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A NOTE ON A FENYMAN-KAC-TYPE FORMULA

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Abstract
In this article, we establish a probabilistic representation for the second-order moment of the solution of stochastic heat equation in $[0,1]\times\mathbb{R}^d$, with multiplicative noise, which is fractional in time and colored in space. This representation is similar to the one given in [8] in the case of an s.p.d.e. driven by a Gaussian noise, which is white in time. Unlike the formula of [8], which is based on the usual Poisson process, our representation is based on the planar Poisson process, due to the fractional component of the noise.

1 Introduction

The classical Feynman-Kac (F-K) formula gives a stochastic representation for the solution of the heat equation with potential, as an exponential moment of a functional of Brownian paths (see e.g. [14]). This representation is a useful tool in stochastic analysis, in particular for the study stochastic partial differential equations (s.p.d.e.'s). We mention briefly several examples in this direction. The F-K formula lied at the origin of the existence, large time asymptotic and intermittency results of [4] and [5], for the solution of the heat equation with random potential on $\mathbb{Z}^d$, respectively $\mathbb{R}^d$. The same method, based on a discretized F-K formula, was used in [17] for obtaining an upper bound for the exponential behavior of the solution of the heat equation with random potential on a smooth compact manifold. The technique of [4] was further refined in [6] in the case of parabolic equations with Lévy noise, for proving the exponential growth of the solution. The F-K formula was used in [9] for solving stochastic parabolic equations, in the context of white noise analysis. A F-K formula for the solution of the stochastic KPP equation is used in [15] for examining the asymptotic behavior of the solution.
The present work has been motivated by the recent article [8], in which the authors obtained an alternative probabilistic representation for the solution of a deterministic p.d.e., as well as a

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representation for the moments of the (mild-sense) solution of a s.p.d.e. perturbed by a Gaussian noise $\dot{F}$, with "formal" covariance:

$$E[\dot{F}_{t,x}\dot{F}_{s,y}] = \delta_0(t-s)f(x-y).$$

More precisely, in [8], $\{F(h), h \in \mathscr{P}\}$ is a zero-mean Gaussian process with covariance $E(F(h)F(g)) = \langle h, g\rangle_{\mathscr{P}}$, where $\mathscr{P}$ is the completion of $\{1_{[0,t]\times A}; t \in [0,1], A \in \mathscr{B}_b(\mathbb{R}^d)\}$ with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathscr{P}}$ given by

$$\langle 1_{[0,t]\times A}, 1_{[0,s]\times B}\rangle_{\mathscr{P}} = (t\wedge s)\int_A\int_B f(x-y)dydx.$$

(Here, $\mathscr{B}_b(\mathbb{R}^d)$ denotes the class of bounded Borel sets in $\mathbb{R}^d$.)
In the particular case of the stochastic heat equation:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\Delta u + u\,\dot{F}, \quad t>0, x\in\mathbb{R}^d \\ u_{0,x} &= u_0(x), \quad x\in\mathbb{R}^d, \end{aligned} \tag{1}$$

the representation for the second-moments of the (mild-sense) solution u is:

$$E[u_{t,x}u_{t,y}] = e^t E_{x,y}\left[w(t-\tau_{N_t}, B^1_{\tau_{N_t}})w(t-\tau_{N_t}, B^2_{\tau_{N_t}})\prod_{j=1}^{N_t} f(B^1_{\tau_j} - B^2_{\tau_j})\right], \tag{2}$$

(with the convention that on $\{N_t = 0\}$, the product is defined to be 1), where $B^1 = (B^1_t)_{t\geq 0}$ and $B^2 = (B^2_t)_{t\geq 0}$ are independent d-dimensional Brownian motions starting from x, respectively y, $N = (N_t)_{t\geq 0}$ is an independent Poisson process with rate 1 and points $\tau_1 < \tau_2 < \ldots$, and

$$w(t,x) = \int_{\mathbb{R}^d} p_t(x-y)u_0(y)dy, \quad \text{where } p_t(x) = \frac{1}{(2\pi t)^{d/2}}e^{-|x|^2/(2t)}.$$

Note that the representation (2) does not rely on the entire Brownian path, but only on its values at the (random) points $\tau_1, \tau_2, \ldots, \tau_{N_t}$. This property has allowed the authors of [8] to generalize the representation to a large class of s.p.d.e.'s, including the wave equation.
To see where the idea for this representation comes from, we recall briefly the salient points leading to (2). If the solution of (1) exists, then it is unique and admits the Wiener chaos expansion: (see e.g. Proposition 4.1 of [8])

$$u_{t,x} = w(t,x) + \sum_{n=1}^{\infty} I_n(f_n(\cdot,t,x)), \tag{3}$$

where $I_n(f_n(\cdot,t,x)) = \int_{([0,1]\times\mathbb{R}^d)^n} f_n(t_1,x_1,\ldots,t_n,x_n)dF_{t_1,x_1}\ldots dF_{t_n,x_n}$, and $f_n \in \mathscr{P}^{\otimes n}$ is a symmetric function given by:

$$f_n(t_1,x_1,\ldots,t_n,x_n,t,x) = \frac{1}{n!}w(t_{\rho(1)},x_{\rho(1)})\prod_{j=1}^{n} p_{t_{\rho(j+1)}-t_{\rho(j)}}(x_{\rho(j+1)} - x_{\rho(j)}). \tag{4}$$

Here, ρ is a permutation of $\{1,\ldots,n\}$ such that $t_{\rho(1)} < t_{\rho(2)} < \ldots < t_{\rho(n)}$, $t_{\rho(n+1)} = t$ and $x_{\rho(n+1)} = x$. By the orthogonality of the terms in the series (3),

$$E[u_{t,x}u_{t,y}] = w(t,x)w(t,y) + \sum_{n=1}^{\infty} J_n(t,x,y),$$

where

$$J_n(t,x,y) = n!\langle f_n(\cdot,t,x), f_n(\cdot,t,y)\rangle^2_{\mathscr{P}^{\otimes n}} = \int_{T_n(t)} F(t_1,\ldots,t_n)d\mathbf{t}, \quad \text{and}$$

$$F(t_1,\ldots,t_n) = \int_{\mathbb{R}^{2nd}} \prod_{j=1}^{n} p_{t_{j+1}-t_j}(x_{j+1}-x_j) \prod_{j=1}^{n} p_{t_{j+1}-t_j}(y_{j+1}-y_j) \\ w(t_1,x_1)w(t_1,y_1)\prod_{j=1}^{n} f(x_j-y_j)d\mathbf{y}d\mathbf{x}d\mathbf{t}.$$

Here, we denote $\mathbf{t}=(t_1,\ldots,t_n), \mathbf{x}=(x_1,\ldots,x_n)$, $\mathbf{y}=(y_1,\ldots,y_n)$ and $T_n(t)=\{(t_1,\ldots,t_n);\ 0<t_1<\ldots<t_n<t\}$.
A crucial observation of [8] is that, for any $F:[0,t]^n \to \mathbb{R}_+$ measurable,

$$\int_{T_n(t)} F(t_1,\ldots,t_n)d\mathbf{t} = e^t E^N[F(t-\tau_n,\ldots,t-\tau_1)1_{\{N_t=n\}}]. \tag{5}$$

This is a key idea, which yields a probabilistic representation for $J_n(t,x,y)$, based on the jump times of the Poisson process. This idea is new in the literature, although it appeared implicitly in the earlier works [10], [13], [16]. Secondly, for any $0<t_1<\ldots<t_n<t$, $F(t-t_n,\ldots,t-t_1)$ is represented as:

$$F(t-t_n,\ldots,t-t_1) = \\ E^{B^1,B^2}\left[w(t-t_n,B^1_{t_n})w(t-t_n,B^2_{t_n})\prod_{j=1}^{n} f(B^1_{t_j}-B^2_{t_j})\right]. \tag{6}$$

Relation (2) follows from these observations, using the independence between N and B^1, B^2.

In this article, we generalize these ideas to the case of the stochastic heat equation driven by a fractional-colored noise. More precisely, we consider the following equation:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\Delta u + u \diamond \dot{W}, \quad t\in[0,1], x\in\mathbb{R}^d \\ u_{0,x} &= u_0(x), \quad x\in\mathbb{R}^d, \end{aligned} \tag{7}$$

where $u_0 \in C_b(\mathbb{R}^d)$ is non-random, $\diamond$ denotes the Wick product, and $\dot{W}$ is a Gaussian noise whose covariance is formally given by:

$$E[\dot{W}_{t,x}\dot{W}_{s,y}] = \eta(t,s)f(x-y),$$

with

$$\eta(t,s) = \alpha_H|t-s|^{2H-2}, \quad f(x) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x}\mu(d\xi).$$

Here, $H\in(1/2,1)$, $\alpha_H = H(2H-1)$ and μ is a tempered measure on $\mathbb{R}^d$. (Note that $\eta(t,s)$ is the covariance kernel of the fractional Brownian motion.)
The noise $\dot{W}$ is defined rigourously as in [2], by considering a zero-mean Gaussian process $W=\{W(h); h\in\mathscr{HP}\}$ with covariance

$$E(W(h)W(g)) = \langle h,g\rangle_{\mathscr{HP}},$$

where $\mathscr{HP}$ is the the completion of $\{1_{[0,t]\times A}; t \in [0,1], A \in \mathscr{B}_b(\mathbb{R}^d)\}$ with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathscr{HP}}$ given by:

$$\langle 1_{[0,t]\times A}, 1_{[0,s]\times B} \rangle_{\mathscr{HP}} = \int_0^t \int_0^s \int_A \int_B \eta(u,v) f(x-y) dy dx dv du.$$

(See also [7] for a martingale treatment of the case $\eta(t,s) = \delta_0(t-s)$, which corresponds to $H = 1/2$.)

As in [3], the solution of equation (7) is interpreted in the mild sense, using the Skorohod integral with respect to W. More precisely, an adapted square-integrable process $u = \{u_{t,x}; (t,x) \in [0,1] \times \mathbb{R}^d\}$ is a **solution to** (7) if for any $(t,x) \in [0,1] \times \mathbb{R}^d$, the process $\{p_{t-s}(x-y) u_{s,y} 1_{[0,t]}(s); (s,y) \in [0,1] \times \mathbb{R}^d\}$ is Skorohod integrable, and

$$u_{t,x} = p_t u_0(x) + \int_0^T \int_{\mathbb{R}^d} p_{t-s}(x-y) u_{s,y} 1_{[0,t]}(s) \delta W_{s,y}.$$

The existence of a solution u (in the space of square-integrable processes) depends on the roughness of the noise, introduced by H and the kernel f. We mention briefly several cases which have been studied in the literature. If $f = \delta_0$, the solution exists for $d = 1,2$ (see [12]). If $f(x) = \prod_{i=1}^d \alpha_{H_i} |x_i|^{2H_i - 2}$, the solution exists for $d < 2/(2H-1) + \sum_{i=1}^d H_i$ (see [11]). Recently, it was shown in [3] that if f is the Riesz kernel of order α, or the Bessel kernel of order $\alpha < d$, then the solution exists for $d \leq 2 + \alpha$, whereas if f is the heat kernel or the Poisson kernel, then the solution exists for any d.

Our main result establishes a probabilistic representation for the second moment of the solution, similar to (2), and based on the planar Poisson process.

Theorem 1.1. *Suppose that equation (7) has a solution $u = \{u_{t,x}; (t,x) \in [0,1] \times \mathbb{R}^d\}$ in $[0,1] \times \mathbb{R}^d$. Then, for any $(t,x) \in [0,1] \times \mathbb{R}^d$ and $(s,y) \in [0,1] \times \mathbb{R}^d$,*

$$E[u_{t,x} u_{s,y}] = w(t,x) w(s,y) + e^{ts}$$

$$\sum_{i_1,\ldots,i_n} E_{x,y} \left[w(t-\tau^*, B^1_{\tau^*}) w(s-\rho^*, B^2_{\rho^*}) \prod_{j=1}^{N_{t,s}} \eta(t-\tau_{i_j}, s-\rho_{i_j}) f(B^1_{\tau_{i_j}} - B^2_{\rho_{i_j}}) I_{A_{i_1,\ldots,i_n}} \right],$$

where the sum is taken over all distinct indices $i_1, \ldots, i_n$,

- *$B^1 = (B^1_t)_{t\in[0,1]}$ and $B^2 = (B^2_t)_{t\in[0,1]}$ are independent d-dimensional standard Brownian motions, starting from x, respectively y,*
- *$N = (N_{t,s})_{(t,s)\in[0,1]^2}$ is an independent planar Poisson process with rate 1 and points $\{P_i\}_{i\geq 1}$, with $P_i = (\tau_i, \rho_i)$,*
- *$A_{i_1,\ldots,i_n}(t,s)$ is the event that N has points $P_{i_1}, \ldots, P_{i_n}$ in $[0,t] \times [0,s]$, $\tau^* = \max\{\tau_{i_1}, \ldots, \tau_{i_n}\}$ and $\rho^* = \max\{\rho_{i_1}, \ldots, \rho_{i_n}\}$.*

2 Proof of Theorem 1.1

We begin by recalling some basic facts about the planar Poisson process. If $N=(N_{t,s})_{(t,s)\in[0,1]^2}$ is a 2-parameter process, and $R=(a,b]\times(c,d]$ is a rectangle in $[0,1]^2$, we define $N_R=N_{a,c}+N_{b,d}-N_{a,d}-N_{b,c}$.

We say that $N=(N_{t,s})_{(t,s)\in[0,1]^2}$ is a **planar Poisson process** of rate $\lambda>0$, if it satisfies the following conditions:

(i) N vanishes on the axes, i.e. $N_{t,0}=N_{0,t}=0$ for all $t\in[0,1]$.

(ii) N_R has a Poisson distribution with mean $\lambda|R|$, for any rectangle R;

(iii) $N_{R_1},\ldots,N_{R_k}$ are independent, for any disjoint rectangles $R_1,\ldots,R_k$.

The following construction of the planar Poisson process is well-known (see e.g. [1]). Let X a Poisson random variable with mean λ, and $\{P_i\}_{i\geq 1}$ be an independent sequence of i.i.d. random vectors, uniformly distributed on $[0,1]^2$. We denote $P_i=(\tau_i,\rho_i)$. For any $(t,s)\in[0,1]^2$, define

$$N_{t,s}=\sum_{i=1}^{X} I\{\tau_i\leq t,\rho_i\leq s\}.$$

Then $N=(N_{t,s})_{(t,s)\in[0,1]^2}$ is a planar Poisson process with rate λ.

For any $n\geq 1$ and for any distinct positive integers $i_1,\ldots,i_n$, let $A_{i_1,\ldots,i_n}(t,s)$ be the event that N has points $P_{i_1},\ldots,P_{i_n}$ in $[0,t]\times[0,s]$. Then

$$\{N_{t,s}=n\}=\bigcup_{i_1,\ldots,i_n\geq 1\ \text{distinct}} A_{i_1,\ldots,i_n}(t,s). \tag{8}$$

The following result is probably well-known. We include it for the sake of completeness.

Lemma 2.1. *The conditional distribution of*

$$((t-\tau_{i_1},s-\rho_{i_1}),\ldots,(t-\tau_{i_n},s-\rho_{i_n})) \quad \textit{given} \quad A_{i_1,\ldots,i_n}(t,s)$$

is uniform over $([0,t]\times[0,s])^n$.

Proof: Let $I=\{i_1,\ldots,i_n\}$ and $i^*=\max\{i_1,\ldots,i_n\}$. Then

$$A_I(t,s)=\bigcup_{k\geq n, I\subset\{1,\ldots,k\}}\{X=k\}\bigcap\left(\bigcap_{i\in I}\{\tau_i\leq t,\rho_i\leq s\}\right)\bigcap\left(\bigcap_{i\leq k, i\notin I}\{\tau_i>t \text{ or } \rho_i>s\}\right)$$

and

$$P(A_I(t,s))=\sum_{k\geq n\vee i^*} e^{-\lambda}\frac{\lambda^k}{k!}(ts)^n(1-ts)^{k-n}.$$

For any Borel sets $\Gamma_1,\ldots,\Gamma_n$ in $[0,t]\times[0,s]$, we have

$$P\left(\bigcap_{j=1}^{n}\{(t-\tau_{i_j},s-\rho_{i_j})\in\Gamma_j\}\bigcap A_I(t,s)\right)=\sum_{k\geq n\vee i^*} e^{-\lambda}\frac{\lambda^k}{k!}\left(\prod_{j=1}^{n}|\Gamma_j|\right)(1-ts)^{k-n},$$

and hence

$$P\left(\bigcap_{j=1}^{n}\{(t-\tau_{i_j},s-\rho_{i_j})\in\Gamma_j\}\mid A_I(t,s)\right)=\frac{1}{(ts)^n}\prod_{j=1}^{n}|\Gamma_j|.$$

□

As a consequence, for any measurable function $F:([0,t]\times[0,s])^n\to\mathbf{R}_+$,

$$E[F(t-\tau_{i_1},s-\rho_{i_1},\dots,t-\tau_{i_n},s-\rho_{i_n})\mid A_{i_1,\dots,i_n}(t,s)]= \frac{1}{(ts)^n}\int_{[0,t]^n}\int_{[0,s]^n}F(t_1,s_1,\dots,t_n,s_n)d\mathbf{s}d\mathbf{t}, \tag{9}$$

where $\mathbf{t}=(t_1,\dots,t_n)$ and $\mathbf{s}=(s_1,\dots,s_n)$.

Suppose that $\lambda=1$. Then $(ts)^n=n!\,e^{ts}P(N_{t,s}=n)$ and

$$E[F(t-\tau_{i_1},s-\rho_{i_1},\dots,t-\tau_{i_n},s-\rho_{i_n})\mid A_{i_1,\dots,i_n}(t,s)]= \frac{1}{n!e^{ts}P(N_{t,s}=n)}\int_{[0,t]^n}\int_{[0,s]^n}F(t_1,s_1,\dots,t_n,s_n)d\mathbf{s}d\mathbf{t}.$$

Using (8) and (9), we obtain that for any $F:([0,t]\times[0,s])^n\to\mathbf{R}_+$ measurable,

$$\int_{[0,t]^n}\int_{[0,s]^n}F(t_1,s_1,\dots,t_n,s_n)d\mathbf{s}d\mathbf{t}= n!\,e^{ts}\sum_{i_1,\dots,i_n\geq 1 \text{ distinct}}E^N[F(t-\tau_{i_1},s-\rho_{i_1},\dots,t-\tau_{i_n},s-\rho_{i_n})I_{A_{i_1,\dots,i_n}(t,s)}]. \tag{10}$$

Relation (10) is the analogue of (5), needed in the fractional case.

Suppose now that equation (7) has a solution $u=\{u_{t,x};(t,x)\in[0,1]\times\mathbb{R}^d\}$. Then the solution is unique and admits the Wiener chaos expansion (3), where

$$I_n(f_n(\cdot,t,x))=\int_{([0,1]\times\mathbb{R}^d)^n}f_n(t_1,x_1,\dots,t_n,x_n)dW_{t_1,x_1}\dots dW_{t_n,x_n}$$

is the multiple Wiener integral with respect to W, and $f_n\in\mathscr{H}\mathscr{P}^{\otimes n}$ is the symmetric function given by (4). (See (7.4) of [11], or (4.4) of [12], or Proposition 3.2 of [3]).

Using (3), and the orthogonality of the Wiener chaos spaces, we conclude that for any $(t,x)\in[0,1]\times\mathbb{R}^d$ and $(s,y)\in[0,1]\times\mathbb{R}^d$,

$$\begin{aligned} E[u_{t,x}u_{s,y}] &= E(u_{t,x})E(u_{s,y})+\sum_{n\geq 1}E[I_n(f_n(\cdot,t,x))I_n(f_n(\cdot,s,y))] \\ &= w(t,x)w(s,y)+\sum_{n\geq 1}\frac{1}{n!}\alpha_n(t,x,s,y), \end{aligned} \tag{11}$$

where $\alpha_n(t,x,s,y):=(n!)^2\langle f_n(\cdot,t,x),f_n(\cdot,s,y)\rangle_{\mathscr{H}\mathscr{P}^{\otimes n}}$ for any $n\geq 1$. Note that

$$\alpha_n(t,x,s,y)=\int_{[0,t]^n}\int_{[0,s]^n}\prod_{j=1}^n\eta(t_j,s_j)\langle G_{\mathbf{t};x},G_{\mathbf{s};y}\rangle_{\mathscr{P}(\mathbb{R}^d)^{\otimes n}}d\mathbf{s}d\mathbf{t}, \tag{12}$$

where $\mathscr{P}(\mathbb{R}^d)$ is the the completion of $\{1_A;A\in\mathscr{B}_b(\mathbb{R}^d)\}$ with respect to the inner product $\langle 1_A,1_B\rangle_{\mathscr{P}(\mathbb{R}^d)}=\int_A\int_B f(x-y)dydx$,

$$\begin{aligned} G_{\mathbf{t};x}(x_1,\dots x_n) &= w(t_{\rho(1)},x_{\rho(1)})\prod_{j=1}^n p_{t_{\rho(j+1)}-t_{\rho}(j)}(x_{\rho(j+1)}-x_{\rho(j)}) \\ G_{\mathbf{s};y}(y_1,\dots y_n) &= w(s_{\sigma(1)},y_{\sigma(1)})\prod_{j=1}^n p_{s_{\sigma(j+1)}-s_{\sigma}(j)}(y_{\sigma(j+1)}-y_{\sigma(j)}), \end{aligned}$$

the permutations ρ and σ are chosen such that:

$$0 < t_{\rho(1)} < t_{\rho(2)} < \ldots < t_{\rho(n)} \quad \text{and} \quad 0 < s_{\sigma(1)} < s_{\sigma(2)} < \ldots < s_{\sigma(n)},$$

$t_{\rho(n+1)} = t$, $s_{\sigma(n+1)} = s$, $x_{\rho(n+1)} = x$ and $y_{\sigma(n+1)} = y$.

Using (10) and (12), we conclude that $\alpha_n(t)$ admits the representation:

$$\alpha_n(t,x,s,y) = n!\, e^{ts} \sum_{i_1,\ldots,i_n \geq 1 \text{ distinct}} E^N \Bigg[\prod_{j=1}^{n} \eta(t-\tau_{i_j}, s-\rho_{i_j}) \langle G_{t-\tau_{i_1},\ldots,t-\tau_{i_n};x}, G_{s-\rho_{i_1},\ldots,s-\rho_{i_n};y} \rangle_{\mathscr{P}(\mathbb{R}^d)^{\otimes n}} I_{A_{i_1,\ldots,i_n}(t,s)} \Bigg]. \tag{13}$$

The next result gives a probabilistic representation of the inner product appearing in (13). This result is the analogue of (6), needed for the treatment of the fractional case.

Lemma 2.2. *For any $t_1, \ldots, t_n \in [0,t]$, $s_1, \ldots, s_n \in [0,s]$, $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$,*

$$\langle G_{t-t_1,\ldots,t-t_n;x}, G_{s-s_1,\ldots,s-s_n;y} \rangle_{\mathscr{P}(\mathbb{R}^d)^{\otimes n}} = E^{B^1,B^2} \left[w(t-t^*, B^1_{t^*}) w(s-s^*, B^2_{s^*}) \prod_{j=1}^{n} f(B^1_{t_j} - B^2_{s_j}) \right],$$

where $t^* = \max\{t_1, \ldots, t_n\}$, $s^* = \max\{s_1, \ldots, s_n\}$, and $B^1 = (B^1_t)_{t\in[0,1]}$ and $B^2 = (B^2_t)_{t\in[0,1]}$ are independent d-dimensional Brownian motions, starting from x, respectively y.

Proof: Let B^1 and B^2 be independent d-dimensional Brownian motions, starting from 0. Note that

$$I := \langle G_{t-t_1,\ldots,t-t_n;x}, G_{s-s_1,\ldots,s-s_n;y} \rangle_{\mathscr{P}(\mathbb{R}^d)^{\otimes n}} = \int_{\mathbb{R}^{2nd}} w(t-t_{\rho(1)}, x_{\rho(1)}) w(s-s_{\sigma(1)}, y_{\sigma(1)})$$

$$\prod_{j=1}^{n} f(x_j - y_j) p_{t_{\rho(j)}-t_{\rho(j+1)}}(x_{\rho(j)} - x_{\rho(j+1)}) p_{s_{\sigma(j)}-s_{\sigma(j+1)}}(y_{\sigma(j)} - y_{\sigma(j+1)}) d\mathbf{y} d\mathbf{x},$$

where the permutations ρ and σ of $\{1, \ldots, n\}$ are chosen such that

$$0 < t_{\rho(n)} < \ldots < t_{\rho(1)} < t \quad \text{and} \quad 0 < s_{\sigma(n)} < \ldots < s_{\sigma(1)} < s,$$

$t_{\rho(n+1)} = s_{\sigma(n+1)} = 0$, $x_{\rho(n+1)} = x$, $y_{\sigma(n+1)} = y$. We use the change of variables

$$x_{\rho(j)} - x_{\rho(j+1)} = z_{n+1-j}, \quad y_{\sigma(j)} - y_{\sigma(j+1)} = w_{n+1-j}, \quad j = 1, \ldots, n.$$

Note that $x_{\rho(j)} = x + \sum_{k=1}^{n+1-j} z_k$, i.e. $x_j = x + \sum_{k=1}^{n+1-\rho^{-1}(j)} z_k$. We get:

$$I = \int_{\mathbb{R}^{2nd}} w(t-t_{\rho(1)}, x + \sum_{k=1}^{n} z_k) w(s-s_{\sigma(1)}, y + \sum_{k=1}^{n} w_k)$$

$$\prod_{j=1}^{n} f\left((x + \sum_{k=1}^{n+1-\rho^{-1}(j)} z_k) - (y + \sum_{k=1}^{n+1-\sigma^{-1}(j)} w_k) \right)$$

$$\prod_{j=1}^{n} p_{t_{\rho(j)}-t_{\rho(j+1)}}(z_{n+1-j}) \prod_{j=1}^{n} p_{s_{\sigma(j)}-s_{\sigma(j+1)}}(w_{n+1-j}) d\mathbf{w} d\mathbf{z}.$$

We now use the fact that $p_{t-s}(x)dx$ is the density of the increment of a d-dimensional Brownian motion over the interval $(s,t]$, these increments over disjoint intervals are independent, and the Brownian motions B^1, B^2 are independent. Therefore, we replace the integral over $\mathbb{R}^{2nd}$ by the expectation E^{B^1,B^2}, and the variables z_k, w_k by $B^1_{t_{\rho(n+1-k)}} - B^1_{t_{\rho(n+1-k+1)}}$, respectively $B^2_{s_{\sigma(n+1-k)}} - B^2_{s_{\sigma(n+1-k+1)}}$, for all $k = 1, \dots, n$. Note that, for any $m = 1, \dots, n$

$$\sum_{k=1}^{m} (B^1_{t_{\rho(n+1-k)}} - B^1_{t_{\rho(n+1-k+1)}}) = B^1_{t_{\rho(n+1-m)}}$$

$$\sum_{k=1}^{m} (B^2_{s_{\sigma(n+1-k)}} - B^2_{s_{\sigma(n+1-k+1)}}) = B^2_{s_{\sigma(n+1-m)}}.$$

Hence,

$$I = E^{B^1,B^2}[w(t - t_{\rho(1)}, x + B^1_{t_{\rho(1)}}) w(s - s_{\sigma(1)}, y + B^2_{s_{\sigma(1)}}) \prod_{j=1}^{n} f((x + B^1_{t_j}) - (y + B^2_{s_j}))].$$

□

Conclusion of the Proof of Theorem 1.1: Since equation (7) has a solution, this solution is unique and admits the Wiener chaos expansion (3). The result follows from (11), (13) and Lemma 2.2. □

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