - 1/2| \leq \varepsilon$ provided $t \in (1 - \delta, 1)$. Hence, almost surely

$$\limsup_{t \uparrow 1} |\eta_t - 1/2| = 0,$$

and we obtain the almost sure convergence of η_t to 1/2.

Proof of property (ii): it is easy to see that for any fixed $\omega \in \Omega$

$$v_p(\eta.(\omega)) = v_p(\eta.(\omega), [0,1]) \geq \sum_{m=1}^{\infty} v_p(\eta.(\omega), J_m), \tag{11}$$

where for $m = 1$

$$v_p(\eta.(\omega), J_1) \geq |1 - 2\eta_0(\omega)|^p + |1 - \eta_0(\omega) - \eta(\Delta_1)(\omega)|^p \geq 1$$

and for $m \geq 2$ we have

$$\begin{aligned} v_p(\eta.(\omega), J_m) &\geq |\eta(\Delta_{m-1})(\omega) - \eta(V_m)(\omega)|^p + |\eta(V_m)(\omega) - \eta(\Delta_m)(\omega)|^p \\ &= |1 - 2\eta(\Delta_{m-1})(\omega)|^p + |1 - \eta(\Delta_{m-1})(\omega) - \eta(\Delta_m)(\omega)|^p \\ &\geq |1 - 2c_{m-1}|^p + |1 - c_{m-1} - c_m|^p \geq 2\,|1 - 2c_m|^p\,. \end{aligned}$$

Plugging in these bounds into (11) we obtain

$$v_p(\eta.(\omega)) \geq 1 + 2\sum_{m=2}^{\infty} |1 - 2c_m|^p = 1 + 2\sum_{m=2}^{\infty} (1 + 2\ln(m+1))^{-p} = +\infty,$$

for every $p > 0$ and $\omega \in \Omega$. Hence $v_p(\eta.) = +\infty$ almost surely for every $p > 0$.

Proof of property (iii): inequality (10) will follow once we obtain a bound on transition probabilities $P(|\eta_t - \eta_s| > a|\eta_s = y)$ for $0 \leq s < t \leq (s+h) \wedge 1$ and $y \in A_\infty$. This will be done using two lemmas: the first will handle the case when both s and t belong to the same subinterval J_m for some $m \geq 1$, and the second will tackle the case $s \in J_m$ and $t \in J_k$ for $k > m$.

Lemma 10 *Suppose η_t is defined by* (7)*. Let $s, t \in J_m$ for some $m \geq 1$. Then for any $y \in A_\infty$* (*see* (8)) *almost surely*

$$\begin{aligned} P(|\eta_t - \eta_s| > a|\eta_s = y) \leq \frac{1}{1-\delta_1}\Big(&\psi_m(s \vee (t - a_m)) - \psi_m(t) \\ &+ 2\delta_m\big\{\psi_m(s - a_m) - \psi_m(t - a_m)\big\}\Big), \end{aligned}$$

where $\psi_m(x) = P(x < V_m)$ is given by

$$\psi_m(x) = \begin{cases} 1, & \text{if } 0 \leq x < a_m + d_m, \\ \dfrac{n_m + 1 - (x - d_m)a_m^{-1}}{n_m}, & \text{if } a_m + d_m \leq x < a_m(n_m+1) + d_m, \\ 0, & \text{if } a_m(n_m+1) + d_m \leq x \leq 1. \end{cases} \tag{12}$$

Proof. For given s and t from J_m, we have $\eta_t \in \{y, 1-y, c_m, 1-c_m\}$ whenever $\eta_s = y$. Moreover,

$$\begin{aligned} P(|\eta_t - \eta_s| > a|\eta_s = y) &= P(\eta_t = 1 - y|\eta_s = y)\mathbf{1}_{\{|1-2y|>a\}} \\ &\quad + P(\eta_t = c_m|\eta_s = y)\mathbf{1}_{\{|c_m - y|>a\}} \\ &\quad + P(\eta_t = 1 - c_m|\eta_s = y)\mathbf{1}_{\{|1-c_m-y|>a\}} \\ &= q_1\mathbf{1}_{\{|1-2y|>a\}} + q_2\mathbf{1}_{\{|c_m-y|>a\}} + q_3\mathbf{1}_{\{|1-c_m-y|>a\}}. \end{aligned} \tag{13}$$

For $s,t \in J_m$ we have $q_1 = 0$ if $y \notin A_{m-1}$ (see (9)), and if $y \in A_{m-1}$

$$\begin{aligned} q_1 &= \frac{P(\eta_t = 1-y, \eta_s = y)}{P(\eta_s = y)} \\ &= \frac{\sum_{x \in A_{m-1}} P(\eta_t = 1-y, \eta_s = y | \eta(\Delta_{m-1}) = x) P(\eta(\Delta_{m-1}) = x)}{\sum_{z \in A_{m-1}} P(\eta_s = y | \eta(\Delta_{m-1}) = z) P(\eta(\Delta_{m-1}) = z)} \\ &= \frac{P(\eta_t = 1-y, \eta_s = y | \eta(\Delta_{m-1}) = y)}{P(\eta_s = y | \eta(\Delta_{m-1}) = y) + P(\eta_s = y | \eta(\Delta_{m-1}) = 1-y)}, \end{aligned}$$

where we used the definition of η_t to get $P(\eta_t = 1-y, \eta_s = y|\eta(\Delta_{m-1}) = x) = 0$, $P(\eta_s = y|\eta(\Delta_{m-1}) = z) = 0$, and, due to symmetry of η_t, $P(\eta(\Delta_{m-1}) = x) = P(\eta(\Delta_{m-1}) = 1-x)$, for $x \in A_{m-1} \setminus \{y\}$, $z \in A_{m-1} \setminus \{y, 1-y\}$ and $y \in A_{m-1}$. Furthermore,

$$\begin{aligned} P(\eta_t = 1-y, \eta_s = y | \eta(\Delta_{m-1}) = y) &= P(s < V_m \le t < V'_m), \\ P(\eta_s = y | \eta(\Delta_{m-1}) = y) &= P(s < V_m) + P(V'_m \le s, X_m = 0), \\ P(\eta_s = y | \eta(\Delta_{m-1}) = 1-y) &= P(V_m \le s < V'_m). \end{aligned}$$

Using the function ψ_m we can rewrite a bound for q_1 with $y \in A_{m-1}$ as follows

$$\begin{aligned} q_1 &= \frac{P(s < V_m \le t < V'_m)}{P(s < V_m) + P(V'_m \le s, X_m = 0) + P(V_m \le s < V'_m)} \\ &= \frac{\psi_m\big(s \vee (t - a_m)\big) - \psi_m(t)}{1 - \delta_m\big\{1 - \psi_m(s - a_m)\big\}} \le \frac{\psi_m\big(s \vee (t - a_m)\big) - \psi_m(t)}{1 - \delta_1}. \end{aligned} \tag{14}$$

Let us now evaluate q_2. For $m = 1, 2, \ldots$, set $A^0_m = \{c_k \,|\, k = 0, 1, \ldots, m\}$ and $A^1_m = \{1 - c_k \,|\, k = 0, 1, \ldots, m\}$. If $|c_m - y| > a$, then by definition of η_t we get $q_2 = 0$ if $y \notin A^0_{m-1}$, and if $y \in A^0_{m-1}$ then

$$\begin{aligned} q_2 &= P(\eta_t = c_m | \eta_s = y) \\ &= \frac{\sum_{x \in A_{m-1}} P(\eta_t = c_m, \eta_s = y | \eta(\Delta_{m-1}) = x) P(\eta(\Delta_{m-1}) = x)}{\sum_{z \in A_{m-1}} P(\eta_s = y | \eta(\Delta_{m-1}) = z) P(\eta(\Delta_{m-1}) = z)} \\ &= \frac{\sum_{x \in \{y, 1-y\}} P(\eta_t = c_m, \eta_s = y | \eta(\Delta_{m-1}) = x)}{1 - \delta_m\big\{1 - \psi_m(s - a_m)\big\}} \\ &\le (1 - \delta_1)^{-1}\big\{P(s < V_m, t \ge V'_m, X_m = 1) + P(V_m \le s < V'_m \le t, X_m = 1)\big\} \\ &= \frac{\delta_m}{1 - \delta_1}\big\{\psi_m(s) - \psi_m(s \vee (t - a_m)) + \psi_m(s - a_m) - \psi_m((t - a_m) \wedge s)\big\} \\ &= \frac{\delta_m}{1 - \delta_1}\big\{\psi_m(s - a_m) - \psi_m(t - a_m)\big\}. \end{aligned} \tag{15}$$

The argument for q_3 is essentially the same: if $|1 - c_m - y| > a$, then by definition of η_t we get $q_3 = 0$ if $y \notin A^1_{m-1}$, and if $y \in A^1_{m-1}$ then

$$q_3 = P(\eta_t = 1 - c_m | \eta_s = y) \le \frac{\delta_m}{1 - \delta_1}\big\{\psi_m(s - a_m) - \psi_m(t - a_m)\big\}. \tag{16}$$

Using the fact that the sets A^0_{m-1} and A^1_{m-1} are disjoint and combining the bounds (14),(15), (16) together with (13) we obtain the claim of this lemma. □

Lemma 11 *Suppose η_t is defined by (7). Let $s \in J_m$ and $t \in J_k$ for some $1 \leq m < k$. Then for any $y \in A_\infty$ almost surely*

$$P(|\eta_t - \eta_s| > a|\eta_s = y) \leq 2\sum_{j=m}^{k-1} \delta_j + \delta_k + \frac{4}{n_m}.$$

In particular, if $\delta_m = 2 \cdot 3^{-m}$ and n_m is given by (4), then

$$P(|\eta_t - \eta_s| > a|\eta_s = y) \leq 12 \cdot 3^{-m}.$$

PROOF. First notice that if $\eta_s = y$, then we always have $\eta_t \in (A_k \setminus A_{m-1}) \cup \{y, 1-y\}$, and so

$$P(|\eta_t - \eta_s| > a \,|\, \eta_s = y) = \sum_{j=m_y}^{k} (v_{0,j} + v_{1,j}) + P(\eta_t = 1-y|\eta_s = y)\mathbf{1}_{\{|1-2y|>a\}},$$

where

$$\begin{aligned}
v_{0,j} &= P(\eta_t = c_j|\eta_s = y)\mathbf{1}_{\{|c_j-y|>a\}},\\
v_{1,j} &= P(\eta_t = 1-c_j|\eta_s = y)\mathbf{1}_{\{|1-c_j-y|>a\}},\\
m_y &= \begin{cases} m, & \text{if } y \in A_{m-1},\\ m+1, & \text{if } y \in A_m \setminus A_{m-1} = \{c_m, 1-c_m\}. \end{cases}
\end{aligned}$$

For $j < k$ we simply use the definition of η_t and properties of X_j to get

$$\begin{aligned}
v_{0,j} &\leq P(X_j = 1|\eta_s = y)\mathbf{1}_{\{|c_j-y|>a\}} \leq \delta_j,\\
v_{1,j} &\leq P(X_j = 1|\eta_s = y)\mathbf{1}_{\{|1-c_j-y|>a\}} \leq \delta_j,
\end{aligned}$$

and for $j = k$ (7) yields

$$\begin{aligned}
v_{0,k} &\leq P(X_k = 1, \eta(\Delta_{k-1}) < 1/2 \,|\eta_s = y)\mathbf{1}_{\{|c_k-y|>a\}} \leq \delta_k \mathbf{1}_{\{|c_k-y|>a, y<1/2\}},\\
v_{1,k} &\leq P(X_k = 1, \eta(\Delta_{k-1}) > 1/2 \,|\eta_s = y)\mathbf{1}_{\{|1-c_k-y|>a\}} \leq \delta_k \mathbf{1}_{\{|1-c_k-y|>a, y>1/2\}},
\end{aligned}$$

since

$$\begin{aligned}
P(\eta(\Delta_{k-1}) < 1/2 \,|\eta_s = y) = P(X_0 = 0 \,|\eta_s = y) = \mathbf{1}_{\{y<1/2\}},\\
P(\eta(\Delta_{k-1}) > 1/2 \,|\eta_s = y) = P(X_0 = 1 \,|\eta_s = y) = \mathbf{1}_{\{y>1/2\}}.
\end{aligned}$$

Also

$$\begin{aligned}
P(\eta_t = 1-y|\eta_s = y) &= P(\eta_t = 1-y, s \in [V_m, V'_m) \,|\eta_s = y)\\
&\quad + P(\eta_t = 1-y, s \notin [V_m, V'_m) \,|\eta_s = y)\\
&\leq P(t \notin [V_k, V'_k), s \in [V_m, V'_m) \,|\eta_s = y)\\
&\quad + P(t \in [V_k, V'_k), s \notin [V_m, V'_m) \,|\eta_s = y)\\
&= P(t \notin [V_k, V'_k))P(s \in [V_m, V'_m) \,|\eta_s = y)\\
&\quad + P(t \in [V_k, V'_k))P(s \notin [V_m, V'_m) \,|\eta_s = y)\\
&\leq P(s \in [V_m, V'_m) \,|\eta_s = y) + P(t \in [V_k, V'_k)),
\end{aligned}$$

where, using similar argument as for q_1 in the proof of Lemma 10,

$$
\begin{aligned}
P(s \in [V_m, V_m')\,|\eta_s = y) &= \frac{P(s \in [V_m, V_m'), \eta_s = y)}{P(\eta_s = y)} \mathbf{1}_{\{y \in A_{m-1}\}} \\
&= \frac{\sum_{x \in A_{m-1}} P(s \in [V_m, V_m'), \eta_s = y | \eta(\Delta_{m-1}) = x) P(\eta(\Delta_{m-1}) = x)}{\sum_{z \in A_{m-1}} P(\eta_s = y | \eta(\Delta_{m-1}) = z) P(\eta(\Delta_{m-1}) = z)} \mathbf{1}_{\{y \in A_{m-1}\}} \\
&= \frac{P(s \in [V_m, V_m') | \eta(\Delta_{m-1}) = 1 - y)}{1 - \delta_m \big\{1 - \psi_m(s - a_m)\big\}} \\
&\le \frac{P(s \in [V_m, V_m'))}{1 - \delta_m} = \frac{\psi_m(s - a_m) - \psi_m(s)}{1 - \delta_m}.
\end{aligned}
$$

Hence, for any $y \in A_m$, by the definitions of δ_m and ψ_j, for $j = m, k$, we get

$$
P(\eta_t = 1 - y | \eta_s = y) \le \frac{\psi_m(s - a_m) - \psi_m(s)}{1 - \delta_m} + \psi_k(t - a_k) - \psi_k(t) \le \frac{3}{n_m} + \frac{1}{n_k} \le \frac{4}{n_m}.
$$

Combining the above bounds we get the first inequality of this lemma. To get the second simply sum the tail of the geometric series with $\delta_{m+1}/\delta_m = 3^{-1}$, $m \ge 1$. Furthermore, since $3^m > 2m$ for any $m \ge 0$, the definitions of δ_m and n_m imply $\delta_{m-1} \ge 4/n_m$, and the second inequality of this lemma follows. $\square$

Back to the proof of (iii) Lemmas 10 and 11 show that we need only bounds on various differences of the function ψ_j for $j = m, k$ to get a bound on transition probabilities. And the bound (10) will follow from the careful treatment of the supremum with respect to the index m. Therefore, we first provide the following inequalities which follow easily from the definition of $\psi_m(x)$:

$$
\begin{aligned}
|\psi_m(s \vee (t - a_m)) - \psi_m(t)| &\le \frac{1}{n_m} \mathbf{1}_{\{t - s \ge a_m\}} + \frac{t - s}{a_m n_m} \mathbf{1}_{\{t - s < a_m\}} = \frac{a_m \wedge |t - s|}{a_m n_m}, \\
|\psi_m(s - a_m)) - \psi_m(t - a_m)| &\le 1 \wedge \frac{|t - s|}{a_m n_m}, \quad \text{for any} \quad 0 \le s < t \le 1.
\end{aligned}
$$

Thus for any $0 \le s < t \le 1$ almost surely

$$
\begin{aligned}
\sup_{m:\, s,t \in J_m} \sup_{y \in A_\infty} P(|\eta_t - \eta_s| > a | \eta_s = y) &= \sup_{m:\, s,t \in J_m} \sup_{y \in A_{m-1}} P(|\eta_t - \eta_s| > a | \eta_s = y) \\
&\le \frac{1}{1 - \delta_1} \sup_{m:\, t - s \le \delta_m} \left\{ \frac{a_m \wedge |t - s|}{a_m n_m} + \delta_m \Big(1 \wedge \frac{|t - s|}{a_m n_m}\Big) \right\} =: G_1(t - s),
\end{aligned}
\tag{17}
$$

Define $\bar{m}(x) := \min\{m \ge 1 |\, x \ge a_m\}$. Since $a_m \downarrow 0$ as $m \to \infty$, $\bar{m}(x)$ is finite for any $x > 0$. Also for $m \ge \bar{m}(t - s)$ we have $t - s \ge a_m$, and $t - s < a_m$ is true for all $m < \bar{m}(t - s)$. Now notice that

$$
\sup_m \frac{\delta_m}{a_m n_m} = \sup_m \frac{3(n_m + 3)}{n_m} = 3\big(1 + \frac{3}{n_1}\big) =: K_1,
$$

so that

$$
\begin{aligned}
G_1(t - s) &\le \frac{1}{1 - \delta_1} \Big[\max\Big\{ \sup_{m < \bar{m}(t-s)} \frac{t - s}{a_m n_m}, \sup_{m \ge \bar{m}(t-s)} \frac{1}{n_m} \Big\} + K_1 (t - s) \Big] \\
&\le \frac{1}{1 - \delta_1} \Big[\max\Big\{ K_1 \frac{t - s}{\delta_{\bar{m}(t-s)}}, \frac{1}{n_{\bar{m}(t-s)}} \Big\} + K_1 (t - s) \Big] \\
&\le \frac{K_1 (t - s)}{1 - \delta_1} \Big\{ \frac{1}{\delta_{\bar{m}(t-s)}} + 1 \Big\} \le K_2 (t - s) 3^{\bar{m}(t-s)},
\end{aligned}
$$

where K_2 can be taken to be $2K_1(1-\delta_1)^{-1}$ and the last inequality follows from the choice of $\bar{m}(t-s)$ and δ_m. Furthermore,

$$t-s < a_{\bar{m}(t-s)-1} = \left\{3^{3^{\bar{m}(t-s)-1}} + \frac{1}{2}3^{\bar{m}(t-s)+1}\right\}^{-1} \leq 3^{-3^{\bar{m}(t-s)-1}},$$

thus,

$$\begin{aligned} G_1(t-s) &\leq 3K_2(t-s)\log_3\left((t-s)^{-1}\right) \\ &\leq 3K_2\left(\frac{1}{e\ln 3} \wedge h\log_3\frac{1}{h}\right) \leq \frac{3K_2}{\ln 3}h\left(1 \vee \ln\frac{1}{h}\right), \end{aligned} \tag{18}$$

for any $0 < t-s < h$, since the function $g(x) = x\log_3(1/x), x \in (0,1]$, has the absolute maximum at $x = 1/e$ equal to $(e\ln 3)^{-1}$ and is increasing for $x \in (0, 1/e)$.
Now let's look at the cases when $s \in J_m$ and $t \in J_k$ for $m < k$. Then it is clear that if $0 \leq s < t \leq 1$ and $\eta_t \neq \eta_s$, then

$$\begin{aligned} t-s &\geq \big(d_{m+1} - \big(d_m + (n_m+1)a_m\big)\big)\mathbf{1}_{\{k=m+1\}} + \big(\Delta_{k-1} - \Delta_m\big)\mathbf{1}_{\{k>m+1\}} \\ &= \left(\frac{\delta_{m+1}}{3} + \frac{\delta_m}{3}\frac{n_m+5}{n_m+3}\right)\mathbf{1}_{\{k=m+1\}} + \left(\sum_{j=m+1}^{k-1}\delta_j\right)\mathbf{1}_{\{k>m+1\}} \\ &\geq \min\left\{\frac{\delta_m + \delta_{m+1}}{3}, \delta_{m+1}\right\} = \delta_{m+1}. \end{aligned}$$

Therefore, by Lemma 11

$$\begin{aligned} \sup_{m:s\in J_m}\ \sup_{k:k>m,t\in J_k}\ \sup_{y\in A_\infty} P(|\eta_t - \eta_s| > a | \eta_s = y) &\leq \sup_{m:t-s\geq\delta_{m+1}} 2\delta_{m-1} \\ &\leq 2\delta_{\tilde{m}(t-s)-2} \leq 9h, \end{aligned} \tag{19}$$

if $0 < t-s < h$, and where for any $x > 0$ we set $\tilde{m}(x) = \min\{m \geq 1 : x \geq \delta_m\}$. Combining (17), (18) and (19) we get for any $h \in (0,1)$ and $a > 0$

$$\alpha(h,a) \leq \max\left\{\frac{3K_2}{\ln 3}h\left(1 \vee \ln\frac{1}{h}\right), 9h\right\} \leq Kh\left(1 \vee \ln\frac{1}{h}\right),$$

with $K = \max\{3K_2/\ln 3, 9\}$. This concludes the proof of the theorem.

Concluding remarks

One can choose larger n_m's to reduce the factor $1 \vee \ln\frac{1}{h}$ in (10) to $1 \vee \ln\ln\frac{1}{h}$, e.g. by taking

$$\log_3(n_m) = 3^{3^m} - m - 1 + \log_3 2,$$

or something even smaller, but getting rid of this logarithmic factor completely poses a problem. The reason is simple: even though the functions ψ_m are Lipschitz continuous, they are not uniformly Lipschitz for $m \geq 1$ (the Lipschitz constants are of the order $O(\delta_m^{-1})$). So the bound of q_1 (see (14)) cannot be essentially improved, unless one is willing to increase the speed of convergence to $1/2$ of the sequence c_m which would restrict the number of terms one needs to consider in (17) (see also (13)) and would also allow for the p-variation of η_t to be almost surely finite for some $p > 0$.

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