Elect. Comm. in Probab. **13** (2008), 272–279

ELECTRONIC COMMUNICATIONS in PROBABILITY

INTERNAL DIFFUSION-LIMITED AGGREGATION ON NON-AMENABLE GRAPHS

WILFRIED HUSS[1]
Department of Mathematical Structure Theory, TU Graz, Steyrergasse 30/III, A - 8010 Graz
email: huss@finanz.math.tugraz.at

Submitted January 31, 2008, *accepted in final form* April 28, 2008

AMS 2000 Subject classification: Primary: 60K35, Secondary: 82B24, 82B41
Keywords: Interacting particle systems - Random Walks on graphs

Abstract
The stochastic growth model Internal Diffusion Limited Aggregation was defined in 1991 by Diaconis and Fulton. Several shape results are known when the underlying state space is the d-dimensional lattice, or a discrete group with exponential growth. We prove an extension of the shape result of Blachère and Brofferio for Internal Diffusion Limited Aggregation on a wide class of Markov chains on non-amenable graphs.

1 Introduction

Let X be a infinite, locally finite connected graph, and let $P = \{p(x, y)\}_{x,y\in X}$ be the transition matrix of an irreducible random walk on X that is adapted to the graph structure, i.e. $p(x, y)$ is positive if and only if x is a neighbour of y in X. We use X to denote both the graph as well as its vertex set, since the neighbourhood relation of X is also encoded in the transition matrix P. We write $\{S^j(n)\}_{j\in\mathbb{N}}$ for the trajectories of independent realizations of the random walk (X, P), with a common starting point $S^j(0) = o$.
The Internal Diffusion Limited Aggregation (iDLA) is a stochastic process of increasing subsets $\{A(t)\}_{t\in\mathbb{N}}$ of X, which are defined by the following rule:

$$\begin{aligned} A(0) &= \{o\} \\ \mathbb{P}[A(t+1) = A(t) \cup \{x\} |\, A(t)] &= \mathbb{P}^o[S^{t+1}(\sigma_{t+1}) = x]. \end{aligned}$$

Here $\sigma_t = \inf\{k \geq 0 : S^t(n) \notin A(t-1)\}$, is the time of the first exit of the random walk from the set $A(t-1)$.
This means that at time t a random walk S^t is started at the root o, and evolves as long as it stays inside the iDLA-cluster $A(t-1)$. When S^t leaves $A(t-1)$ for the first time, the random walk stops, and the point outside of the cluster that is visited by S^t is added to the new cluster $A(t)$.

[1]RESEARCH SUPPORTED BY THE FWF PROGRAM FWF-P18703-N18

This growth model was introduced by Diaconis and Fulton [1] in 1991. Several shape results and estimates for the fluctuations around the limiting shape for $\mathbb{Z}^d$ as the underlying graph have been obtained by Lawler, Bramson and Griffeath [2], Lawler [3] and Blachère [4]. In 2004 Blachère [5] obtained a shape result for the iDLA model with symmetric random walk on discrete groups of polynomial growth, although with less precise bounds than in the case of $\mathbb{Z}^d$. Finally in 2007 Blachère and Brofferio [6] proved a similar shape result for iDLA on finitely generated groups with exponential growth.
In all cases where iDLA dynamics has been studied so far, a common behaviour for the limiting shape of the clusters emerged. Namely, the boundary of the limiting shape can be described as the level lines of the Green function, that is the sets of the form

$$\{x \in X : \ G(o, x) = N\},$$

where N is a positive constant.
This correspondence has been made particularly clear by Blachère and Brofferio [6], with the introduction of the *hitting distance*, a left invariant metric on finitely generated groups, which is defined as

$$d_H(x, y) = -\ln F(x, y),$$

where $F(x, y)$ is the probability that y is hit in finite time by a random walk starting in x. Further work on the hitting distance and its connections with random walk entropy and the Martin boundary has been done by Blachère, Haïssinsky and Mathieu [7].
In this note we extend the result of Blachère and Brofferio to a class of random walks on general non-amenable graphs.

2 Random walks on non-amenable graphs

Let X be a locally finite, connected graph. We denote by $\sim$ the neighbourhood relation of X. Let $P = \{p(x, y)\}_{x,y\in X}$ be the transition matrix of an irreducible Markov chain on X. We denote by $S(n)$ the trajectories and by

$$p^{(n)}(x, y) \stackrel{\text{def.}}{=} \mathbb{P}^x[S(n) = y]$$

the n-step transition probabilities of the Markov chain (X, P).
We first recall some basic notions from the theory of random walks. The *spectral radius* is defined as

$$\rho(P) \stackrel{\text{def.}}{=} \limsup_{n\to\infty} p^{(n)}(x, y)$$

and is independent of $x, y \in X$, because of irreducibility. It is known that

$$p^{(n)}(x, x) \leq \rho(P)^n, \text{ for all } x \in X. \tag{1}$$

The Green function

$$G(x, y) = \sum_{n=0}^{\infty} p^{(n)}(x, y)$$

denotes the expected number of visits in y, of the random walk starting in x.
For a random walk $S(n)$, let τ_y be the first hitting time of the point y:

$$\tau_y \stackrel{\text{def.}}{=} \inf\{n \geq 0 : \ S(n) = y\},$$

with the convention $\tau_y = \infty$, if the random walk never visits y. The hitting probability of y, for a random walk starting in x, is denoted by

$$F(x,y) \overset{\text{def.}}{=} \mathbb{P}^x\left[\tau_y < \infty\right].$$

A simple calculation shows

$$G(x,y) = F(x,y)G(y,y).$$

Definition 2.1. *A random walk is called* uniformly irreducible*, if there are constants $\varepsilon_0 > 0$ and $N < \infty$ such that*

$$x \sim y \text{ implies } p^{(n)}(x,y) \geq \varepsilon_0 \text{ for some } n \leq N.$$

Definition 2.2. *A Markov chain (X,P) is called* reversible*, if there exists a measure $m : X \to (0,\infty)$, such that*

$$m(x)p(x,y) = m(y)p(y,x), \text{ for all } x,y \in X.$$

If m is bounded, i.e. there exists a $C \in (0,\infty)$, such that

$$C^{-1} \leq m(x) \leq C, \text{ for all } x \in X, \tag{2}$$

then the Markov chain is called strongly reversible*.*

Remark 2.3. *The simple random walk is strongly reversible if and only if the vertex degree of X is bounded. And in that case it is also uniformly irreducible.*

The following inequality from [8, Lemma 8.1] gives us a generalization of (1) for arbitrary n-step transition probabilities in the case of strongly reversible Markov chains.

Lemma 2.4. *If (X,P) is strongly reversible then $p^{(n)}(x,y) \leq C\rho(P)^n$, with C as in Definition 2.2.*

Proposition 2.5. *For (X,P) uniformly irreducible, there exists $\varepsilon_0 > 0$ such that, for all neighbours $x \sim y$:*

$$F(x,y) \geq \varepsilon_0.$$

Proof. Recall from Definition 2.1, that there exist $\varepsilon_0 > 0$ and $n \leq N$ such that

$$\varepsilon_0 \leq p^{(n)}(x,y) = \mathbb{P}^x\left[S(n) = y\right] \leq \mathbb{P}^x\left[\tau_y < \infty\right] = F(x,y).$$

□

The converse is not true in general and it is easy to construct counterexamples. Using some additional assumptions we can show the following result.

Proposition 2.6. *Let (X,P) be a strongly reversible random walk with $\rho(P) < 1$, and such that there exists a constant $\varepsilon_0 > 0$ such that $F(x,y) \geq \varepsilon_0$ for all neighbours $x \sim y \in X$. Then the random walk is uniformly irreducible.*

Proof. Suppose that the random walk is not uniformly irreducible. This means that for all $\delta > 0$, there exist neighbouring points $x_\delta \sim y_\delta$ such that, for all $n \in \mathbb{N}:\ p^{(n)}(x_\delta, y_\delta) < \delta$.
So, we can construct sequences $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$, with $x_i \sim y_i$ and

$$p^{(n)}(x_i, y_i) < \frac{1}{i}, \text{ for all } n \in \mathbb{N}. \tag{3}$$

For $n = 0$ this implies $x_i \neq y_i$.
Lemma 2.4 gives a second bound for the n-step transition probabilities:

$$p^{(n)}(x_i, y_i) \leq C\rho(P)^n. \tag{4}$$

Now we look at the Green function

$$G(x_i, y_i) = F(x_i, y_i)G(y_i, y_i) \geq F(x_i, y_i) \geq \varepsilon_0. \tag{5}$$

But, on the other hand, using (3) and (4)

$$G(x_i, y_i) = \sum_{n=1}^{\infty} p^{(n)}(x_i, y_i) \leq \sum_{n=1}^{\infty} \min\left\{\frac{1}{i},\, C\rho(P)^n\right\}.$$

Defining $N_i = \left\lfloor -\frac{\ln(i\cdot C)}{\ln\rho(P)} \right\rfloor$, we can split the above sum in two parts:

$$\begin{aligned} G(x_i, y_i) &\leq \sum_{n=1}^{N_i} \frac{1}{i} + C \sum_{n=N_i+1}^{\infty} \rho(P)^n \\ &\leq -\frac{\ln(i\cdot C)}{i\cdot \ln\rho(P)} + \frac{C}{1-\rho(P)}\rho(P)^{N_i+1}. \end{aligned}$$

This goes to 0 for $i \to \infty$, because N_i goes to infinity and $\rho(P) < 1$, and gives a contradiction to the lower bound of the Green function in equation (5). □

For a subset $K \subset X$ set $\partial_E(X) \overset{\text{def.}}{=} \{(x,y) \in X^2 : x \sim y, x \in K, y \notin K\}$. The *edge-isoperimetric constant* of X is defined as

$$\iota_E(X) \overset{\text{def.}}{=} \inf\left\{\frac{|\partial_E K|}{|K|} : K \subset X \text{ finite}, K \neq \emptyset\right\}.$$

A Graph X is called *amenable* if $\iota_E(x) = 0$ and *non-amenable* if $\iota_E(x) > 0$. The following relation between amenability and the spectral radius of simple random walk is well known [9, 10]. See [8, Theorem 10.3] for the proof of a more general version of this theorem.

Theorem 2.7. *X is non-amenable if and only if $\rho(P) < 1$. Here P is the transition operator of simple random walk on X.*

The following theorem (see [8, Theorem 10.3]) will be needed later. For a reversible Markov chain we define the real Hilbert space $\ell^2(X, m)$ with inner product

$$\langle f, g\rangle = \sum_{x\in X} f(x)g(x)m(x).$$

Theorem 2.8. *The following statements are equivalent for reversible Markov chains (X, P).*

(a) *The spectral radius $\rho(P)$ is strictly smaller than 1.*

(b) *The Green function defines a bounded linear operator on $\ell^2(X, m)$ by*

$$Gf(x) = \sum_{y \in X} G(x, y) f(y).$$

From now on we only consider Markov chains (X, P) that are uniformly irreducible, strongly reversible and have $\rho(P) < 1$. In this setting we can follow [6] and define the "*hitting distance*". For all $x, y \in X$:

$$d_H(x, y) \overset{\text{def.}}{=} -\ln F(x, y).$$

If X is the Cayley graph of a finitely generated group, and P a symmetric left invariant random walk on X then d_H is a left invariant metric on X, see [6]. For arbitrary graphs this does not hold anymore, but we can still use d_H to define balls of radius Kn, where $K = -\ln \varepsilon_0$ (with ε_0 as in Definition 2.1):

$$\mathcal{B}_n(x) \overset{\text{def.}}{=} \{z \in X : \, d_H(x, z) \leq Kn\},$$

and its boundaries as

$$\partial \mathcal{B}_n(x) \overset{\text{def.}}{=} \{y \notin \mathcal{B}_n(x) : \, \exists z \sim y \wedge z \in \mathcal{B}_n(x)\}.$$

The constant K is needed in the definition of the balls to ensure that the balls with radius n and $n+1$ are properly nested, i.e:

$$\partial \mathcal{B}_n(x) \subseteq \mathcal{B}_{n+1}(x). \tag{6}$$

Indeed, let $y \in \partial \mathcal{B}_n(x)$ and $z \in \mathcal{B}_n(x)$ such that $y \sim z$. Then

$$\begin{aligned} d_H(x, y) &= -\ln F(x, y) \leq -\ln F(x, z) F(z, y) = d(x, z) - \ln F(z, y) \\ &\leq Kn - \ln \varepsilon_0 = K(n+1). \end{aligned}$$

This implies that $y \in \mathcal{B}_{n+1}(x)$.

Proposition 2.9. *The balls $\mathcal{B}_n(x)$ are finite.*

Proof. It suffices to show that $\lim_{n \to \infty} F(x, y_n) = 0$, for all sequences $\{y_n\}_{n \in \mathbb{N}}$ with infinitely many distinct elements. The functions $\{e_x\}_{x \in X}$ with $e_x(z) = \delta_x(z) m(x)^{-1/2}$ form an orthonormal basis of $\ell^2(X, m)$. By Theorem 2.8 the Green function defines a bounded linear operator. From Bessel's inequality it follows that for any sequence $y_n \in X$, which has infinitely many distinct elements, $\langle G e_{y_n}, e_x \rangle \to 0$ as $n \to \infty$.
Because $\langle G e_{y_n}, e_x \rangle = G(x, y_n) \sqrt{\frac{m(x)}{m(y_n)}}$ and by strong reversibility $C^{-1} \leq \sqrt{\frac{m(x)}{m(y_n)}} \leq C$, this is equivalent to $G(x, y_n) \to 0$. Since $F(x, y_n) = \frac{G(x, y_n)}{G(y_n, y_n)} \leq G(x, y_n)$ the finiteness of $\mathcal{B}_n(x)$ follows. □

To simplify the notation we write $\mathcal{B}_n = \mathcal{B}_n(o)$ and $\partial \mathcal{B}_n = \partial \mathcal{B}_n(o)$. Denote by $V(n)$ the size of the ball of radius n:

$$V(n) = |\mathcal{B}_n|.$$

Proposition 2.10. *For all finite subsets $A \subset X$ the following estimates hold:*

$$\sum_{x\in A} F(x,y) \le \mathcal{J} \cdot c_\rho^{-1} \ln|A|, \tag{7}$$

$$\sum_{y\in A} F(x,y) \le \mathcal{J} \cdot c_\rho^{-1} \ln|A|, \tag{8}$$

with $c_\rho = -\ln \rho(P)$ and some constant $\mathcal{J} > 0$, which does not depend on A.

Proof. We have

$$F(x,y) = \frac{G(x,y)}{G(y,y)} \le G(x,y).$$

Therefore, using the estimate of Lemma 2.4:

$$\begin{aligned}\sum_{x\in A} F(x,y) &\le \sum_{x\in A}\sum_{n=0}^{\infty} p^{(n)}(x,y) = \sum_{n=0}^{\infty}\sum_{x\in A} p^{(n)}(x,y)\\ &\le \sum_{n\le c_\rho^{-1}\ln|A|}\ \sum_{x\in A} \frac{m(y)}{m(x)} p^{(n)}(y,x) + \sum_{n> c_\rho^{-1}\ln|A|}\ \sum_{x\in A} C\cdot\rho(P)^n.\end{aligned}$$

For $M = \sup\left\{\frac{m(x)}{m(y)} : x,y \in X\right\}$ we get:

$$\begin{aligned}\sum_{x\in A} F(x,y) &\le M\cdot c_\rho^{-1}\ln|A| + C|A| \int_{c_\rho^{-1}\ln|A|}^{\infty} e^{-c_\rho t} dt\\ &= M\cdot c_\rho^{-1}\ln|A| + C\cdot c_\rho^{-1}\\ &\le J\cdot c_\rho^{-1}\ln|A|.\end{aligned}$$

The second estimate can be derived in the same way, but without the need to apply reversibility. □

The next proposition gives some estimates of the size of the balls.

Proposition 2.11. *There exist constants $\mathcal{C}_l, \mathcal{C}_u > 0$, such that*

$$\mathcal{C}_l e^{Kn} \le V(n) \le \mathcal{C}_u n e^{Kn}, \text{ for all } n \ge n_0.$$

Proof. Since for $x \notin \mathcal{B}_{n-1}$ the distance $d_H(o,x) > K(n-1)$:

$$\sum_{x\in\partial\mathcal{B}_{n-1}} F(o,x) \le |\partial\mathcal{B}_{n-1}|\cdot e^{-K(n-1)}. \tag{9}$$

Every random walk that leaves the ball $\mathcal{B}_{n-1}$ has to hit at least one point of $\partial\mathcal{B}_{n-1}(n)$. Because the balls are finite the random walk leaves $\mathcal{B}_{n-1}$ with probability 1, hence

$$\sum_{x\in\partial\mathcal{B}_{n-1}} F(o,x) \ge 1.$$

Using (9) and (6), it implies that

$$e^{K(n-1)} \le |\partial\mathcal{B}_{n-1}| \le |\mathcal{B}_n| = V(n).$$

Choosing $\mathcal{C}_l = e^{-K}$, gives the lower bound of the proposition.
To prove the upper bound, consider the following inequality

$$|\mathcal{B}_n| \cdot e^{-Kn} \leq \sum_{x \in \mathcal{B}_n} F(o,x) \leq \mathcal{J} \cdot c_\rho^{-1} \cdot \ln|\mathcal{B}_n|, \tag{10}$$

which follows from the definition of the balls $\mathcal{B}_n$ and Proposition 2.10. With the finiteness of the balls this implies that $|\mathcal{B}_n| = O(e^{K'n})$. Further, (10) gives for some constants $\tilde{C}$ and $\mathcal{C}_u$:

$$\begin{aligned}
|\mathcal{B}_n| &\leq \mathcal{J} \cdot c_\rho^{-1} e^{Kn} \ln|\mathcal{B}_n| \\
&\leq \mathcal{J} \cdot c_\rho^{-1} e^{Kn} \ln\left(\tilde{C} e^{K'n}\right) \\
&\leq \mathcal{J} \cdot c_\rho^{-1} e^{Kn} \left(K'n + \ln \tilde{C}\right) \\
&\leq \mathcal{C}_u n e^{Kn}.
\end{aligned}$$

□

3 Internal diffusion limited aggregation

We can now formulate our main result, which connects the shape of the internal DLA clusters to the balls with respect to the hitting distance.

Theorem 3.1. *Let the sequence of random subsets $\{A(n)\}_n \in \mathbb{N}$ be the internal DLA process on a strongly reversible, uniformly irreducible Markov chain with spectral radius strictly smaller than 1. Then for any $\varepsilon > 0$ and all constants $\mathcal{C}_I \geq \frac{1+\varepsilon}{K}$ and $\mathcal{C}_O > \sqrt{3}$,*

$$\mathbb{P}\left[\exists n_\varepsilon \text{ s.t. } \forall n \geq n_\varepsilon : \mathcal{B}_{n-\mathcal{C}_I \ln n} \subseteq A(V(n)) \subseteq \mathcal{B}_{n+\mathcal{C}_O\sqrt{n}}\right] = 1.$$

The proof of Theorem 3.1 is the same as the proof the equivalent statement for groups with exponential growth [6, Theorem 3.1]. We just need to replace [6, Proposition 2.3] with Proposition 2.10 and Proposition 2.11, which give an equivalent statement in our setting.
For the estimate of the outer error of Theorem 3.1, the upper bound in Proposition 2.11 is tighter than the analog statement in [6, Proposition 2.3] because of non-amenability (see also [6, Remark 2.4]), which in turn leads to a smaller constant $\mathcal{C}_O$.

Acknowlegments I would like to thank Sara Brofferio and Sébastien Blachère for helpful conversations.

References

[1] P. Diaconis and W. Fulton, A growth model, a game, an algebra, Lagrange inversion, and characteristic classes, *Rend. Sem. Mat. Univ. Pol. Torino* **49** (1991) 95–119. MR1218674

[2] G. F. Lawler, M. Bramson, and D. Griffeath, Internal diffusion limited aggregation, *Ann. Probab.* **20** (1992) 2117–2140. MR1188055

[3] G. F. Lawler, Subdiffusive fluctuations for internal diffusion limited aggregation, *Ann. Probab.* **23** (1995) 71–86. MR1330761

[4] S. Blachère, *Agrégation limitée par diffusion interne et temps de coupure sur les groupes discrets à croissance polynomiale.* PhD thesis, L'Université Paul Sabatier, 2000.

[5] S. Blachère, Internal diffusion limited aggregation on discrete groups having polynomial growth, in *Random Walks and Geometry*, Proceedings (Erwin Schroedinger Insitute, Vienna 2001). de Gruyter, Berlin, 2004.

[6] S. Blachère and S. Brofferio, Internal diffusion limited aggregation on discrete groups having exponential growth, *Probab. Th. Rel. Fields* **137 (3-4)** (2007). MR2278460

[7] S. Blachère, P. Haïssinsky, and P. Mathieu, Asymptotic entropy and Green speed for random walks on countable groups, *Ann. Probab.* **36** (2008) n.3 1134–1152.

[8] W. Woess, *Random walks on infinite graphs and groups.* Cambridge Univ. Press, Cambridge, 2000. MR1743100

[9] J. Dodziuk, Difference equations, isoperimetric inequality, and transience of certain random walks, *Trans. Amer. Math. Soc.* **284** (1984) 787–794. MR0743744

[10] J. Dodziuk and W. S. Kendall, Combinatorial Laplacians and isoperimetric inequality, *From Local Times to Global Geometry, Control and Physics (ed. K. D. Elworthy), Pitman Res. Notes Math. Ser.* **150** (1986) 68–74. MR0894523