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MAXIMUM OF DYSON BROWNIAN MOTION AND NON-COLLIDING SYSTEMS WITH A BOUNDARY

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Abstract

We prove an equality-in-law relating the maximum of GUE Dyson's Brownian motion and the non-colliding systems with a wall. This generalizes the well known relation between the maximum of a Brownian motion and a reflected Brownian motion.

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1 Introduction and Results

Dyson's Brownian motion model of GUE (Gaussian unitary ensemble) is a stochastic process of positions of m particles, $X(t) = (X_1(t), \ldots, X_m(t))$ described by the stochastic differential equation,

$$dX_i = dB_i + \sum_{\substack{1\le j\le m\\ j\ne i}} \frac{dt}{X_i - X_j}, \quad 1 \le i \le m, \tag{1.1}$$

where $B_i, 1 \le i \le m$ are independent one dimensional Brownian motions[6]. The process satisfies $X_1(t) < X_2(t) < \cdots < X_m(t)$ for all $t > 0$. We remark that the process X can be started from the origin, i.e., one can take $X_i(0) = 0, 1 \le i \le m$. See [11].

One can introduce similar non-colliding system of m particles with a wall at the origin [8, 9, 17]. The dynamics of the positions of the m particles $X^{(C)} = (X_1^{(C)}, \ldots, X_m^{(C)})$ satisfying $0 < X_1(t) < X_2(t) < \cdots < X_m(t)$ for all $t > 0$ are described by the stochastic differential equation,

$$dX_i^{(C)} = dB_i + \frac{dt}{X_i^{(C)}} + \sum_{\substack{1\le j\le m\\ j\ne i}} \left(\frac{1}{X_i^{(C)} - X_j^{(C)}} + \frac{1}{X_i^{(C)} + X_j^{(C)}} \right) dt, \ 1 \le i \le m. \tag{1.2}$$

This process is referred to as Dyson's Brownian motion of type C. It can be interpreted as a system of m Brownian particles conditioned to never collide with each other or the wall.

One can also consider the case where the wall above is replaced by a reflecting wall[9]. The dynamics of the positions of the m particles $X^{(D)} = (X_1^{(D)}, \ldots, X_m^{(D)})$ satisfying $0 \le X_1(t) < X_2(t) < \cdots < X_m(t)$ for all $t > 0$, is described by the stochastic differential equation,

$$dX_i^{(D)} = dB_i + \frac{1}{2}\mathbf{1}_{(i=1)} dL(t) + \sum_{\substack{1\le j\le m\\ j\ne i}} \left(\frac{1}{X_i^{(D)} - X_j^{(D)}} + \frac{1}{X_i^{(D)} + X_j^{(D)}} \right) dt, \ 1 \le i \le m, \tag{1.3}$$

where $L(t)$ denotes the local time of $X_1^{(D)}$ at the origin. This process will be referred to as Dyson's Brownian motion of type D. Some authors consider a process defined by the s.d.e.s (1.3) without the local time term. In this case the first component of the process is not constrained to remain non-negative, and the process takes values in the Weyl chamber of type D, $\{|x_1| < x_2 < x_3 \ldots < x_m\}$. The process we consider with a reflecting wall is obtained from this by replacing the first component with its absolute value, with the local time term appearing as a consequence of Tanaka's formula.

It is known the processes $X^{(C)}$ and $X^{(D)}$ can be obtained using the Doob h-transform, see [8, 9]. Let $(P_t^{0,(C)}; t \ge 0)$, resp. $(P_t^{0,(D)}; t \ge 0)$, be the transition semigroup for m independent Brownian motions killed on exiting $\{0 < x_1 < x_2 \ldots < x_m\}$, resp. the transition semigroup for m independent Brownian motions reflected at the origin killed on exiting $\{0 \le x_1 < x_2 \ldots < x_m\}$. From the Karlin-McGregor formula, the corresponding densities can be written as

$$\det\{\phi_t(x_i - x_j') - \phi_t(x_i + x_j')\}_{1\le i,j\le m}, \tag{1.4}$$

resp.,

$$\det\{\phi_t(x_i - x_j') + \phi_t(x_i + x_j')\}_{1\le i,j\le m}, \tag{1.5}$$

where $\phi_t(z) = \frac{1}{\sqrt{2\pi t}} e^{-z^2/(2t)}$. Let

$$\begin{aligned} h^{(C)}(x) &= \prod_{i=1}^{m} x_i \prod_{1\le i<j\le m} (x_j^2 - x_i^2), \\ h^{(D)}(x) &= \prod_{1\le i<j\le m} (x_j^2 - x_i^2). \end{aligned} \tag{1.6}$$

For notational simplicity we suppress the index C, D for the semigroups and in h in the following. Then one can show that $h(x)$ is invariant for the P_t^0 semigroup and we may define a Markov semigroup by

$$P_t(x, dx') = h(x')P_t^0(x, dx')/h(x). \tag{1.7}$$

This is the semigroup of the Dyson non-colliding system of Brownian motions of type C and D. Similarly to the X process, the processes $X^{(C)}$ and $X^{(D)}$ can also be started from the origin (see [4] or use Lemma 4 in [9] and apply the same arguments as in [11]).

In GUE Dyson's Brownian motion of n particles, let us take the initial conditions to be $X_i(0) = 0, 1 \le i \le n$. The quantity we are interested in is the maximum of the position of the top particle for a finite duration of time, $\max_{0\le s\le t} X_n(s)$. In the sequel we write sup instead of max to conform with common usage in the literature. Let m be the integer such that $n = 2m$ when n is even and $n = 2m - 1$ when n is odd. Consider the non-colliding systems of $X^{(C)}$, resp. $X^{(D)}$, of m particles starting from the origin, $X_i^{(C)}(0) = 0, 1 \le i \le m$, resp. $X_i^{(D)}(0) = 0, 1 \le i \le m$.

Our main result of this note is

Theorem 1. *Let X and $X^{(C)}, X^{(D)}$ start from the origin. Then for each fixed $t \ge 0$, one has*

$$\sup_{0\le s\le t} X_n(s) \stackrel{d}{=} \begin{cases} X_m^{(C)}(t), & \text{for } n = 2m, \\ X_m^{(D)}(t), & \text{for } n = 2m-1. \end{cases} \tag{1.8}$$

To prove the theorem we introduce two more processes Z_j and Y_j. In the Z process, $Z_1 \le Z_2 \le \ldots \le Z_n$, Z_1 is a Brownian motion and Z_{j+1} is reflected by Z_j, $1 \le j \le n-1$. Here the reflection means the Skorokhod construction to push Z_{j+1} up from Z_j. More precisely,

$$\begin{aligned} Z_1(t) &= B_1(t), \\ Z_j(t) &= \sup_{0\le s\le t} (Z_{j-1}(s) + B_j(t) - B_j(s)), \quad 2 \le j \le n, \end{aligned} \tag{1.9}$$

where $B_i, 1 \le i \le n$ are independent Brownian motions, each starting from 0. The process is the same as the process $(X_1^1(t), X_2^2(t), \ldots, X_n^n(t); t \ge 0)$ studied in section 4 of [18]. The representation (1.9) was given earlier in [2]. In the Y process, $0 \le Y_1 \le Y_2 \le \ldots \le Y_n$, the interactions among Y_i's are the same as in the Z process, i.e., Y_{j+1} is reflected by Y_j, $1 \le j \le n-1$, but Y_1 is now a Brownian motion reflected at the origin (again by Skorokhod construction). Similarly to (1.9),

$$\begin{aligned} Y_1(t) &= B_1(t) - \inf_{0\le s\le t} B_1(s) = \sup_{0\le s\le t} (B_1(t) - B_1(s)), \\ Y_j(t) &= \sup_{0\le s\le t} (Y_{j-1}(s) + B_j(t) - B_j(s)), \quad 2 \le j \le n. \end{aligned} \tag{1.10}$$

From the results in [11, 5, 18], we know

$$(X_n(t); t \ge 0) \stackrel{d}{=} (Z_n(t); t \ge 0) \tag{1.11}$$

and hence

$$\sup_{0\le s\le t} X_n(s) \stackrel{d}{=} \sup_{0\le s\le t} Z_n(s). \tag{1.12}$$

In this note we show

Proposition 2. *The following equalities in law hold between processes:*

$$\begin{aligned}(Y_{2m}(t); t\ge 0) &\stackrel{d}{=} (X_m^{(C)}(t); t\ge 0),\\ (Y_{2m-1}(t); t\ge 0) &\stackrel{d}{=} (X_m^{(D)}(t); t\ge 0),\end{aligned} \tag{1.13}$$

$m \in \mathbb{N}$.

The proof of this proposition is given in Section 2. The idea behind it is that the processes $(Y_i)_{i\ge 1}$, $(X_j^{(C)})_{j\ge 1}$ and $(X_j^{(D)})_{j\ge 1}$ could be realized on a common probability space consisting of Brownian motions satisfying certain interlacing conditions with a boundary [18, 19]. Such a system is expected to appear as a scaling limit of the discrete processes considered in [3, 19]. In this enlarged process, the processes $Y_n(t)$ and $X_m^{(C)}(t)$ or $X_m^{(D)}(t)$ just represent two different ways of looking at the evolution of a specific particle and so the statement of Proposition 2 follows immediately. Justification of such an approach is however quite involved, and we prefer to give a simple independent proof. See also [5] for another representation of $X_m^{(C)}$ and $X_m^{(D)}$ in terms of independent Brownian motions.
Then to prove (1.8) it is enough to show

Proposition 3. *For each fixed t we have*

$$\sup_{0\le s\le t} Z_n(s) \stackrel{d}{=} Y_n(t). \tag{1.14}$$

This is shown in Section 3. For $n=1$ case, this is well known from the Skorokhod construction of reflected Brownian motion [12]. The $n>1$ case can also be understood graphically by reversing time direction and the order of particles. This relation could also be established as a limiting case of the last passage percolation. In fact the identities in our theorem was first anticipated from the consideration of a diffusion scaling limit of the totally asymmetric exclusion process with 2 speeds [1] (in particular the last part of sections 1,2 and section 7).
Before closing the section, we remark that similar maximization properties of Dyson's Brownian motion have been considered for other boundary conditions in [15, 10, 7].

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2 Proof of Proposition 2

In this section we prove the relation between $X^{(C)}$ or $X^{(D)}$ and Y, (1.13). The following lemma is a generalization of the Rogers-Pitman criterion [13] for a function of a Markov process to be Markovian. Note that it gives us a method to deduce an equality in law between two processes that need not themselves be Markov- as indeed is the case in Propostion 2

Lemma 4. *Suppose that $\{X(t) : t \geq 0\}$ is a Markov process with state space E, evolving according to a transition semigroup $(P_t; t \geq 0)$ and with initial distribution μ. Suppose that $\{Y(t) : t \geq 0\}$ is a Markov process with state space F, evolving according to a transition semigroup $(Q_t; t \geq 0)$ and with initial distribution ν. Suppose further that L is a Markov transition kernel from E to F, such that $\mu L = \nu$ and the intertwining $P_t L = L Q_t$ holds. Now let $f : E \to G$ and $g : F \to G$ be maps into a third state space G, and suppose that*

$$L(x, \cdot) \text{ is carried by } \{y \in F : g(y) = f(x)\} \text{ for each } x \in E.$$

Then we have

$$\{f(X(t)) : t \geq 0\} \stackrel{d}{=} \{g(Y(t)) : t \geq 0\},$$

in the sense of finite dimensional distributions.

Proof of Lemma 4. For any bounded function α on G let $\Gamma_1\alpha$ be the function $\alpha \circ f$ defined on E and let $\Gamma_2\alpha$ be the function $\alpha \circ g$ defined on F. Then it follows from the condition that $L(x, \cdot)$ is carried by $\{y \in F : g(y) = f(x)\}$ that whenever h is a bounded function defined on F then

$$L(\Gamma_2\alpha \times h) = \Gamma_1\alpha \times Lh, \tag{2.1}$$

which is shorthand for $\int L(x, dy)\Gamma_2\alpha(y)h(y) = \Gamma_1\alpha \times Lh$. For any bounded test functions $\alpha_0, \alpha_1, \cdots, \alpha_n$ defined on G, and times $0 < t_1 < \cdots < t_n$, we have, using the previous equation and the intertwining relation repeatedly,

$$\begin{aligned}
&\mathbb{E}[\alpha_0(g(Y(0)))\alpha_1(g(Y(t_1)))\ldots\alpha_n(g(Y(t_n)))] \\
&= \nu(\Gamma_2\alpha_0 \times Q_{t_1}(\Gamma_2\alpha_1 \times Q_{t_2-t_1}(\cdots(\Gamma_2\alpha_{n-1} \times Q_{t_n-t_{n-1}}\Gamma_2\alpha_n)\cdots))) \\
&= \mu L(\Gamma_2\alpha_0 \times Q_{t_1}(\Gamma_2\alpha_1 \times Q_{t_2-t_1}(\cdots(\Gamma_2\alpha_{n-1} \times Q_{t_n-t_{n-1}}\Gamma_2\alpha_n)\cdots))) \\
&= \mu(\Gamma_1\alpha_0 \times P_{t_1}(\Gamma_1\alpha_1 \times P_{t_2-t_1}(\cdots(\Gamma_1\alpha_{n-1} \times P_{t_n-t_{n-1}}\Gamma_1\alpha_n)\cdots))) \\
&= \mathbb{E}[\alpha_0(f(X(0)))\alpha_1(f(X(t_1)))\ldots\alpha_n(f(X(t_n)))]
\end{aligned} \tag{2.2}$$

which proves the equality in law. □

We let $(Y(t) : t \geq 0)$ be the process Y of n reflected Brownian motions with a wall introduced in the previous section. It is clear from the construction (1.10) that the process Y is a time homogeneous Markov process. We denote its transition semigroup by $\big(Q_t; t \geq 0\big)$. It turns out that there is an explicit formula for the corresponding densities. Recall $\phi_t(z) = \frac{1}{\sqrt{2\pi t}}e^{-z^2/(2t)}$. Let us define $\phi_t^{(k)}(y) = \frac{d^k}{dy^k}\phi_t(y)$ for $k \geq 0$ and $\phi_t^{(-k)}(y) = (-1)^k \int_y^\infty \frac{(z-y)^{k-1}}{(k-1)!}\phi_t(z)dz$ for $k \geq 1$.

Proposition 5. *The transition densities $q_t(y, y')$ from $y = (y_1, \ldots, y_n)$ at $t = 0$ to $y' = (y_1', \ldots, y_n')$ at t of the Y process can be written as*

$$q_t(y, y') = \det\{a_{i,j}(y_i, y_j')\}_{1 \leq i,j \leq n} \tag{2.3}$$

where $a_{i,j}$ is given by

$$a_{i,j}(y, y') = (-1)^{i-1}\phi_t^{(j-i)}(y + y') + (-1)^{i+j}\phi_t^{(j-i)}(y - y'). \tag{2.4}$$

The same type of formula was first obtained for the totally asymmetric simple exclusion process by Schütz [16]. The formula for the Z process was given as a Proposition 8 in [18], see also [14].

$$\begin{array}{|ccccccc}
z_1^1 & & & & & & \\
 & z_1^2 & & & & & \\
z_1^3 & & z_2^3 & & & & \\
 & z_1^4 & & z_2^4 & & & \\
z_1^5 & & z_2^5 & & z_3^5 & & \\
 & \vdots & & \vdots & & \ddots & \\
z_1^n & & z_2^n & & z_3^n & \dots & z_m^n
\end{array}$$

Figure 1: The set $\mathbb{K}$. The triangle represents the intertwining relations of the variables z and the vertical line on the left indicates $z_1^{2k+1} \geq 0$, see (2.5),(2.6). The set of variables on the bottom line is denoted by $b(z)$ and the one on the upper right line by $e(z)$.

Proof of Proposition 5. For a fixed y', define $G(y,t)$ to be (2.3) as a function of y and t. We check that G satisfies (i) the heat equation, (ii) the boundary conditions $\frac{\partial G}{\partial y_1}|_{y_1=0} = 0$, $\frac{\partial G}{\partial y_i}|_{y_i=y_{i-1}} = 0$, $i = 2,3,\dots,n$ and (iii) the initial conditions $G(y,t=0) = \prod_{i=1}^n \delta(y_i - y_i')$.
(i) holds since $\phi_t^{(k)}(y)$ for each k satisfies the heat equation. (ii) follows from the relations, $\frac{\partial}{\partial y} a_{1j}(y,y')|_{y=0} = \phi_t^{(j)}(y') + (-1)^{j+1}\phi_t^{(j)}(-y') = 0$ and $\frac{\partial}{\partial y} a_{ij}(y,y') = -a_{i-1,j}(y,y')$. For (iii) we notice that the first term in (2.4) goes to zero as $t \to 0$ for $y, y' > 0$ and the statement for the remaining part is shown in Lemma 7 in [18]. $\square$

For $n = 2m$, resp. $n = 2m-1$ we take $(X(t), t \geq 0)$ to be Dyson Brownian motion of type C, resp. of type D. The transition semigroup $\left(P_t; t \geq 0\right)$ of this process is given by (1.7).
Let $\mathbb{K}$ denote the set with n layers $z = (z^1, z^2, \dots, z^n)$ where $z^{2k} = (z_1^{2k}, z_2^{2k}, \dots, z_k^{2k}) \in \mathbb{R}_+^k$, $z^{2k-1} = (z_1^{2k-1}, z_2^{2k-1}, \dots, z_k^{2k-1}) \in \mathbb{R}_+^k$ and the intertwining relations,

$$z_1^{2k-1} \leq z_1^{2k} \leq z_2^{2k-1} \leq z_2^{2k} \leq \dots \leq z_k^{2k-1} \leq z_k^{2k} \tag{2.5}$$

and

$$0 \leq z_1^{2k+1} \leq z_1^{2k} \leq z_2^{2k+1} \leq z_2^{2k} \leq \dots \leq z_k^{2k} \leq z_{k+1}^{2k+1} \tag{2.6}$$

hold (Fig. 1). Let $n = 2m$ or $n = 2m-1$ for some integer m. We define a kernel L^0 from $E = \{0 \leq x_1 \leq \dots \leq x_m\}$ to $F = \{0 \leq y_1 \leq \dots \leq y_n\}$. For $z \in \mathbb{K}$, define $b(z) = z^n = (z_1^n, \dots, z_m^n) \in E$, $e(z) = (z_1^1, z_1^2, z_2^3, z_2^4, \dots, z_m^n) \in F$ and $\mathbb{K}(x) = \{z \in \mathbb{K}; b(z) = x \in E\}, \mathbb{K}[y] = \{z \in \mathbb{K}; e(z) = y \in F\}$. The kernel L^0 is defined by

$$L^0 g(x) = \int_F L^0(x, dy) g(y) = \int_{\mathbb{K}(x)} g(e(z)) dz. \tag{2.7}$$

where the integrals are taken with respect to Lebesgue measure but integrations with respect to z on the RHS is for $b(z) = x$ fixed.
The function h defined at (1.6) is equal to the Euclidean volume of $\mathbb{K}(x)$. Consequently we may define L to be the Markov kernel $L(x, dy) = L^0(x, dy)/h(x)$. In the remaining part of this section we show

Proposition 6.

$$LQ_t = P_t L. \tag{2.8}$$

Now if we apply Lemma 4 with $f(x)=x_m$, $g(y)=y_n$ and the initial conditions starting from the origin we obtain (1.13).

Proof of Proposition 6. The kernels $P_t(x,\cdot)$ and $L(x,\cdot)$ are continuous in x. Thus we may consider x in the interior of E, and it is enough to prove

$$(L^0 Q_t)(x,dy)=(P_t^0 L^0)(x,dy). \tag{2.9}$$

From the definition of the kernel L^0, this is equivalent to showing

$$\int_{\mathbb{K}(x)} q_t(e(z),y)dz=\int_{\mathbb{K}[y]} p_t^0(x,b(z))dz \tag{2.10}$$

where q_t and p^0 are densities corresponding to Q_t and P_t^0. Integrations with respect to z are on the LHS with $b(z)=x$ fixed and on the RHS with $e(z)=y$ fixed.
Let us consider the case where $n=2m$. Using the determinantal expressions for q_t and p_t^0 we show that both sides of (2.10) are equal to the determinant of size $2m$ whose (i,j) matrix element is $a_{2i,j}(0,y_j)$ for $1\le i\le m, 1\le j\le 2m$ and $a_{2m,j}(x_{i-m},y_j)$ for $m+1\le i\le 2m, 1\le j\le 2m$.
The integrand of the LHS of (2.10) is

$$q_t(e(z),y)=\det\{a_{i,j}(e(z)_i,y_j)\}_{1\le i,j\le 2m} \tag{2.11}$$

with $b(z)=x$. We perform the integral with respect to $z^1,\dots,z^{2m-1}$ in this order. After the integral up to $z^{2l-1}, 1\le l\le m$, we get the determinant of size $2m$ whose (i,j) matrix element is $a_{2i,j}(0,y_j)$ for $1\le i\le l$, $a_{2l,j}(z_{i-l}^{2l},y_j)$ for $l+1\le i\le 2l$ and $a_{i,j}(e(z)_i,y_j)$ for $2l+1\le i\le 2m$. Here we use a property of $a_{i,j}$,

$$a_{i,j}(y,y')=\int_y^\infty a_{i-1,j}(u,y')du, \tag{2.12}$$

and do some row operations in the determinant. The case for $l=m$ gives the desired expression.
The integrand of the RHS of (2.10) is

$$p_t^0(x,z^{2m})=\det(a_{2m,2m}(x_i,z_j^{2m}))_{1\le i,j\le m} \tag{2.13}$$

with the condition $e(z)=y$. We perform the integrals with respect to $(z_1^{2m},\dots,z_{m-1}^{2m})$, $(z_1^{2m-1},\dots,z_{m-1}^{2m-1}),\dots,z_1^4,z_1^3$ in this order. We use properties of $a_{i,j}$,

$$a_{i,j}(y,y')=-\int_{y'}^\infty a_{i,j+1}(y,u)du, \tag{2.14}$$

$$a_{2i,2j}(x,0)=0,\quad a_{2i,2i-1}(0,y)=1,\quad a_{2i,j}(0,y)=0,\ 2i\le j. \tag{2.15}$$

After each integration corresponding to a layer of $\mathbb{K}$ we simplify the determinant using column operations. We also expand the size of the determinant after an integration corresponding to $(z_1^{2l},\dots,z_{l-1}^{2l})$ for $1\le l\le m$, by adding a new first row

$$(\underbrace{1,1,\dots,1}_{l},\underbrace{0,0,\dots,0}_{2m-2l+1})=$$
$$(a_{2l,2l-1}(0,z_1^{2l-1}),\dots,a_{2l,2l-1}(0,z_l^{2l-1}),a_{2l,2l}(0,e(z)_{2l}),\dots,a_{2l,2m}(0,e(z)_{2m})) \tag{2.16}$$

together with a new column. After the integrals up to $(z_1^{2l-1}, \dots, z_{l-1}^{2l-1})$ have been performed, we obtain the determinant of size $2m - l + 1$,

$$\begin{vmatrix} a_{2(l+i-1),2(l-1)}(0, z_j^{2(l-1)}) & a_{2(l+i-1),j+l-1}(0, e(z)_{j+l-1}) \\ a_{2m,2(l-1)}(x_{i-m+l-1}, z_j^{2(l-1)}) & a_{2m,j+l-1}(x_{i-m+l-1}, e(z)_{j+l-1}) \end{vmatrix}. \tag{2.17}$$

Here $1 \le i \le m - l + 1$ (resp. $m - l + 2 \le i \le 2m - l + 1$) for the upper expression (resp. the lower expression) and $1 \le j \le l - 1$ (resp. $l \le j \le 2m - l + 1$) for the left (resp. right) expression. For $l = 1$ this reduces to the same determinant as for the LHS.
The case $n = 2m - 1$ is almost identical. Similar arguments show that both sides of (2.9) are equal to a determinant of size $2m - 1$ whose (i, j) matrix element is $a_{2i,j}(0, y_j)$ for $1 \le i \le m - 1, 1 \le j \le 2m - 1$ and $a_{2m-1,j}(x_{i-m+1}, y_j)$ for $m + 1 \le i \le 2m - 1, 1 \le j \le 2m - 1$. □

3 Proof of Proposition 3

Using (1.10) repeatedly, one has

$$Y_n(t) = \sup_{0 \le t_1 \le \ldots \le t_n \le t} \sum_{i=1}^{n} (B_i(t_{i+1}) - B_i(t_i)) \tag{3.1}$$

with $t_{n+1} = t$. By renaming $t - t_{n-i+1}$ by t_i and changing the order of the summation, we have

$$Y_n(t) = \sup_{0 \le t_1 \le \ldots \le t_n \le t} \sum_{i=1}^{n} (B_{n-i+1}(t - t_{i+1}) - B_{n-i+1}(t - t_i)). \tag{3.2}$$

Since $\tilde{B}_i(s) := B_{n-i+1}(t) - B_{n-i+1}(t - s) \stackrel{d}{=} B_i(s)$,

$$Y_n(t) \stackrel{d}{=} \sup_{0 \le t_1 \le \ldots \le t_n \le t} \sum_{i=1}^{n} (B_i(t_i) - B_i(t - t_{i-1})) = \sup_{0 \le s \le t} Z_n(t). \tag{3.3}$$

Graphically the above proof corresponds to reversing the time direction and the order of particles.

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