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DISTRIBUTION OF A RANDOM FUNCTIONAL OF A FERGUSON-DIRICHLET PROCESS OVER THE UNIT SPHERE

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Abstract

Jiang, Dickey, and Kuo [12] gave the multivariate c-characteristic function and showed that it has properties similar to those of the multivariate Fourier transformation. We first give the multivariate c-characteristic function of a random functional of a Ferguson-Dirichlet process over the unit sphere. We then find out its probability density function using properties of the multivariate c-characteristic function. This new result would generalize that given by [11].

1 Introduction

Ferguson [5] introduced the Ferguson-Dirichlet process and studied its applications to nonparametric Bayesian inference. He also showed that when the prior distribution is a Ferguson-Dirichlet process with parameter μ, then the posterior distribution, given the sample $s_1, s_2, \ldots, s_n$, is also a Ferguson-Dirichlet process having parameter $\mu + \sum_{j=1}^{n} \delta_{s_j}$, where δ_{s_j} denotes point mass at s_j. The most natural use of random functionals of a Ferguson-Dirichlet process is to make Bayesian inferences concerning the parameters of a statistical population. Hence, the expression for the probability density function of any random functional of a Ferguson-Dirichlet process can be employed both for prior and posterior Bayesian analyses. Further applications related to the random functional can be seen in [3] and other references. For example, random means and random variances of a Ferguson-Dirichlet process can be used for smooth Bayesian nonparametric density estimation (see [15]) and for quality control problems (see [4] for further discussions), respectively.

Research on the distribution of a random functional of a Ferguson-Dirichlet process has been ongoing for decades. A partial list of papers in this area are [2, 3, 8, 9, 11, 12, 14, 16, 17]. In particular, [11] gave the distribution of a random functional of a Ferguson-Dirichlet process over the unit circle. In this paper, we shall use the multivariate c-characteristic function, a tool given

by [12], to generalize the result to the case over the unit sphere in three-dimension.
In Section 2, we first review the definition of the multivariate c-characteristic function and some of its properties. We then compute a multivariate c-characteristic function of an interesting distribution. The multivariate c-characteristic function of the random mean of a Ferguson-Dirichlet process over the unit sphere is given in Section 3. Using the uniqueness property of the multivariate c-characteristic function, we then determine the distribution of the random mean of a Ferguson-Dirichlet process over the unit sphere. Conclusions are given in Section 4.

2 Multivariate c-characteristic function

Jiang [10] first gave a univariate c-characteristic function. Jiang, Dickey, and Kuo [12] generalized it to a multivariate c-characteristic function, which can be very useful when a distribution is difficult to deal with by traditional characteristic function. See [12] for detailed results. First, we state the definition of the multivariate c-characteristic function.

Definition 1. *If $\boldsymbol{u} = (u_1, \dots, u_L)'$ is a random vector on a subset S of $A = [-a_1, a_1] \times \cdots \times [-a_L, a_L]$, its multivariate c-characteristic function is defined as*

$$g(\boldsymbol{t}; \boldsymbol{u}, c) = E[(1 - i\boldsymbol{t} \cdot \boldsymbol{u})^{-c}],\ |\boldsymbol{t}| < a^{-1},$$

where $c > 0$, $a = \sqrt{\sum_{i=1}^{L} a_i^2}$, $\boldsymbol{t}' = (t_1, \dots, t_L)$, $|\boldsymbol{t}| = \sqrt{\sum_{i=1}^{L} t_i^2}$, and $\boldsymbol{t} \cdot \boldsymbol{u}$ is the inner product of $\boldsymbol{t}$ and $\boldsymbol{u}$.

The above assumptions that c is positive and $\boldsymbol{u}$ has a bounded support are needed in [12, Lemma 2.2], which shows that, for any positive c, there is a one-to-one correspondence between $g(\boldsymbol{t}; \boldsymbol{u}, c)$ and the distribution of $\boldsymbol{u}$.
Next, we give the multivariate c-characteristic function of an interesting distribution in the next lemma.

Lemma 2. *Let $\boldsymbol{u} = (u_1, u_2, u_3)'$ be a distribution on the inside of a unit ball, i.e., $\{(u_1, u_2, u_3) \mid u_1^2 + u_2^2 + u_3^2 < 1\}$, with the probability density function*

$$f(\boldsymbol{u}) = \frac{-e}{4\pi^2 r}(1+r)^{-(1+r)/2}(1-r)^{-(1-r)/2}\left(-\pi \sin\frac{\pi r}{2} + \ln\frac{1-r}{1+r}\cos\frac{\pi r}{2}\right),$$

where $r = |\boldsymbol{u}|$. Then the multivariate 1-characteristic function of $\boldsymbol{u}$ is

$$g(\boldsymbol{t}; \boldsymbol{u}, 1) = \exp\left(\sum_{n=1}^{\infty} \frac{(-t_1^2 - t_2^2 - t_3^2)^n}{2n(2n+1)}\right). \tag{1}$$

Proof. Let $C = \{(u_1, u_2, u_3) \mid u_1^2 + u_2^2 + u_3^2 < 1\}$. Eq. (1) is equivalent to the following identity

$$\int_C (1 - i\boldsymbol{t} \cdot \boldsymbol{u})^{-1} f(\boldsymbol{u})\, d\boldsymbol{u} = \exp\left(\sum_{n=1}^{\infty} \frac{(-t_1^2 - t_2^2 - t_3^2)^n}{2n(2n+1)}\right).$$

To prove the above identity, we establish the following four equations first. From [7, p. 105], we have

$$\int_0^{2\pi} (a\cos\alpha + b\sin\alpha)^n\, d\alpha = \begin{cases} \frac{(1/2, n/2) 2(a^2+b^2)^{n/2}\pi}{(n/2)!}, & n \text{ is even}, \\ 0, & n \text{ is odd}, \end{cases} \tag{2}$$

where a and b are real numbers and $(a,k) = a(a+1)\cdots(a+k-1)$. We also can obtain the following equation from [6, Eq. 3.621.5],

$$\int_0^\pi \sin^{a-1} x \cos^{b-1} x \, dx = \begin{cases} \frac{B(a/2,b/2)}{2}, & \text{Re } a > 0,\ b > 0 \text{ is odd}, \\ 0, & \text{Re } a > 0,\ b > 0 \text{ is even}. \end{cases} \tag{3}$$

Using integration by parts, we have the following identity,

$$\begin{aligned} &\int_0^1 r^{2n+1}(1+r)^{-(1+r)/2}(1-r)^{-(1-r)/2}\left(-\pi \sin\frac{\pi r}{2}\right) dr \\ = \ & -\int_0^1 r^{2n+1}(1+r)^{-(1+r)/2}(1-r)^{-(1-r)/2} \ln\frac{1-r}{1+r}\cos\frac{\pi r}{2}\, dr \\ &-\int_0^1 2(2n+1)r^{2n}(1+r)^{-(1+r)/2}(1-r)^{-(1-r)/2}\cos\frac{\pi r}{2}. \end{aligned} \tag{4}$$

Using [13, Lemma 8 and Example 2], we can obtain the following equality:

$$\exp\left(-\int_{-1}^1 \ln(1-itx)\frac{1}{2}\,dx\right) = \int_{-1}^1 (1-itx)^{-1}\frac{e}{\pi}(x+1)^{-(x+1)/2}(1-x)^{-(1-x)/2}\cos\frac{\pi x}{2}\,dx.$$

Since

$$\exp\left(-\int_{-1}^1 \ln(1-itx)\frac{1}{2}\,dx\right) = \exp\left(\sum_{n=1}^\infty \frac{(-t^2)^n}{2n(2n+1)}\right)$$

and

$$\begin{aligned} &\int_{-1}^1 (1-itx)^{-1}\frac{e}{\pi}(x+1)^{-(x+1)/2}(1-x)^{-(1-x)/2}\cos\frac{\pi x}{2}\,dx \\ = \ & \sum_{n=0}^\infty \int_{-1}^1 \frac{e i^n t^n}{\pi} x^n (x+1)^{-(x+1)/2}(1-x)^{-(1-x)/2}\cos\frac{\pi x}{2}\,dx, \end{aligned}$$

and by the fact that the function $(x+1)^{-(x+1)/2}(1-x)^{-(1-x)/2}\cos\frac{\pi x}{2}$ is symmetric at $x=0$, we have

$$\exp\left(\sum_{n=1}^\infty \frac{(-t^2)^n}{2n(2n+1)}\right) = \frac{2e}{\pi}\sum_{n=0}^\infty (-t^2)^n \int_0^1 x^{2n}(x+1)^{-(x+1)/2}(1-x)^{-(1-x)/2}\cos\frac{\pi x}{2}\,dx. \tag{5}$$

Setting

$$g(r) = \frac{-er}{4\pi^2}(1+r)^{-(1+r)/2}(1-r)^{-(1-r)/2}\left(-\pi\sin\frac{\pi r}{2} + \ln\frac{1-r}{1+r}\cos\frac{\pi r}{2}\right),$$

and using the spherical coordinate transformation, we have

$$
\begin{aligned}
&\int_C (1 - i\boldsymbol{t}\cdot\boldsymbol{u})^{-1} f(\boldsymbol{u})\, d\boldsymbol{u} \\
&= \int_0^1\int_0^{2\pi}\int_0^{\pi} (1 - it_1 r\cos\theta\sin\phi - it_2 r\sin\theta\sin\phi - it_3 r\cos\phi)^{-1}\sin\phi\, g(r)\, d\phi\, d\theta\, dr \\
&= \int_0^1 \sum_{n=0}^{\infty}(ir)^n g(r)\int_0^{2\pi}\int_0^{\pi}(t_1\cos\theta\sin\phi + t_2\sin\theta\sin\phi + t_3\cos\phi)^n \sin\phi\, d\phi\, d\theta\, dr \\
&= \int_0^1 \sum_{n=0}^{\infty}(ir)^n g(r)\int_0^{2\pi}\int_0^{\pi}\sum_{k=0}^{n}\binom{n}{k}(t_1\cos\theta + t_2\sin\theta)^k t_3^{n-k}\sin^{k+1}\phi\cos^{n-k}\phi\, d\phi\, d\theta\, dr \\
&= \int_0^1 \sum_{n=0}^{\infty}\frac{4\pi(-t_1^2 - t_2^2 - t_3^2)^n r^{2n}}{2n+1} g(r)\, dr \qquad (6) \\
&= \frac{2e}{\pi}\sum_{n=0}^{\infty}(-t_1^2 - t_2^2 - t_3^2)^n \int_0^1 r^{2n}(1+r)^{-(1+r)/2}(1-r)^{-(1-r)/2}\cos\frac{\pi r}{2}\, dr \qquad (7) \\
&= \exp\left(\sum_{n=1}^{\infty}\frac{(-t_1^2 - t_2^2 - t_3^2)^n}{2n(2n+1)}\right). \qquad (8)
\end{aligned}
$$

Identity (6) can be obtained by Eqs. (2) and (3). Identities (7) and (8) follow from Eq. (4) and Eq. (5), respectively. □

3 Distribution of a random functional of a Ferguson-Dirichlet process over the unit sphere

Ferguson [5] first defined the Ferguson-Dirichlet process. Let μ be a finite non-null measure on (Y,A), where Y is a Borel set in Euclidean space $\mathbb{R}^n$ and A is the σ-field of Borel subsets of Y, and let U be a stochastic process indexed by elements of A. We say that U is a Ferguson-Dirichlet process with parameter μ, if for every finite measurable partition $\{B_1,\ldots,B_m\}$ of Y, the random vector $(U(B_1),\ldots,U(B_m))$ has a Dirichlet distribution with parameter $(\mu(B_1),\ldots,\mu(B_m))$, where $\mu(B_j) > 0$ for all $j = 1,\ldots,m$. A random vector $\boldsymbol{v} = (v_1,\ldots,v_m)'$ is said to have a Dirichlet distribution with parameter $\boldsymbol{b} = (b_1,\ldots,b_m)'$ where each $b_j > 0$, if $\boldsymbol{v}$ has the probability density function

$$f(\boldsymbol{v};\boldsymbol{b}) = \frac{\Gamma(b_1+\cdots+b_m)}{\prod_{j=1}^m \Gamma(b_j)}\prod_{j=1}^m v_j^{b_j-1},$$

for all $\boldsymbol{v}$ in the probability simplex $\{\boldsymbol{v} \mid \text{each } v_j \geq 0,\ v_1+\cdots+v_m = 1\}$.

First, we give a trivariate c-characteristic function expression of any trivariate random functional of a Ferguson-Dirichlet process over a Borel set Y in Euclidean space in the next lemma.

Lemma 3. *Let $\boldsymbol{w} = \int_Y \boldsymbol{h}(\boldsymbol{x})\, dU(\boldsymbol{x})$ be a random functional where $\boldsymbol{h}(\boldsymbol{x}) = (h_1(\boldsymbol{x}),h_2(\boldsymbol{x}),h_3(\boldsymbol{x}))'$ is a bounded measurable function defined on a Borel set Y in Euclidean space $\mathbb{R}^n$, and U is a Ferguson-Dirichlet process with parameter μ on (Y,A). Then the trivariate c-characteristic function of $\boldsymbol{w}$ can* *be expressed as*

$$g(\boldsymbol{t};\boldsymbol{w},c)=\exp\left(-\int_Y \ln(1-i\boldsymbol{t}\cdot\boldsymbol{h}(\boldsymbol{x}))\,d\mu(\boldsymbol{x})\right),\ \textit{where } c=\mu(Y).$$

Proof. For any $k\geq 2$, let $\{B_{k1},B_{k2},\ldots,B_{kk}\}$ be a partition of Y, $\boldsymbol{b}_{kj}\in B_{kj}$, $v_k=\max\{\text{volume}(B_{kj})\mid 1\leq j\leq k\}$, and $\lim_{k\to\infty}v_k=0$. Then $(U(B_{k1}),\ldots,U(B_{kk}))$ follows a Dirichlet distribution with parameter $(\mu(B_{k1}),\ldots,\mu(B_{kk}))$. In addition, $\sum_{j=1}^k U(B_{kj})=1$ for all $k\geq 2$. Define $\boldsymbol{g}_k(\boldsymbol{x})=\sum_{j=1}^k \boldsymbol{h}(\boldsymbol{b}_{kj})\delta_{B_{kj}}(\boldsymbol{x})$ and $\boldsymbol{w}_k=\int_Y \boldsymbol{g}_k(\boldsymbol{x})\,dU(\boldsymbol{x})$, where $\delta_{B_{kj}}(\boldsymbol{x})$ is 1, for $\boldsymbol{x}\in B_{kj}$; and is 0, otherwise. Then $\lim_{k\to\infty}\boldsymbol{g}_k(\boldsymbol{x})=\boldsymbol{h}(\boldsymbol{x})$ for all $\boldsymbol{x}\in Y$, and $\boldsymbol{w}_k=\sum_{j=1}^k \boldsymbol{g}_k(\boldsymbol{b}_{kj})U(B_{kj})$. The trivariate c-characteristic function of $\boldsymbol{w}_k$ can be expressed as

$$\begin{aligned}
g(\boldsymbol{t};\boldsymbol{w}_k,c) &= E(1-i\boldsymbol{t}\cdot\boldsymbol{w}_k)^{-c}\\
&= E\left(1-i\sum_{j=1}^k[\boldsymbol{t}\cdot\boldsymbol{g}_k(\boldsymbol{b}_{kj})]U(B_{kj})\right)^{-c}\\
&= E\left(\sum_{j=1}^k U(B_{kj})[1-i\boldsymbol{t}\cdot\boldsymbol{g}_k(\boldsymbol{b}_{kj})]\right)^{-c}\\
&= \mathscr{R}_{-c}(\mu(B_{k1}),\ldots,\mu(B_{kk});1-i\boldsymbol{t}\cdot\boldsymbol{g}_k(\boldsymbol{b}_{k1}),\ldots,1-i\boldsymbol{t}\cdot\boldsymbol{g}_k(\boldsymbol{b}_{kk}))\\
&= \prod_{j=1}^k(1-i\boldsymbol{t}\cdot\boldsymbol{g}_k(\boldsymbol{b}_{kj}))^{-\mu(B_{kj})},
\end{aligned}$$

where $\mathscr{R}$ is a Carlson's multiple hypergeometric function ([1]), and the last equality can be obtained by [1, formula 6.6.5]. Therefore, the limit of the trivariate c-characteristic function of $\boldsymbol{w}_k$'s, as k approaches ∞, is

$$\begin{aligned}
\lim_{k\to\infty} g(\boldsymbol{t};\boldsymbol{w}_k,c) &= \exp\left(\lim_{k\to\infty}\sum_{j=1}^k -\mu(B_{kj})\ln(1-i\boldsymbol{t}\cdot\boldsymbol{g}_k(\boldsymbol{b}_{kj}))\right)\\
&= \exp\left(-\int_Y \ln(1-i\boldsymbol{t}\cdot\boldsymbol{h}(\boldsymbol{x}))\,d\mu(\boldsymbol{x})\right).
\end{aligned}$$

In addition, by the Dominated Convergence Theorem, we have $\lim_{k\to\infty}\boldsymbol{w}_k=\boldsymbol{w}$. By [12, Theorem 2.4], we conclude that

$$g(\boldsymbol{t};\boldsymbol{w},c)=\exp\left(-\int_Y \ln(1-i\boldsymbol{t}\cdot\boldsymbol{h}(\boldsymbol{x}))\,d\mu(\boldsymbol{x})\right).$$

□

In the rest of this section, we study the random functional $\boldsymbol{u}=\int_X \boldsymbol{x}\,dU(\boldsymbol{x})$, where X is the unit sphere in $\mathbb{R}^3$. We use Lemma 3 in the following theorem to first establish the trivariate c-characteristic function of $\boldsymbol{u}$.

Theorem 4. *Let $X=\{(x_1,x_2,x_3)\mid x_1^2+x_2^2+x_3^2=1\}$, and U be a Ferguson-Dirichlet process over X with uniform measure μ as its parameter, where $\mu(X)=c$. Then the trivariate c-characteristic* *function of the random functional $\boldsymbol{u} = \int_X \boldsymbol{x}\, dU(\boldsymbol{x})$ can be expressed as*

$$g(\boldsymbol{t};\boldsymbol{u},c) = \exp\left(\sum_{n=1}^{\infty} \frac{c}{2n(2n+1)}(-t_1^2 - t_2^2 - t_3^2)^n\right), \text{ where } \boldsymbol{t} = (t_1, t_2, t_3)'.$$

Proof. First, we give the following two equations, which are about Appell's notations and can be shown easily.

$$\Gamma(a+n) = \Gamma(a)(a,n), \tag{9}$$

$$(a,2n) = 2^{2n}\left(\frac{a}{2}, n\right)\left(\frac{a+1}{2}, n\right). \tag{10}$$

By Lemma 3, we have

$$\begin{aligned}
& g(\boldsymbol{t};\boldsymbol{u},c) \\
&= \exp\left(\frac{-c}{4\pi}\int_X \ln(1 - i\boldsymbol{t}\cdot\boldsymbol{x})\, d\boldsymbol{x}\right) \\
&= \exp\left(\frac{-c}{4\pi}\int_0^{\pi}\int_0^{2\pi} \ln(1 - it_1\cos\theta_1 - it_2\sin\theta_1\cos\theta_2 - it_3\sin\theta_1\sin\theta_2)\sin\theta_1\, d\theta_2\, d\theta_1\right) \\
&= \exp\left(\frac{c}{4\pi}\sum_{n=1}^{\infty}\frac{i^n}{n}\int_0^{\pi}\int_0^{2\pi}(t_1\cos\theta_1 + t_2\sin\theta_1\cos\theta_2 + t_3\sin\theta_1\sin\theta_2)^n\sin\theta_1\, d\theta_2\, d\theta_1\right) \\
&= \exp\left(\frac{c}{4\pi}\sum_{n=1}^{\infty}\frac{i^n}{n}\sum_{k=0}^{n}\binom{n}{k}\int_0^{\pi}\int_0^{2\pi}(t_1\cos\theta_1)^k\sin^{n-k+1}\theta_1(t_2\cos\theta_2 + t_3\sin\theta_2)^{n-k}\, d\theta_2\, d\theta_1\right) \\
&= \exp\left(\frac{c}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n}\sum_{k=0}^{n}\binom{2n}{2k}\frac{(1/2, n-k)(t_2^2 + t_3^2)^{n-k}}{(n-k)!}t_1^{2k}B(n-k+1, k+1/2)\right) \\
&= \exp\left(\sum_{n=1}^{\infty}\frac{c}{2n(2n+1)}(-t_1^2 - t_2^2 - t_3^2)^n\right).
\end{aligned}$$

The fifth identity can be obtained by Eqs. (2) and (3). The last identity follows from Eqs. (9) and (10). □

By [12, Lemma 2.2], Lemma 2, and Theorem 4, we can obtain the following corollary.

Corollary 5. *The probability density function of $\boldsymbol{u} = \int_X \boldsymbol{x}\, dU(\boldsymbol{x})$, where U is a Ferguson-Dirichlet process over the unit sphere X with uniform probability measure as its parameter, is*

$$f(\boldsymbol{u}) = \frac{-e}{4\pi^2 r}(1+r)^{-(1+r)/2}(1-r)^{-(1-r)/2}\left(-\pi\sin\frac{\pi r}{2} + \ln\frac{1-r}{1+r}\cos\frac{\pi r}{2}\right),$$

where $r = \sqrt{u_1^2 + u_2^2 + u_3^2}$ and $u_1^2 + u_2^2 + u_3^2 < 1$.

4 Conclusions

In this paper, we obtain the trivariate c-characteristic function expression for a random functional of a Ferguson-Dirichlet process over any finite three-dimensional space. We also obtain the probability density function of the random functional of a Ferguson-Dirichlet process with uniform probability measure parameter over the unit sphere. This generalizes [11, Theorem 2].

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References

[1] Carlson, B.C. (1977). *Special Functions of Applied Mathematics*. Academic Press, New York. MR0590943

[2] Cifarelli, D.M. and Regazzini, E. (1990). Distribution functions of means of a Dirichlet process. *Ann. Statist.* **18**, 429–442. MR1041402. Correction (1994): *Ann. Statist.* **22**, 1633–1634. MR1311994

[3] Diaconis, P. and Kemperman, J. (1996). Some new tools for Dirichlet priors, in: J.M. Bernardo, J.O. Berger, A.P. Dawid, and A.F.M. Smith eds. *Bayesian Statistics* 5, pp. 97–106. Oxford University Press. MR1425401

[4] Epifani, I., Guglielmi, A., and Melilli, E. (2006). A stochastic equation for the law of the random Dirichlet variance. *Statist. Probab. Lett.* **76**, 495–502. MR2266603

[5] Ferguson, T.S. (1973). A Bayesian analysis of some nonparametric problems. *Ann. Statist.* **1**, 209–230. MR0350949

[6] Gradshteyn, I.S. and Ryzhik, I.M. (2000). *Table of Integrals, Series, and Products*. Academic Press, New York, 6th ed. MR1773820

[7] Gröbner, W. and Hofreiter, W. (1973). *Integraltafel*, Vol. 2. Springer-Verlag, New York, 5th ed.

[8] Hannum, R.C., Hollander, M., and Langberg, N.A. (1981). Distributional results for random functionals of a Dirichlet process. *Ann. Probab.* **9**, 665–670. MR0630318

[9] Hjort, N.L. and Ongaro, A. (2005). Exact inference for random Dirichlet means. *Stat. Inference Stoch. Process.* **8**, 227–254. MR2177313

[10] Jiang, J. (1988). Starlike functions and linear functions of a Dirichlet distributed vector. *SIAM J. Math. Anal.* **19**, 390–397. MR0930035

[11] Jiang, T.J. (1991). Distribution of random functional of a Dirichlet process on the unit disk. *Statist. Probab. Lett.* **12**, 263–265. MR1130367

[12] Jiang, T.J., Dickey, J.M., and Kuo, K.-L. (2004). A new multivariate transform and the distribution of a random functional of a Ferguson-Dirichlet process. *Stochastic Process. Appl.* **111**, 77–95. MR2049570

[13] Jiang, T.J. and Kuo, K.-L. (2006). On the random functional of the Ferguson-Dirichlet process. *2006 Proceeding of the Section on Bayesian Statistical Science of the American Statistical Association*, pp. 52–59.

[14] Lijoi, A. and Regazzini, E. (2004). Means of a Dirichlet process and multiple hypergeometric functions. *Ann. Probab.* **32**, 1469–1495. MR2060305

[15] Lo, A.Y. (1984). On a class of Bayesian nonparametric estimates: I. density estimates. *Ann. Statist.* **12**, 351–357. MR0733519

[16] Regazzini, E., Guglielmi, A., and Di Nunno, G. (2002). Theory and numerical analysis for exact distributions of functionals of a Dirichlet process. *Ann. Statist.* **30**, 1376–1411. MR1936323

[17] Yamato, H. (1984). Characteristic functions of means of distributions chosen from a Dirichlet process. *Ann. Probab.* **12**, 262–267. MR0723745