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THE ESCAPE MODEL ON A HOMOGENEOUS TREE.

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Abstract

There are two types of particles interacting on a homogeneous tree of degree $d+1$. The particles of the first type colonize the empty space with exponential rate 1, but cannot take over the vertices that are occupied by the second type. The particles of the second type spread with exponential rate λ. They colonize the neighboring vertices that are either vacant or occupied by the representatives of the opposite type, and annihilate the particles of the type 1 as they reach them. There exists a critical value $\lambda_c = (2d-1)+\sqrt{(2d-1)^2-1}$ such that the first type survives with positive probability for $\lambda < \lambda_c$, and dies out with probability one for $\lambda > \lambda_c$. We also find the growth profile which characterizes the rate of growth of the type 1 in the space-time on the event of survival.

1 Introduction

We consider a model of a predator-prey type which we call the Escape model. There are two entities growing on the vertices of a homogeneous tree $\mathcal{T}_d = \mathcal{T}$ of degree $d+1$. The entities may be thought of as biological species, political parties or manufacturers competing on a market. The second entity dominates the first in the sense that the representatives of the second entity can take over the vertices occupied by the representatives of the first entity but not vice versa. It is also assumed that the second entity grows faster. We are interested in the possibility of the long-term coexistence of the species which occurs when the first species survives.

There are several multi-type interacting particle systems for which either mutual coexistence or mutual unbounded growth were studied. In [3] and [4], Häggström and Pemantle introduced a two-type Richardson model on the integer lattice. Neuhauser [7] has considered a multi-type contact process. A two-type competition model was studied in [5]. In the models mentioned above, the particles interacted on the integer lattice $\mathbb{Z}^d$. In many cases, the integer lattice is a natural choice since it geometrically corresponds to the locations of the species representatives in space. However if we are dealing with several types of infections (or political ideas) in a population of people in a city, then the infections would spread between individuals who have contact with each other. Hence, a tree might be a better approximation to the graph of interactions than an integer lattice.

In the Escape model, at each time t each site of the tree is occupied by at most one representative of either of the two entities. We refer to the representatives of the entities as particles of types 1 and 2 respectively. If a site is not occupied, it is said to be *vacant*. The dynamics of the process is specified by exponential transition rates. A vacant site gets colonized by a particle of type 1 with exponential rate equal to the number of nearest neighbors of type 1. The sites that are either vacant or occupied by type 1 flip to 2 with the rate $\lambda > 1$ times the number of neighbors of type 2. If a vertex is occupied by a particle of type 2, the particle stays at the vertex forever. We assume that at time zero there are finitely many particles of each type. The sets of sites occupied by particles of types 1 and 2 at time $t \geq 0$ are denoted by $A(t)$ and $B(t)$ respectively. The event $\cap_{t\geq 0}\{A(t) \neq \emptyset\}$ is referred to as the event of *survival* of type 1. If at time zero the cluster $A(0)$ is surrounded by $B(0)$, then all particles of type 1 eventually die out with probability one. Thus, we are only interested in the initial configurations for which there exist a vertex x and an infinite geodesic segment $\gamma_{x,\infty}$ such that $x \in A(0)$ and $\gamma_{x,\infty} \cap B(0) = \emptyset$. All such configurations are referred to as *non-trivial* configurations. The question of interest is for what values of λ type 1 survives with positive probability in the long term.

Theorem 1. *There exists a critical value $\lambda_c = (2d-1) + \sqrt{(2d-1)^2 - 1}$ such that, for all $\lambda \in (1, \lambda_c)$ and for all non-trivial finite configurations $(A(0), B(0))$, type 1 survives with positive probability. For all $\lambda \in (\lambda_c, \infty)$, type 1 dies out with probability one.*

For $c > 0$ denote by $M_n(n/c)$ the number of vertices $x \in A(n/c)$ at distance n from the root ρ. From the results of Sections 4 and 5, it follows that

$$\lim \frac{1}{n} \log\left(\mathrm{E}\, M_n(n/c)\right) = -g(c).$$

where

$$g(c) = \begin{cases} (\lambda/c - \log(\lambda/c) - 1) - \log d & : \quad 0 < c \leq 1 \\ (\lambda/c - \log(\lambda/c) - 1) + (1/c - \log(1/c) - 1) - \log d & : \quad 1 < c < \lambda \\ (1/c - \log(1/c) - 1) - \log d & : \quad c \geq \lambda \end{cases}$$

The function $g(c)$ is referred to as the *growth profile* of type 1. The growth profile was introduced in Lalley [6] (in a slightly different form) to study the weakly supercritical contact process on a homogeneous tree. The function $g(c)$ has a unique minimum at c_0, is strictly decreasing on $(0, c_0)$ and strictly increasing on (c_0, ∞). For all $\lambda \in (1, \lambda_c)$, we have $g(c_0) < 0$, and let r_1 and r_2, with $0 < r_1 < r_2$, be the two solutions of $g(c) = 0$.

Theorem 2. *Let $\lambda \in (1, \lambda_c)$. For every $\epsilon > 0$ and all large t, the particles of type 1 are concentrated in the annulus of radii $(r_1 - \epsilon)t$ and $(r_2 + \epsilon)t$ centered at the root. For every $c \in (r_1, r_2)$, almost surely on the event of survival of type 1,*

$$\lim \frac{1}{n} \log\left(M_n(n/c)\right) = -g(c).$$

The growth profile gives the exponential rate of growth of the type 1 particles with time (on the event of survival). For every $c > 0$ such that $g(c) < 0$, at time t the number of type 1 particles at distance ct from the root is of order $\exp\{-cg(c)t + o(t)\}$. For every c with $g(c) > 0$ and all large t, at time t there are no particles of type 1 at distance ct from the root. The critical value λ_c is the smallest $\lambda > 1$ such that $g(c)$ is non-negative for all $c > 0$. To investigate the Escape model, we consider the simplest growth model, the Richardson model, presented in Section 3. In the Escape model, the cluster of type 2 particles evolves exactly as the infected set in Richardson model with rate $\lambda > 1$. Note also that the set $A(t)$ is dominated by Richardson model with rate 1. By using the comparison of the two independent Richardson processes with respective rates λ and 1, we compute the growth profile in Section 4.

The Escape model can be considered on the integer lattice $\mathbb{Z}^d$. It is easy to determine that the critical value does not depend on the dimension and is equal to 1. Change the rules of interactions in the Escape model so that the second entity is not able to occupy the vacant sites. This model is referred to as the Chase-Escape model. In this case, the particles of type 2 can be thought of as carnivores and the particles of type 1 as herbivores. The herbivores colonize the vacant area, while the carnivores try to chase and kill the herbivores. On the integer lattice $\mathbb{Z}^2$, the Chase-Escape model is very difficult to analyze rigorously. Even the existence of the critical value is not *a priori* obvious. Note that on a homogeneous tree the dynamics of the Chase-Escape and Escape models become essentially the same. Hence the critical rate λ_c exists and is explicitly computable on $\mathcal{T}$. Furthermore, the critical rate in the Chase-Escape model on the homogeneous tree of degree $2d$ can serve as an upper bound for the critical rate of the model on $\mathbb{Z}^d$. This is because in both graphs every vertex has exactly $2d$ nearest neighbors, but on the tree the escape routes of the prey do not intersect, and hence the prey has better chance of survival.

2 Preliminaries.

2.1 A homogeneous tree

A homogeneous tree $\mathcal{T}_d = \mathcal{T}$ of degree $d+1$ is an infinite tree such that every vertex has exactly $d+1$ nearest neighbors. A distinguished vertex is called *root* and denoted by ρ. For every two vertices x and y of the tree, denote by $|x, y|$ the number of edges in the shortest path from x to y (the path without loops). If x is the root, then we simply write $|y|$. Note that $|\cdot,\cdot|$ is a metric on $\mathcal{T}$.
For every vertex $x \in \mathcal{T}$, a *geodesic segment* $\gamma_{x,\infty}$ is an infinite path in $\mathcal{T}$ beginning at x and having no loops. Define by $\mathcal{T}_+(x)$, the set of all vertices y such that the shortest path connecting y with the root ρ goes through x. Consider a geodesic segment $\gamma_{x,\infty}$ such that $\gamma_{x,\infty} \in \mathcal{T}_+(x)$. Given an integer m, consider also a sequence of vertices $(y_k)_{k\geq 0}$ on the geodesic segment $\gamma_{x,\infty}$ such that $|y_k| = mk + |x|$ for all integers $k \geq 0$. For every $k \geq 1$, y_{k-1} is called the *m-predecessor* of y_k in $\mathcal{T}$.
Let $D(x, r) = \{y \in \mathcal{T} : |x, y| \leq r\}$ be the closed disk of radius $r \in (0, \infty)$ centered at x, and let $C(x, r) = \{y \in \mathcal{T} : |x, y| = r\}$ be the circumference of that disk. If x is the root, then we write D_r and C_r. Note that for all integers $n \geq 1$ the number of vertices in C_n is $(d+1)d^n$.

2.2 Construction of the Escape process.

The Escape process can be built using a percolation structure as follows. For each ordered pair of neighboring vertices x and y in $\mathcal{T}$, define two independent Poisson processes with respective rates 1 and $\lambda - 1$, and respective occurrence times $\{T_n^{x,y} : n \geq 1\}$ and $\{U_n^{x,y} : n \geq 1\}$. Set $T_0^{x,y} = 0$ and $U_0^{x,y} = 0$, and make these Poisson processes independent from pair to pair. Consider $\mathcal{T} \times \mathbb{R}_+$. Arrows are drawn from x to y at the occurrence times $T_n^{x,y}$ and $U_n^{x,y}$. We say that there is a *directed path* in $\mathcal{T} \times \mathbb{R}_+$ from (x_0, s_0) to (x_n, s_{n+1}) if there is a sequence of times $s_0 < s_1 < .. < s_{n+1}$ and sequence of vertices $x_0, x_1, .., x_n$ so that for each j, $1 \leq j \leq n$, there is an arrow from x_{j-1} to x_j at time s_j. A *type* 1 *path* is a directed path that uses only arrows generated by the Poisson processes T_n. For every vertex y we say that $y \in B(t)$ if and only if there is a vertex $x \in B(0)$ and directed path from $(x, 0)$ to (y, t) in the percolation structure. Define $A(t)$ to be the set of vertices y such that $y \notin B(t)$ and there is a type 1 path that ends at (y, t) and starts at $(z, 0)$ for some $z \in A(0)$.

3 The Richardson model on a homogeneous tree

The Richardson process on $\mathcal{T}$ with parameter $\lambda > 0$ is a continuous time Markov process $R(t)$ on the set of finite subsets of $\mathcal{T}$. We say that a vertex x is *infected* (or occupied) at time t if $x \in R(t)$, and is vacant otherwise. The process develops according to the following rules: if a vertex gets infected, it stays infected forever and starts infecting unoccupied neighboring vertices with rate λ, i.e. that the infection times have exponential distributions with parameter λ. The infection times are all independent. Consequently, a vacant site becomes infected with the rate

$$\lambda \cdot (\text{number of infected neighbors}).$$

The model with parameter $\lambda = c$ can be obtained from the model with parameter $\lambda = 1$ by running the later process at speed c. Therefore we treat just the case $\lambda = 1$.
Without loss of generality consider the initial configuration where at time zero the only occupied site is the root $R(0) = \{\rho\}$. The main questions were how fast the infected set grows and what limiting shape the infected set has. For the tree $\mathcal{T}$ the number of vertices in the disk D_n grows exponentially with n (whereas for $\mathbb{Z}^d$ it has polynomial growth). Consequently, the behavior of the Richardson process on $\mathcal{T}$ is different from the behavior of the model on the integer lattice (see [8] and [1]). The infected region $R(t)$ still grows linearly with time, but there are constants a and b, with $a < b$, such that as t goes to infinity we can classify two subregions: a completely infected subregion, having approximately the shape of a ball D_{ta}, and a partially infected subregion, having approximately the shape of a ring $D_{tb} \setminus D_{ta}$. Consequently, there are two speeds: the speed of invasion b, indicating how fast the infection spreads, and the speed of occupation a, governing the rate of growth of the region that is completely covered by the infection.

Proposition 1. *Assume that $\lambda = 1$, and let $d \geq 2$ be an integer. Let*

$$f(c) = \frac{1}{c} - \log\frac{1}{c} - 1 - \log d \tag{1}$$

for $c \in (0, \infty)$, and let a and b, with $0 < a < 1 < b$, be the two roots of the equation

$$f(c) = 0.$$

Then $a = \sup\{a'\}$ *and* $b = \inf\{b'\}$, *where the* sup *and* inf *are taken over all* a' *and* b' *satisfying*

$$\mathrm{P}[\exists \text{ a random } \tau < \infty \text{ such that } \forall t > \tau,\ D_{ta'} \subset R(t) \subset D_{tb'}] = 1.$$

As $d \to \infty$, *we have* $a(d)\log d \to 1$ *and* $b(d)/d \to e$.

Let $N_n(t)$ be the number of vertices at distance n from the root that are infected at time t, and let $F_n(t)$ be the number of vertices in C_n that are *not* infected at time t. We compute asymptotic values of $N_n(t)$ and $F_n(t)$ as n goes to infinity and $t = n/c$ for different values of c.

Proposition 2. *For all* $c \in (1, b)$,

$$\lim \frac{1}{n} \log\left(N_n(n/c)\right) = -f(c) > 0 \text{ a.s.}. \tag{2}$$

For all $c \in (a, 1)$,

$$\lim \frac{1}{n} \log\left(F_n(n/c)\right) = -f(c) > 0 \text{ a.s.}. \tag{3}$$

Proof of Proposition 1. It is enough to verify that for every $\epsilon \in (0, a)$

$$\mathrm{P}[\exists \tau < \infty : \ \forall t > \tau,\ D_{t(a-\epsilon)} \subset R(t) \subset D_{t(b+\epsilon)}] = 1, \tag{4}$$

$$\mathrm{P}[\exists \tau < \infty : \ \forall t > \tau,\ R(t) \subset D_{t(b-\epsilon)}] = 0, \tag{5}$$

$$\mathrm{P}[\exists \tau < \infty : \ \forall t > \tau,\ D_{t(a+\epsilon)} \subset R(t)] = 0. \tag{6}$$

Proof of equation (4). For every vertex x on the tree, let $T(x)$ be the (random) time at which the vertex gets infected. Consider an arbitrary vertex at distance n from the root, and denote it by x_n. Fix $c > 1$. Then we have the following estimate (recall that $\lambda = 1$):

$$\mathrm{P}\left[x_n \in R(n/c)\right] = \mathrm{P}\left[T(x_n) \le \frac{n}{c}\right] = \exp\left\{-n\left(\frac{1}{c} - \log\frac{1}{c} - 1\right) + o(n)\right\}$$

where

$$o(n)/n \to 0 \quad \text{as} \quad n \to \infty.$$

The estimate follows from Cramér's theorem for i.i.d. random variables (see [2]) and from the fact that $T(x_n)$ is distributed as a sum of n i.i.d. exponentials with parameter 1. Observe that

$$\begin{aligned}
\mathrm{E}\, N_n(n/c) &= (\text{the total number of vertices in } C_n) \cdot \mathrm{P}(x_n \in R(n/c)) \\
&= \frac{d+1}{d} d^n \exp\left\{-n\left(\frac{1}{c} - \log\frac{1}{c} - 1\right) + o(n)\right\} \\
&= \exp\{-nf(c) + o(n)\}
\end{aligned} \tag{7}$$

where f was defined in (1).
Similarly, for $c < 1$ we have:

$$\mathrm{P}[x_n \notin R(n/c)] = \mathrm{P}\left[T(x_n) \geq \frac{n}{c}\right] = \exp\left\{-n\left(\frac{1}{c} - \log\frac{1}{c} - 1\right) + o(n)\right\}.$$

Hence

$$\mathrm{E}\, F_n(n/c) = \exp\{-nf(c) + o(n)\}. \tag{8}$$

Observe that f is strictly decreasing on $(0,1)$ and strictly increasing on $(1,\infty)$, with unique minimum at $c = 1$. Moreover $f(1) < 0$ and $f(0+) = f(\infty-) = \infty$. Thus, there are just two roots a and b of the equation $f(c) = 0$, such that $a < 1 < b$. For every $\epsilon > 0$, we have that

$$\mathrm{P}\left[\exists x \in C_n :\ T(x) \leq \frac{n}{b+\epsilon}\right] \leq \mathrm{E}\, N_n(n/(b+\epsilon)).$$

Since $f(b+\epsilon) > 0$, by (7) the upper bound decays exponentially with n. Consequently the left side is summable, and by Borel-Cantelli lemma we have

$$\mathrm{P}\left[\exists N < \infty :\ \forall n > N \text{ and } \forall x \in C_n,\ T(x) > \frac{n}{b+\epsilon}\right] = 1.$$

Finally, observe that this is equivalent to

$$\mathrm{P}\left[\exists \tau < \infty :\ \forall t > \tau,\ R(t) \subset D_{t(b+\epsilon)}\right] = 1 \tag{9}$$

(the events are identical).
Analogously, for all $\epsilon \in (0, a)$,

$$\mathrm{P}\left[\exists \tau < \infty :\ \forall t > \tau,\ D_{t(a-\epsilon)} \subset R(t)\right] = 1. \tag{10}$$

To prove (10), note that

$$\mathrm{P}\left[\exists x \in C_n :\ T(x) > \frac{n}{a-\epsilon}\right] \leq \mathrm{E}\, F_n(n/(a-\epsilon)),$$

and it decays exponentially by (8). Applying Borel-Cantelli lemma one more time, we get

$$\mathrm{P}\left[\exists N < \infty :\ \forall n > N \text{ and } \forall x \in C_n,\ T(x) \leq \frac{n}{a-\epsilon}\right] = 1$$

which implies (10). Obviously, (9) and (10) together are equivalent to (4).
Equations (5) and (6) are direct consequences of Proposition 2. As a heuristic argument, note that (7) implies that, for every $c \in (1, b)$, $\mathrm{E}\, N_n(n/c)$ grows exponentially, and, similarly, (8) implies that for every $c \in (a, 1)$, $\mathrm{E}\, F_n(n/c)$ grows exponentially. These observations suggest that (5) and (6) should be true.
The limits

$$\lim_{d\to\infty} a(d) \log d = 1,$$

$$\lim_{d\to\infty} b(d)/d = e$$

immediately follow from the fact that a and b are the roots of

$$\frac{1}{c} - \log\frac{1}{c} - 1 - \log d = 0.$$

The phenomenon is easily anticipated. Since for larger d's there are more directions for the infection to spread around, it is natural that the invasion speed is strictly increasing with d. For the same reason, the occupation speed decreases to zero (the number of vertices in D_n grows unboundedly with d). $\square$

Proof of Proposition 2. We only prove (2). The proof of (3) is identical. First we claim that for any $\epsilon > 0$

$$\limsup \frac{1}{n}\log\left(N_n(n/c)\right) \leq -f(c) + \epsilon$$

almost surely. By Markov's inequality and (7),

$$\mathrm{P}\left[\frac{1}{n}\log\left(N_n(n/c)\right) > -f(c) + \epsilon\right] = \mathrm{P}\left[N_n(n/c) > \exp\left\{n(-f(c)+\epsilon)\right\}\right] \leq$$

$$\leq \mathrm{E}\left[N_n(n/c)\right]\cdot\exp\left\{nf(c) - n\epsilon\right\} = \exp\left\{-n\epsilon + o(n)\right\},$$

and the claim follows by Borel-Cantelli lemma.
Let $\alpha = \frac{1}{c}$. To finish the proof, it is enough to show that, for an arbitrarily small $\epsilon > 0$, with probability 1

$$\liminf \frac{1}{n}\log\left(N_n(\alpha n)\right) \geq -f\left(1/\alpha\right) - \epsilon \tag{11}$$

Recall that $1 < c < b$, so $f\left(1/\alpha\right) < 0$. Assume that ϵ is small enough to satisfy $f\left(1/\alpha\right) + \epsilon < 0$. To make the notation less complicated, let

$$\mu = \mu(\epsilon) = \exp\left\{-\left(f\left(1/\alpha\right) + \epsilon\right)\right\}.$$

By the continuity of f, there exists an $\epsilon_1 > 0$ such that

$$f\left(1/(\alpha - 2\epsilon_1)\right) < f\left(1/\alpha\right) + \epsilon < 0.$$

Then, for every fixed $w > 1$, there exists an integer $m > 0$ large enough such that

$$\mathrm{E}\,N_m((\alpha - 2\epsilon_1)m) > w\cdot\exp\left\{-m\left(f\left(1/\alpha\right) + \epsilon\right)\right\}$$

$$= w\cdot\mu^m > 1. \tag{12}$$

Choose m to satisfy (12). Fix an arbitrary vertex x of the tree and consider a geodesic segment $\gamma_{x,\infty} \in \mathcal{T}_+(x)$. Consider a sequence of vertices $(y_i)_{i\geq 0}$ on the geodesic segment $\gamma_{x,\infty}$ such that $|y_i| = mi + |x|$ for all integers $i \geq 0$. Note that infection times $T(y_i)$ are increasing in i. For every pair of non-negative integers n_1 and n_2 such that $n_1 < n_2$, say that y_{n_2} is an *m-descendant* of y_{n_1} if, for all integers $i \in [n_1,\ n_2)$,

$$T(y_{i+1}) - T(y_i) < \ (\alpha - 2\epsilon_1)m.$$

Define

$$\mathcal{Z}_k(x) = \left\{z \in C_{km+|x|} :\ z \text{ is an } m\text{-descendant of } x\right\},$$

$$Z_k(x) = \text{ cardinality of } \mathcal{Z}_k(x).$$

Note that $(Z_k(x))_{k\geq 0}$ is a Galton-Watson process with mean offspring number

$$\mathrm{E}\, Z_1(x) > w \cdot \mu^m > 1. \tag{13}$$

Claim 1. *For every vertex x, almost surely on the event of $(Z_k(x))_{k\geq 0}$ survival, there exists a (random) $K < +\infty$ such that, for all $k > K$,*

$$N_{mk+|x|}((\alpha - \epsilon_1)(mk + |x|)) \geq Z_k(x) \geq \mu^{mk+|x|} \cdot d^m. \tag{14}$$

Proof. The first inequality follows from the fact that for sufficiently large k's

$$(\alpha - \epsilon_1)(mk + |x|) \geq (\alpha - 2\epsilon_1)mk + T(x).$$

To obtain the second inequality, observe that, for all large enough k, $w^k \geq d^m \mu^{|x|}$, and hence, $\mathrm{E}\, Z_k(x) > (\mu^m \cdot w)^k \geq \mu^{mk+|x|} \cdot d^m$. A standard theorem from the elementary theory of Galton-Watson processes states that if $\mathrm{E}\, Z_1(x) > 1$ and the variance of $Z_1(x)$ is finite, then on the event of survival $Z_k^{(x)}/(\mathrm{E}\, Z_1^{(x)})^k$ converges almost surely to a positive random variable. Thus, the second inequality is obtained by direct application of (13). □

Therefore, on the event of non-extinction of $(Z_k(x))_{k\geq 0}$, (14) is true and implies (11) for n's of the form $mk + |x|$. To establish the result for all positive integers, consider $mk + |x| < n < m(k+1) + |x|$. Observe that K might be also chosen large enough that, for all $k > K$, we have

$$(\alpha - \epsilon_1)(m(k + 1) + |x|\ ) < \alpha(mk + |x|\ ).$$

By (14) at time $(\alpha - \epsilon_1)(m(k + 1) + |x|)$ we have at least

$$\mu^{m(k+1)+|x|} \cdot d^m$$

infected vertices on level $m(k + 1) + |x|$. Since each particle can generate at most d offspring, it follows that, for each integer n satisfying $mk + |x| < n < m(k + 1) + |x|$, there are at least

$$\mu^{m(k+1)+|x|}\ d^{n-(mk+|x|)}$$

infected predecessors in C_n at time $(\alpha - \epsilon_1)(m(k + 1) + |x|)$. Obviously,

$$\begin{aligned} N_n(\alpha n) &> N_n(\alpha(mk + |x|)) \\ &> N_n((\alpha - \epsilon_1)(m(k + 1) + |x|)) > \mu^{m(k+1)+|x|} > \mu^n. \end{aligned}$$

Therefore we proved that, almost surely on the event of the survival of $(Z_k(x))_{k\geq 0}$, (11) is true. To show that (11) holds with probability one, observe that, for each integer $j > 0$, there are $(d + 1)d^{mj-1}$ Galton-Watson processes $(Z_k(x))_{k\geq 0}$ with $|x| = mj$. Let S_j be the event of non-extinction for at least one of the processes. Since the Galton-Watson processes are independent, the probability of S_j tends to 1 as j tends to $+\infty$. Moreover, for each $j > 0$, $S_j \subset S_{j+1}$ which guarantees the almost sure result. □

4 Two independent Richardson models.

To motivate the proof of Theorem 1 consider two *independent* Richardson processes $R_1(t)$ and $R_\lambda(t)$ with respective rates 1 and $\lambda > 1$. The processes have initial configurations $R_1(0) = R_\lambda(0) = \{\rho\}$, and are built on a homogeneous tree $\mathcal{T}$.
For $c \in (0,\infty)$, estimate the expected number of vertices in $(R_1(n/c) \setminus R_\lambda(n/c)) \cap C_n$, that is the number of vertices at distance n from the root that are occupied by R_1 but not by R_λ at time n/c. Let x_n be a vertex with $|x_n| = n$, $u_n(n/c) = \mathrm{P}\left[x_n \in \{R_1(n/c) \setminus R_\lambda(n/c)\}\right]$, and $V_n(n/c) = \#\left\{x_n : x_n \in R_1(n/c) \setminus R_\lambda(n/c)\right\}$.
Case 1: For every $c \in (0,1]$,

$$u_n(n/c) = \mathrm{P}\left[x_n \notin R_\lambda(n/c)\right]\mathrm{P}\left[x_n \in R_1(n/c)\right] =$$

$$= \exp\left\{-n\left(\frac{\lambda}{c} - \log\frac{\lambda}{c} - 1\right) + o(n)\right\}.$$

Recall that the number of vertices in C_n is $\frac{d+1}{d}d^n$ and define

$$g_1(c) = \left(\frac{\lambda}{c} - \log\frac{\lambda}{c} - 1\right) - \log d.$$

Then, for all $c \in (0,1]$,

$$\mathrm{E}\, V_n(n/c) = \exp\left\{-n \cdot g_1(c) + o(n)\right\}.$$

Case 2: For every $c \in (1,\lambda)$

$$u_n(n/c) = \mathrm{P}\left[x_n \notin R_\lambda(n/c)\right]\mathrm{P}\left[x_n \in R_1(n/c)\right] =$$

$$\exp\left\{-n\left(\frac{\lambda}{c} - \log\frac{\lambda}{c} - 1\right) + o(n)\right\}\exp\left\{-n\left(\frac{1}{c} - \log\frac{1}{c} - 1\right) + o(n)\right\}.$$

Let

$$g_2(c) = \left(\frac{\lambda}{c} - \log\frac{\lambda}{c} - 1\right) + \left(\frac{1}{c} - \log\frac{1}{c} - 1\right) - \log d.$$

Then, for all $c \in (1,\lambda)$,

$$\mathrm{E}\, V_n(n/c) = \exp\left\{-n \cdot g_2(c) + o(n)\right\}.$$

Case 3: For every $c \in [\lambda,\infty)$,

$$u_n(n/c) = \exp\left\{-n\left(\frac{1}{c} - \log\frac{1}{c} - 1\right) + o(n)\right\}.$$

Thus,

$$\mathrm{E}\, V_n(n/c) = \exp\left\{-n \cdot g_3(c) + o(n)\right\}$$

where $g_3(c) = \left(\frac{1}{c} - \log\frac{1}{c} - 1\right) - \log d = f(c)$.
Define function $g(c)$ on $(0,\infty)$ by combining $g_1(c), g_2(c)$ and $g_3(c)$ on their domains. Note that for every $c > 0$ and non-negative integers m and n

$$u_m(m/c)u_n(n/c) \leq u_{m+n}((m+n)/c).$$

Hence

$$\mathrm{E}\, V_n(n/c) \le \exp\{-n \cdot g(c)\}\,. \tag{15}$$

The function $g(c)$ is continuously differentiable, strictly decreasing on $(0,(\lambda+1)/2)$ and strictly increasing on $((\lambda+1)/2,\infty)$ with the unique minimum at $c_0(\lambda) = \frac{\lambda+1}{2}$. Furthermore, $g_\lambda(c_0) = \log \frac{(\lambda+1)^2}{4\lambda d}$. It follows that

$$g(c_0) > 0 \text{ if } \lambda > (2d-1) + \sqrt{(2d-1)^2 - 1},$$

$$g(c_0) < 0 \text{ if } 1 < \lambda < (2d-1) + \sqrt{(2d-1)^2 - 1}.$$

Proposition 3. *Fix $\lambda > \lambda_c = (2d-1) + \sqrt{(2d-1)^2 - 1}$. Almost surely, for all sufficiently large t,*

$$R_1(t) \subset R_\lambda(t).$$

Proof. According to Proposition 1, for any $\epsilon_1 > 0$ and all large t,

$$R_\lambda(t) \supset D_{t(a\lambda-\epsilon_1)}.$$

Thus, we are only interested to see what happens in the region $D^c_{t(a\lambda-\epsilon_1)}$. Note that by (15), for every $\epsilon > 0$ and large n,

$$\mathrm{E}\left[\#\left\{x_n : x_n \in R_1(n/c) \setminus R_\lambda(n/c-1)\right\}\right] \le \exp\{-n \cdot g(c) + n\epsilon\}\,.$$

For large $t > 0$ and integers $i \ge 1$, let $c_i = i/t$. By Markov's inequality,

$$\mathrm{P}\left[(R_1(t) \setminus R_\lambda(t-1)) \cap D^c_{t(a\lambda-\epsilon_1)} \ne \emptyset\right] \le \sum_{i=[t(a\lambda-\epsilon_1)]}^{\infty} \exp\{-i \cdot g(c_i) + i\epsilon\} \le$$

$$\le \sum_{i=[t(a\lambda-\epsilon_1)]}^{\infty} \exp\{-ig(c_0) + i\epsilon\} \le C \exp\{-t(a\lambda-\epsilon_1)\cdot(g(c_0)-\epsilon)\}\,.$$

A routine application of Borel-Cantelli lemma for integer values of t implies the result. ☐

5 The Escape model.

Proof of Theorem 2. Since $A(t)$ can not grow faster than the infected set in the Richardson model with rate 1, from the results of Section 4 (use (15)) it follows that for any initial configuration $(A(0), B(0))$ and all sufficiently large n

$$\mathrm{E}\, M_n(n/c) \le \exp\{-n \cdot g(c) + n\epsilon\}\,.$$

Thus, using the same lines of argument as in Proposition 3, it may be shown that for all large t

$$A(t) \subset D^c_{t(r_1-\epsilon)},$$

$$A(t) \subset D_{t(r_2+\epsilon)}.$$

Furthermore, for every $c \in (r_1, r_2)$ and any $\epsilon > 0$

$$\limsup \frac{1}{n} \log\left(M_n(n/c)\right) \leq -g(c) + \epsilon.$$

(Similarly to the proof of Proposition 2, apply Markov's inequality to get

$$\mathrm{P}\left[\frac{1}{n} \log\left(M_n(n/c)\right) > -g(c) + \epsilon\right] = \mathrm{P}\left[M_n(n/c) > \exp\left\{n(-g(c)+\epsilon)\right\}\right] \leq$$

$$\leq \mathrm{E}\left[M_n(n/c)\right] \cdot \exp\left\{ng(c) - n\epsilon\right\} \leq \exp\left\{-n\epsilon\right\}$$

and the claim follows by Borel-Cantelli lemma.) Next, we show that on the event of survival

$$\liminf \frac{1}{n} \log\left(M_n(n/c)\right) \geq -g(c) - \epsilon. \tag{16}$$

Note that for every non-trivial configuration $(A(0), B(0))$, for all large t and all $x \in A(t)$ we have $\mathcal{T}_+(x) \cap B(t) = \emptyset$. Furthermore, almost surely on the event of type 1 survival, for every integer m there exist $0 < t < \infty$ and a vertex x with m-predecessor y such that $x \in A(t)$ and $B(t) \cap \mathcal{T}_+(y) = \emptyset$. Define $\mathcal{Z}_1(x)$ to be a subset of $C_{|x|+m} \cap \mathcal{T}_+(x)$ such that $z_1 \in \mathcal{Z}_1(x)$ if and only if there is a *type* 1 *path* from (x, t) to $(z_1, t + m/c)$ and there are no directed paths from (y, t) to $(x, t + m/c)$. In general, for $k \geq 2$, $\mathcal{Z}_k(x)$ is defined as a subset of vertices in $C_{|x|+mk} \cap \mathcal{T}_+(x)$ such that $z_k \in \mathcal{Z}_k(x)$ if and only if

1. $z_{k-1} \in \mathcal{Z}_{k-1}(x)$ where z_{k-1} is the m-predecessor of z_k;
2. there is a *type* 1 *path* from $(z_{k-1}, t + (k-1)m/c)$ to $(z_k, t + km/c)$;
3. there are no directed paths from $(z_{k-2}, t + (k-1)m/c)$ to $(z_{k-1}, t + km/c)$ where z_{k-2} is the m-predecessor of z_{k-1} .

Let $Z_k(x)$ be the cardinality of the set $\mathcal{Z}_k(x)$. It is clear from the definition that $(Z_k(x))_{k\geq 0}$ is a Galton-Watson process with the mean offspring number

$$\mathrm{E}[Z_1(x)] = \exp\{-mg(c) + o(m)\}.$$

Thus, for all sufficiently large m, $\mathrm{E}[Z_1(x)] > 1$. Note that $M_{|x|+mk}(t + km/c)$ dominates $Z_k(x)$, and hence, on the event of non-extinction of $(Z_k(x))_{k\geq 0}$, (16) holds. Observe that for every m, almost surely on the event of survival of the type 1, there are infinitely many vertices x at which the Galton-Watson processes $(Z_k(x))_{k\geq 0}$ can be originated. Hence, (16) holds almost surely on the event of non-extinction of the first type. This finishes the proof of Theorem 2. □

Proof of Theorem 1. Fix $\lambda \in (1, \lambda_c)$, and consider any non-trivial initial configuration. With positive probability, there exists a vertex x and a Galton-Watson process $(Z_k(x))_{k\geq 0}$ (constructed in the proof of Theorem 2) with $\mathrm{E}\, Z_1(x) > 1$. Hence, the Galton-Watson process survives with positive probability and so does type 1.
Consider the case $\lambda \in (\lambda_c, \infty)$. Since $A(t)$ is dominated by the Richardson model with rate 1, by Proposition 3 in Section 4 type 1 dies out almost surely. □

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References

[1] Cox, J.T. and Durrett, R (1981). *Some limit theorems for percolation processes with necessary and sufficient conditions.* Annals of Probability Vol. 9, 583-603.

[2] Dembo, A. and Zeitouni, O. (1993). *Large Deviations Techniques and Applications.* Jones and Bartlett.

[3] Häggström, O. and Pemantle, R. (1998). *First Passage Percolation and a Model for Competing Spatial Growth.* Journal of Applied Probability Vol. 35, 683–692.

[4] Häggström, O. and Pemantle, R. (2000). *Absence of mutual unbounded growth for almost all parameter values in the two-type Richardson model.* Stochastic Processes and their Applications 90, 207-222.

[5] Kordzakhia, G., Lalley, S. (2005). *A two-species competition model on* $\mathbb{Z}^d$. Stochastic Processes and their Applications Vol. 115, 781-796.

[6] Lalley, S. (1999). *Growth profile and invariant measures for the weakly supercritical contact process on a homogeneous tree.* Annals of Probability Vol. 27, 206-225.

[7] Neuhauser, C. (1992). *Ergodic theorems for the multitype contact process.* Probability Theory and Related Fields 91, 467-506.

[8] Richardson, D. (1973). *Random growth in tessellation.* Proceedings of Cambridge Philosophical Society 74, 515-528.