Elect. Comm. in Probab. **14** (2009), 317–326

ELECTRONIC COMMUNICATIONS in PROBABILITY

ON THE RE-ROOTING INVARIANCE PROPERTY OF LÉVY TREES

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Submitted February 22, 2009, *accepted in final form* July 4, 2009

AMS 2000 Subject classification: 60J80, 60G51
Keywords: continuous tree, stable tree, re-rooting, Lévy process

Abstract

We prove a strong form of the invariance under re-rooting of the distribution of the continuous random trees called Lévy trees. This extends previous results due to several authors.

1 Introduction

Continuous random trees have been studied extensively in the last fifteen years and have found applications in various areas of probability theory and combinatorics. The prototype of these random trees is the CRT (Continuum Random Tree) which was introduced and studied by Aldous in a series of papers [2, 3, 4] in the early nineties. Amongst other things, Aldous proved that the CRT is the scaling limit of a wide class of discrete random trees (in particular it is the scaling limit of uniformly distributed planar trees with n edges, or of Cayley trees on n vertices, when n tends to infinity) and that it can be coded by a normalized Brownian excursion. The latter fact makes it possible to derive a number of explicit distributions for the CRT.

The class of Lévy trees, generalizing the CRT, was introduced and discussed in the monograph [9], and further studied in [10]. Lévy trees are continuous analogues of Galton-Watson branching trees and, in some sense, they are the only continuous random trees that can arise as scaling limits of sequences of Galton-Watson trees conditioned to be large: See Chapter 2 in [9] for a thorough discussion of these scaling limits. Recently, Weill [22] also proved that Lévy trees are precisely those continuous random trees that enjoy the regeneration property, saying informally that the subtrees originating from a fixed level in the tree are independent and distributed as the full tree. Among Lévy trees, the special class of stable trees, which arise as scaling limits of sequences of conditioned Galton-Watson trees with the same offspring distribution [8], is of particular importance.

In the formalism of [10], or in [11], continuous random trees are viewed as random variables taking values in the space of all compact rooted real trees, which is itself a closed subset of the space of all (isometry classes of) pointed compact metric spaces, which is equipped with the

Gromov-Hausdorff distance [12]. Since the latter space is Polish, this provides a convenient setting for the study of distributional properties of continuous random trees.

In a way analogous to the coding of discrete planar trees by Dyck paths, one can code a compact real tree by a contour function (see Theorem 2.1 in [10]). Roughly speaking, with any continuous function $g : [0,\sigma] \to \mathbb{R}_+$ such that $g(0) = g(\sigma) = 0$ one associates a real tree $\mathcal{T}_g$ called the tree coded by g, which is formally obtained as the quotient of the interval $[0,\sigma]$ for an equivalence relation $\sim_g$ defined in terms of g – See Section 2 below. By definition the root of $\mathcal{T}_g$ is the equivalence class of 0, and the tree $\mathcal{T}_g$ is equipped with a "uniform" measure $\mathbf{m}$, which is just the image of Lebesgue measure under the canonical projection $[0,\sigma] \to [0,\sigma]/\sim_g = \mathcal{T}_g$. This construction applies to the CRT, which is the tree coded by a normalized Brownian excursion (thus $\sigma = 1$ in that case), and more generally to Lévy trees, for which the coding function is the so-called height process, which is a functional of a Lévy process without negative jumps: See [17, 9, 10] and Section 2 below.

The main purpose of the present note is to discuss a remarkable invariance property of Lévy trees under re-rooting. In the case of the CRT, Aldous [3] already observed that the law of the CRT is invariant under uniform re-rooting: If we pick a vertex of the CRT according to the uniform measure $\mathbf{m}$, the CRT re-rooted at this vertex has the same distribution as the CRT. As was noted by Aldous, this property corresponds to the invariance of the law of the Brownian excursion under a simple path transformation. An analogue of this property for Lévy trees was derived in Proposition 4.8 of [10]. It turns out that much more is true. Marckert and Mokkadem [20] (see also Theorem 2.3 in [18]) proved that, for any fixed $s \in [0,1]$, the CRT re-rooted at the vertex s (or rather the equivalence class of s) has the same distribution as the CRT. The main result of the present work (Theorem 2.2) shows that this strong form of the re-rooting invariance property remains valid for Lévy trees. In fact, we obtain the corresponding result for the height process coding the Lévy tree, which gives a more precise statement than just the invariance of the distribution of the tree (compare the first and the second assertion of Theorem 2.2).

In the case of the CRT, the invariance under re-rooting can be deduced from a similar property for approximating discrete random trees. This approach was used both by Aldous and by Marckert and Mokkadem. In the case of Lévy trees, this method seems much harder to implement and we prefer to argue directly on the coding function, using fine properties of the underlying Lévy process. In a sense, our arguments are in the same spirit as the proof of Theorem 2.3 in [18], but the latter paper used very specific properties of Brownian motion, which no longer hold in our general setting.

Part of the motivation for the present work came from the recent paper by Haas, Pitman and Winkel [15]. This paper proves that, among the continuous fragmentation trees, stable trees are the only ones whose distribution is invariant under uniform re-rooting. Note that continuous random trees modelling the genealogy of self-similar fragmentations were investigated by Haas and Miermont [13]. The present work gives more insight in the probabilistic properties of the Lévy trees, which are still the subject of active research (see in particular the recent paper [1]). The re-rooting invariance of the CRT turned out to play a very important role in the study of certain conditionings considered in [18], which are closely related to the continuous limit of random planar maps [20, 16]. We hope that our results about Lévy trees will have similar applications to related models.

The paper is organized as follows. Section 2 recalls the basic notation and the construction of the Lévy tree from the height process, and also shows how our main result can be deduced from the technical Proposition 2.1. At the end of Section 2, we briefly discuss some applications of the invariance under re-rooting. Finally, Section 3 gives the proof of Proposition 2.1.

2 Notation and statement of the main results

We consider the general framework of [9, 10]. Let $X = (X_t)_{t\geq 0}$ be a real Lévy process without negative jumps, which starts from 0 under the probability measure $\mathbf{P}$. Without loss of generality, we may and will assume that X is the canonical process on the Skorokhod space $\mathbb{D}(\mathbb{R}_+,\mathbb{R})$. The Laplace exponent of X is denoted by $\psi(\lambda)$:

$$\mathbf{E}[\exp(-\lambda X_t)] = \exp(t\psi(\lambda)), \quad t,\lambda \geq 0.$$

We assume that X has first moments and that $\mathbf{E}[X_1] \leq 0$. Equivalently, this means that we exclude the case where X drifts to $+\infty$. Then ψ can be written in the form

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} (e^{-\lambda r} - 1 + \lambda r)\,\pi(dr),$$

where $\alpha,\beta \geq 0$ and the Lévy measure π is a σ-finite measure on $(0,\infty)$ such that $\int_{(0,\infty)}(r \wedge r^2)\,\pi(dr) < \infty$. Let

$$I_t = \inf_{0\leq s\leq t} X_s, \quad t \geq 0,$$

denote the minimum process of X. Then the process $X - I$ is a strong Markov process in $[0,\infty)$ and the point 0 is regular and recurrent for this Markov process (see Sections VI.1 and VII.1 in [6]).

We furthermore assume that

$$\int_1^\infty \frac{du}{\psi(u)} < \infty. \tag{1}$$

This implies that at least one of the following two conditions holds:

$$\beta > 0 \quad \text{or} \quad \int_{(0,1)} r\,\pi(dr) = \infty. \tag{2}$$

In particular ([6], Corollary VII.5), the paths of X are of infinite variation almost surely, and the point 0 is regular for $(0,\infty)$ with respect to X, and thus also with respect to $X - I$. By a simple duality argument, this implies that $\int_0^\infty \mathbf{1}_{\{X_s = I_s\}} ds = 0$.

By Theorem VII.1 in [6], the continuous increasing process $(-I_t)_{t\geq 0}$ is a local time at 0 for $X - I$. We denote by N the associated Itô excursion measure, so that N is a σ-finite measure on the space $\mathbb{D}(\mathbb{R}_+,\mathbb{R})$. We also denote by $\sigma := \inf\{t > 0 : X_t = 0\}$ the duration of the excursion under N.

Let us now introduce the height process. For every $0 \leq s \leq t$, set

$$I_t^s = \inf_{s\leq r\leq t} X_r.$$

One can then prove (see Chapter 1 of [9]) that, for every $t \geq 0$, the limit

$$H_t := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{I_t^s < X_s < I_t^s + \varepsilon\}}\, ds \tag{3}$$

exists in $\mathbf{P}$-probability and in N-measure. The reason why it is useful to consider the process H also under the excursion measure N is the fact that $\mathbf{P}$ a.s. H_t only depends on the excursion of $X - I$ that straddles time t: In the integral appearing in the right-hand side of (3), we can restrict our attention to values of s belonging to the excursion interval of $X-I$ away from 0 that straddles t. See the discussion in Section 3.2 of [10].

Thanks to our assumption (1), the process $(H_t)_{t\geq 0}$ has a continuous modification under $\mathbf{P}$ and under N: This means that we can redefine the process $(H_t)_{t\geq 0}$ on the Skorokhod space in such a way that all its sample paths are continuous and the limit (3) still holds for every $t\geq 0$. Notice that $H_0=0$, $\mathbf{P}$ a.s. and N a.e., and that $H_t=0$ for every $t\geq\sigma$, N a.e. The time-reversal invariance property of H states that the two processes $(H_t)_{t\geq 0}$ and $(H_{(\sigma-t)_+})_{t\geq 0}$ have the same distribution under N (see Corollary 3.1.6 in [9]).

The ψ-*Lévy tree* is by definition the rooted real tree $\mathcal{T}_H$ coded by the continuous function $(H_t)_{t\geq 0}$ under the measure N, in the sense of Theorem 2.1 in [10]. This means that $\mathcal{T}_H=[0,\sigma]\,/\sim_H$, where the random equivalence relation $\sim_H$ is defined on $[0,\sigma]$ by

$$s\sim_H t \quad \text{iff} \quad H_s=H_t=\min_{s\wedge t\leq r\leq s\vee t} H_r\ .$$

The canonical projection from $[0,\sigma]$ onto $\mathcal{T}_H$ is denoted by p_H, and the distance on $\mathcal{T}_H$ is given by

$$d_H(p_H(s),p_H(t))=d_H(s,t)=H_s+H_t-2\min_{s\wedge t\leq r\leq s\vee t} H_r$$

(notice that this only depends on $p_H(s)$ and $p_H(t)$, and not on the particular choice of the representatives s and t). By definition the tree $\mathcal{T}_H$ is rooted at $\rho=p_H(0)=p_H(\sigma)$.

In order to discuss re-rooting, let us introduce the following notation. Fix $s\in[0,\sigma]$ and set

$$H^{[s]}_t=\begin{cases} d_H(s,s+t) & \text{if } 0\leq t<\sigma-s\\ d_H(s,s+t-\sigma) & \text{if } \sigma-s\leq t\leq\sigma\end{cases}$$

and $H^{[s]}_t=0$ if $t>\sigma$. By Lemma 2.2 in [10], the tree $\mathcal{T}_{H^{[s]}}$ is then canonically identified with the tree $\mathcal{T}_H$ re-rooted at the vertex $p_H(s)$.

We can now state the key technical proposition that will lead to our main theorem. We denote by $C_0(\mathbb{R}_+,\mathbb{R}_+)$ the space of all continuous functions with compact support from $\mathbb{R}_+$ into $\mathbb{R}_+$, so that $(H_t)_{t\geq 0}$, or $(H^{[s]}_t)_{t\geq 0}$, can be viewed under N as a random element of $C_0(\mathbb{R}_+,\mathbb{R}_+)$.

Proposition 2.1. *For every nonnegative measurable function F on $\mathbb{R}_+\times C_0(\mathbb{R}_+,\mathbb{R}_+)$, and every nonnegative measurable function g from $\mathbb{R}_+$ into $\mathbb{R}_+$,*

$$N\Big(g(\sigma)\int_0^\sigma ds\,F(s,H^{[s]})\Big)=N\Big(g(\sigma)\int_0^\sigma ds\,F(s,H)\Big).$$

The proof of Proposition 2.1 is given in the next section, but we will immediately deduce our main theorem from this proposition. Denote by $\kappa(da)$ the "law" of σ under N. By a standard disintegration theorem, we can find a measurable collection $(N^{(a)})_{a\in(0,\infty)}$ of probability measures on $\mathbb{D}(\mathbb{R}_+,\mathbb{R})$ such that:

(a) $N=\int_0^\infty \kappa(da)\,N^{(a)}$;

(b) for every $a\in(0,\infty)$, $N^{(a)}$ is supported on $\{\sigma=a\}$.

In the stable case where $\psi(u)=c\,u^\gamma$ for some $c>0$ and $\gamma\in(1,2]$, the measures $N^{(a)}$ can be chosen in such a way that $N^{(a)}$ is the law of $(a^{1/\gamma}X_{t/a})_{t\geq 0}$ under $N^{(1)}$. A scaling argument then shows that the continuous modification of H can be chosen so that (3) holds in $N^{(a)}$-probability for every $a>0$. So we may, and will, assume that the latter properties hold in the stable case.

By definition, the *stable tree* is the tree $\mathcal{T}_H$ under the probability measure $N^{(1)}$, when $\psi(u)=u^\gamma$ for some $\gamma\in(1,2]$. The CRT is the case $\gamma=2$.

Theorem 2.2. *The following properties hold for κ-almost every $a > 0$. For every $s_0 \in [0, a)$:*

(i) *The processes $(H_t^{[s_0]})_{t\geq 0}$ and $(H_t)_{t\geq 0}$ have the same distribution under $N^{(a)}$.*

(ii) *The law under $N^{(a)}$ of the tree $\mathcal{T}_H$ re-rooted at $p_H(s_0)$ coincides with the law of $\mathcal{T}_H$.*

In the stable case, the preceding properties hold for every $a > 0$.

Remark. Theorem 2.2 generalizes property (20) in [3] and Proposition 4.9 in [20], which are both concerned with Aldous' CRT, as well as Theorem 11(i) in [15], which deals with uniform re-rooting of the stable tree. Theorem 2.2 also strengthens Proposition 4.8 of [10], which considers only uniform re-rooting and property (ii).

Proof: Using properties (a) and (b), we immediately get that, for every fixed function F satisfying the properties in Proposition 2.1, the equality

$$N^{(a)}\Big(\int_0^a ds\, F(s, H^{[s]})\Big) = N^{(a)}\Big(\int_0^a ds\, F(s, H)\Big)$$

holds $\kappa(da)$ a.e. A simple separability argument implies that the preceding identity holds simultaneously for all choices of F, except for values of a belonging to a set of zero κ-measure. We apply this with $F(s,H) = \varepsilon^{-1}\mathbf{1}_{[s_0,s_0+\varepsilon]}(s)G(H)$ where G is continuous and bounded on $C_0(\mathbb{R}_+,\mathbb{R}_+)$. Letting ε go to 0, and noting that $H^{[s]}$ depends continuously on s, it follows that for κ-almost every $a > 0$, we have for every such function G, for every $s_0 \in [0, a)$,

$$N^{(a)}(G(H^{[s_0]})) = N^{(a)}(G(H)).$$

Property (i) readily follows. Property (ii) is then a consequence of the fact that $\mathcal{T}_{H^{[s_0]}}$ is isometric to the tree $\mathcal{T}_H$ re-rooted at $p_H(s_0)$ ([10], Lemma 2.2).
In the stable case, a scaling argument shows that the properties of the theorem hold for every $a > 0$ as soon as they hold for one value of $a > 0$. □

Let us briefly comment on applications of the invariance under re-rooting. Aldous [3, 4] observed that a convenient way to describe the distribution of a continuous random tree is via its "finite-dimensional marginals". Let $\mathcal{T}$ be a rooted continuous random tree (for instance the stable tree) and suppose that we are given a probability distribution $\mathbf{m}$ on $\mathcal{T}$, which is called the mass measure (in the case of the stable tree, $\mathbf{m}$ is the image of Lebesgue measure over $[0,1]$ under the projection p_H). Let $p \geq 1$ be an integer and suppose that $V_1, \ldots, V_p$ are p vertices chosen independently at random on $\mathcal{T}$ according to the mass measure $\mathbf{m}$. The subtree $\mathcal{T}(V_1, \ldots, V_p)$ spanned by $V_1, \ldots, V_p$ is the union of the line segments beween the root and the vertices $V_1, \ldots, V_p$. It is a finite real tree, meaning that it consists of the union of a finite number of segments, with $p + 1$ labeled vertices: the root is labeled 0 and each vertex V_k is labeled k. The invariance under uniform re-rooting implies that the distribution of this labeled tree is invariant under every permutation of the labels $0, 1, \ldots, p$. For instance, in the case when $\mathcal{T}$ is the stable tree and $p = 2$, we get that if U and U' are two independent random variables uniformly distributed over $[0,1]$, and if $m_H(U,U') = \min_{U\wedge U'\leq s\leq U\vee U'} H_s$, the law under $N^{(1)}$ of the triplet

$$\Big(H_U - m_H(U,U')\,,\, H_{U'} - m_H(U,U')\,,\, m_H(U,U')\Big)$$

is exchangeable. Even in the case of the CRT, where H is the normalized Brownian excursion, it is not so easy to give a direct derivation of this property. The exchangeability of the preceding

triplet is of course a consequence of Theorem 3.3.3 in [9], see also Proposition 2.1 (ii) in [14]. Alternatively, such results follow from discrete tree growth, see e.g. [21] for a well-known growth procedure for the CRT, and [19] or [7] for extensions to the stable case. Exchangeability can also be derived from elementary urn schemes, once convergence to the stable tree has been shown.
Other applications of the invariance under re-rooting are concerned with tree-indexed processes (see in particular [18]). In order to provide a simple example, consider again the stable tree $\mathcal{T}_H$, under the probability measure $N^{(1)}$ as above, and its mass measure $\mathbf{m}$. Let $(Z_a, a \in \mathcal{T}_H)$ be Brownian motion indexed by $\mathcal{T}_H$, which may defined as follows. Conditionally given $\mathcal{T}_H$, this is the centered Gaussian process such that $Z_\rho = 0$ and $E[(Z_a - Z_b)^2] = d_H(a,b)$ for every $a, b \in \mathcal{T}_H$. The occupation measure $\mathscr{I}$ of Z is the random measure on $\mathbb{R}$ defined by:

$$\langle \mathscr{I}, g \rangle = \int \mathbf{m}(da)\, g(Z_a).$$

In the particular case of the CRT, the measure $\mathscr{I}$ is known as (one-dimensional) ISE – See Aldous [5]. As a consequence of the invariance under uniform re-rooting, one easily checks that the quantity $\mathscr{I}(]0,\infty[)$, which represents the total mass to the right of the origin, is uniformly distributed over $[0,1]$. This had already been observed in [5] in the case of the CRT.

3 Proof of the main proposition

In this section we prove Proposition 2.1. We start by recalling a lemma from [10] that plays a key role in our proof. We first need to introduce some notation. It will be convenient to use the notion of a finite path. A (one-dimensional) finite path is just a continuous mapping $\mathrm{w} : [0,\zeta] \longrightarrow \mathbb{R}$, where the number $\zeta = \zeta_{(\mathrm{w})} \geq 0$ is called the lifetime of w. The space $\mathscr{W}$ of all finite paths is equipped with the distance d defined by $d(\mathrm{w},\mathrm{w}') = \|\mathrm{w}-\mathrm{w}'\| + |\zeta_{(\mathrm{w})} - \zeta_{(\mathrm{w}')}|$, where $\|\mathrm{w}-\mathrm{w}'\| = \sup_{t\geq 0} |\mathrm{w}(t\wedge\zeta_{(\mathrm{w})}) - \mathrm{w}'(t\wedge\zeta_{(\mathrm{w}')})|$.
Let M_f denote the space of all finite measures on $\mathbb{R}_+$. For every $\mu \in M_f$, let $\mathrm{supp}(\mu)$ denote the topological support of μ and set $S(\mu) = \sup(\mathrm{supp}(\mu)) \in [0,+\infty]$. We let M_f^* be the subset of M_f consisting of all measures μ such that $S(\mu) < \infty$ and $\mathrm{supp}(\mu) = [0, S(\mu)]$. By convention the measure $\mu = 0$ belongs to M_f^* and $S(0) = 0$.
Let $\psi^*(u) = \psi(u) - \alpha u$, and let (U^1, U^2) be a two-dimensional subordinator with Laplace functional

$$E[\exp(-\lambda U_t^1 - \lambda' U_t^2)] = \exp\Big(-t\,\frac{\psi^*(\lambda) - \psi^*(\lambda')}{\lambda - \lambda'}\Big).$$

Note that U^1 and U^2 have the same distribution and are indeed subordinators with drift β and Lévy measure $\pi([x,\infty))dx$. For every $a > 0$, we let $\mathbb{M}_a$ be the probability measure on $(M_f^*)^2$ which is the distribution of $(\mathbf{1}_{[0,a]}(t)\,dU_t^1, \mathbf{1}_{[0,a]}(t)\,dU_t^2)$. The fact that this defines a distribution on $(M_f^*)^2$ follows from (2). Moreover $\mathbb{M}_a(d\mu\, d\nu)$ a.s., we have $S(\mu) = S(\nu) = a$.
Let $\mu \in M_f^*$ and denote by $|\mu| = \mu([0,\infty))$ the total mass of μ. For every $r \in [0,|\mu|]$, we denote by $k_r\mu$ the unique element of M_f^* such that

$$k_r\mu([0,x]) = \mu([0,x]) \wedge (|\mu| - r)$$

for every $x \geq 0$. We then set, for every $0 \leq t \leq T_{|\mu|} := \inf\{s \geq 0 : X_s = -|\mu|\}$,

$$H_t^\mu = S(k_{-I_t}\mu) + H_t.$$

We view $(H^\mu_t)_{0\le t\le T_{|\mu|}}$ as a random element of $\mathscr{W}$, with lifetime $\zeta_{(H^\mu)} = T_{|\mu|}$, and we let $Q_\mu(d\mathrm{w})$ be the probability measure on $\mathscr{W}$ which is the distribution of H^μ under $\mathbf{P}$. Notice that $\mathrm{w}(0) = S(\mu)$ and $\mathrm{w}(\zeta_{(\mathrm{w})}) = 0$, $Q_\mu(d\mathrm{w})$ a.s.

Finally, under the excursion measure N, we set for every $s \in [0,\sigma]$,

$$\begin{aligned} H^{+,s}_t &= H_{s+t}\,, \qquad 0\le t\le \sigma - s\,,\\ H^{-,s}_t &= H_{s-t}\,, \qquad 0\le t\le s\,, \end{aligned}$$

and we view $H^{+,s}$ and $H^{-,s}$ as random elements of $\mathscr{W}$ with respective lifetimes $\sigma - s$ and s.

The following result is Lemma 3.4 in [10].

Lemma 3.1. *For any nonnegative measurable function Φ on $\mathscr{W}^2$,*

$$N\Big(\int_0^\sigma ds\,\Phi(H^{+,s},H^{-,s})\Big) = \int_0^\infty da\,e^{-\alpha a}\int \mathbb{M}_a(d\mu d\nu)\iint Q_\mu(d\mathrm{w})Q_\nu(d\mathrm{w}')\Phi(\mathrm{w},\mathrm{w}').$$

Before stating the next lemma we introduce two simple transformations of finite paths. Let $\mathrm{w}\in\mathscr{W}$. We define two other finite paths $\overline{\mathrm{w}}$ and $\widetilde{\mathrm{w}}$ both having the same lifetime as w, by setting

$$\begin{aligned} \overline{\mathrm{w}}(t) &= \mathrm{w}(\zeta_{(\mathrm{w})} - t)\,, && 0\le t\le \zeta_{(\mathrm{w})}\,,\\ \widetilde{\mathrm{w}}(t) &= \mathrm{w}(0) + \mathrm{w}(t) - 2\min_{0\le s\le t}\mathrm{w}(s)\,, && 0\le t\le \zeta_{(\mathrm{w})}\,. \end{aligned}$$

If $\mu\in M^*_f$, we also denote by $\overline{\mu}$ the "time-reversed" measure defined as the image of μ under the mapping $t\to S(\mu)-t$.

Lemma 3.2. *Let $\mu\in M^*_f$. The law of $\widetilde{\mathrm{w}}$ under $Q_\mu(d\mathrm{w})$ coincides with the law of $\overline{\mathrm{w}}$ under $Q_{\overline{\mu}}(d\mathrm{w})$.*

Proof: To simplify notation, we write $\tau = T_{|\mu|}$ in this proof. We argue under the probability measure $\mathbf{P}$. Since the function $t\to S(k_{-I_t}\mu)$ is nonincreasing and can only decrease when $X_t = I_t$, which forces $H_t = 0$, it is easy to verify that, a.s. for every $t\in[0,\tau]$,

$$\min_{0\le r\le t} H^\mu_r = S(k_{-I_t}\mu)$$

(consider the last time r before t when $X_r = I_r$). It follows that

$$\widetilde{H}^\mu_t = H^\mu_0 + H^\mu_t - 2\min_{0\le r\le t} H^\mu_r = S(\mu) + H_t - S(k_{-I_t}\mu), \tag{4}$$

for every $t\in[0,\tau]$, a.s.

Now observe that the random function $(H_t, t\in[0,\tau])$ is obtained by concatenating the values of H over all excursion intervals of $X-I$ away from 0 between times 0 and τ. Since $-I$ is a local time at 0 for the process $X-I$, and $\tau = \inf\{t\ge 0 : -I_t = |\mu|\}$, the excursions of $X-I$ between times 0 and τ form a Poisson point process with intensity $|\mu|N$. Furthermore the values of H over each excursion interval only depend on the corresponding excursion of $X-I$, and we know that the law of H under N is invariant under time-reversal. By putting the previous facts together, and using standard arguments of excursion theory, we obtain that the two pairs of processes

$$(H_{\tau-t}, |\mu| + I_{\tau-t})_{0\le t\le\tau} \quad\text{and}\quad (H_t, -I_t)_{0\le t\le\tau}$$

have the same distribution under $\mathbf{P}$.

From this identity in distribution, we deduce that, again under the probability measure $\mathbf{P}$,

$$(S(\mu)+H_t-S(k_{-I_t}\mu))_{0\le t\le\tau} \overset{(d)}{=} (S(\mu)+H_{\tau-t}-S(k_{|\mu|+I_{\tau-t}}\mu))_{0\le t\le\tau}. \tag{5}$$

However, the elementary fact $S(\mu)-S(k_{|\mu|-r}\mu)=S(k_r\overline{\mu})$, which is valid for every $r\in[0,|\mu|]$, implies that, for every $t\in[0,\tau]$,

$$S(\mu)+H_t-S(k_{|\mu|+I_t}\mu)=H_t+S(k_{-I_t}\overline{\mu})=H_t^{\overline{\mu}}. \tag{6}$$

The lemma now follows from (4), (5), (6) and the definition of the measures Q_μ. □

We now turn to the proof of Proposition 2.1. We first observe that, with the notation introduced before Lemma 3.2, we have N a.e. for every $s\in(0,\sigma)$,

$$\begin{aligned}\widetilde{H}_t^{+,s}&=H_t^{[s]}\,, &&0\le t\le\sigma-s\,,\\ \widetilde{H}_t^{-,s}&=H_{\sigma-t}^{[s]}\,, &&0\le t\le s\,.\end{aligned}$$

Let Ψ be a nonnegative measurable function on $\mathscr{W}^2$. Using Lemma 3.1 in the first equality and Lemma 3.2 in the second one, we have

$$\begin{aligned}
&N\Big(\int_0^\sigma ds\,\Psi\Big((H_t^{[s]})_{0\le t\le\sigma-s},(H_{\sigma-t}^{[s]})_{0\le t\le s}\Big)\Big)\\
&\quad=\int_0^\infty da\,e^{-\alpha a}\int\mathbb{M}_a(d\mu d\nu)\iint Q_\mu(d\mathrm{w})Q_\nu(d\mathrm{w}')\,\Psi(\widetilde{\mathrm{w}},\widetilde{\mathrm{w}}')\\
&\quad=\int_0^\infty da\,e^{-\alpha a}\int\mathbb{M}_a(d\mu d\nu)\iint Q_{\overline{\mu}}(d\mathrm{w})Q_{\overline{\nu}}(d\mathrm{w}')\,\Psi(\overline{\mathrm{w}},\overline{\mathrm{w}}')\\
&\quad=\int_0^\infty da\,e^{-\alpha a}\int\mathbb{M}_a(d\mu d\nu)\iint Q_\mu(d\mathrm{w})Q_\nu(d\mathrm{w}')\,\Psi(\overline{\mathrm{w}}',\overline{\mathrm{w}})\\
&\quad=N\Big(\int_0^\sigma ds\,\Psi(\overline{H}^{-,s},\overline{H}^{+,s})\Big)\\
&\quad=N\Big(\int_0^\sigma ds\,\Psi\Big((H_t)_{0\le t\le s},(H_{\sigma-t})_{0\le t\le\sigma-s}\Big)\Big)
\end{aligned}$$

In the third equality, we use the fact that the probability measure $\mathbb{M}_a(d\mu d\nu)$ is symmetric and invariant under the mapping $(\mu,\nu)\to(\overline{\mu},\overline{\nu})$. The fourth equality is Lemma 3.1 again, and the last one is an immediate consequence of the definitions.

Now note that the triplet $(\sigma,\sigma-s,H)$ can be written as a measurable functional Γ of the pair $((H_t)_{0\le t\le s},(H_{\sigma-t})_{0\le t\le\sigma-s})$, and that the triplet $(\sigma,s,H^{[s]})$ is then the same measurable function of the pair $((H_t^{[s]})_{0\le t\le\sigma-s},(H_{\sigma-t}^{[s]})_{0\le t\le s}))$. Therefore, the preceding calculation implies that, for F and g as in the statement of the proposition,

$$N\Big(g(\sigma)\int_0^\sigma ds\,F(s,H^{[s]})\Big)=N\Big(g(\sigma)\int_0^\sigma ds\,F(\sigma-s,H)\Big)=N\Big(g(\sigma)\int_0^\sigma ds\,F(s,H)\Big).$$

This completes the proof. □

Acknowledgement. We thank Bénédicte Haas for keeping us informed of her joint work with Jim Pitman and Matthias Winkel.

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