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MILD SOLUTIONS OF QUANTUM STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract
We introduce the concept of a mild solution for the right Hudson-Parthasarathy quantum stochastic differential equation, prove existence and uniqueness results, and show the correspondence between our definition and similar ideas in the theory of classical stochastic differential equations. The conditions that a process must satisfy in order for it to be a mild solution are shown to be strictly weaker than those for it to be a strong solution by exhibiting a class of coefficient matrices for which a mild unitary solution can be found, but for which no strong solution exists.

1 Introduction

One of the main analytical difficulties in the theory of stochastic differential equations (both classical and quantum) arises whenever the coefficients driving the equation consist of unbounded operators — a requirement that is largely unavoidable in the pursuit of interesting models. For example consider the linear SDE ([DIT],[DaZ]):

$$dX_t = AX_t\,dt + BX_t\,dW_t, \quad X_0 = \xi \tag{1.1}$$

where A is the generator of a strongly continuous semigroup $(S_t)_{t\geq 0}$ on some Hilbert space H, W is a Wiener process taking values in $\mathbb{R}$ (respectively some Hilbert space K, with covariance operator Q), and B is a linear map from $\operatorname{Dom} B \subset \mathsf{H}$ into H (resp. the Hilbert-Schmidt operators $Q^{1/2}(\mathsf{K}) \to \mathsf{H}$). An obvious definition of solution for (1.1) is any process $(X_t)_{t\geq 0}$ that satisfies the corresponding integral equation:

$$X_t = \xi + \int_0^t AX_s\,ds + \int_0^t BX_s\,dW_s,$$

in particular the two integrals on the right hand side must be well-defined, and for this to be true we must have $X_t \in \operatorname{Dom} A \cap \operatorname{Dom} B$ almost surely. So if both A and B are unbounded then any study of (1.1) must incorporate an investigation of how well their domains match up. An alternative route (considered in Chapter 6 of [DaZ]) is to introduce the following weaker notion of solution: a process X is a *mild solution* of (1.1) if it satisfies the following integral equation:

$$X_t = S_t\xi + \int_0^t S_{t-s}BX_s\,dW_s. \tag{1.2}$$

Note that for the above to make sense we no longer require that X_t lie in $\operatorname{Dom} A$ a.s., only that $X_t \in \operatorname{Dom} B$ a.s..
The purpose of this paper is to show that such ideas also have a role to play in the theory of *quantum* stochastic differential equations, in particular when considering the right Hudson-Parthasarathy (HP) equation:

$$dU_t = \sum_{\alpha,\beta=0}^{d} F^\alpha_\beta U_t\,d\Lambda^\beta_\alpha(t), \quad U_0 = 1. \tag{R}$$

Here $[\Lambda^\alpha_\beta]^d_{\alpha,\beta=0}$ is the matrix of fundamental noise processes of HP quantum stochastic calculus ([Mey],[Par]). Each component is a time-indexed family of operators acting on $\mathcal{F}$, the symmetric Fock space over $L^2(\mathbb{R}_+;\mathbb{C}^d)$, and they divide into four distinct groups:

Time:	$\Lambda^0_0(t) = t1,$	Annihilation:	$\Lambda^i_0(t) = A^i(t)$
Creation:	$\Lambda^0_j(t) = A^\dagger_j(t),$	Conservation:	$\Lambda^i_j(t) = N^i_j(t)$

$(i,j = 1,\ldots,d)$. Linear combinations of the creation and annihilation operators give realisations of Brownian motion; including the conservation processes leads to realisations of (compensated) Poisson processes. The coefficient matrix $[F^\alpha_\beta]$ is made up of (unbounded) operators acting on another Hilbert space $\mathfrak{h}$, and the solution process $U = (U_t)_{t\geq 0}$ consists of contraction operators on the tensor product Hilbert space $\mathfrak{h}\otimes\mathcal{F}$. In this paper we use the HP version of quantum stochastic calculus, and an essential part of the definition of the integral of an operator-valued process X against each of these noise processes (denoted $\int_0^t X_s\,d\Lambda^\alpha_\beta(s)$) is that $\bigcap_{t>0}\operatorname{Dom} X_t$ should contain a subspace of the form $\mathcal{D}\odot\mathcal{E}$, the algebraic tensor product of a dense subspace $\mathcal{D}\subset\mathfrak{h}$, and $\mathcal{E}\subset\mathcal{F}$, the linear span of the exponential vectors $\{\varepsilon(f) : f\in L^2(\mathbb{R}_+;\mathbb{C}^d)\}$. Thus the first step in giving rigorous meaning to (R) must be to view each F^α_β as an operator on $\mathfrak{h}\otimes\mathcal{F}$, thereby giving meaning to the term $F^\alpha_\beta U$. This can be done by first taking $F^\alpha_\beta\odot 1$, the algebraic ampliation of F^α_β with the identity operator on $\mathcal{F}$, and then, making the further assumption that each F^α_β is closable, taking the closure of the resulting operator which throughout we will denote by $F^\alpha_\beta\otimes 1$. Then, as defined in [FW], a *strong solution* of (R) on $\mathcal{D}$, a given dense subspace of $\mathfrak{h}$, is any process U such that the integral identity

$$U_t = 1 + \sum_{\alpha,\beta\geq 0}^{d}\int_0^t (F^\alpha_\beta\otimes 1)U_s\,d\Lambda^\beta_\alpha(s)$$

holds on $\mathcal{D}\odot\mathcal{E}$, in particular for each integral to be well-defined we must have

$$\bigcup_{t>0} U_t(\mathcal{D}\odot\mathcal{E}) \subset \bigcap_{\alpha,\beta\geq 0}\operatorname{Dom} F^\alpha_\beta\otimes 1. \tag{1.3}$$

Clearly this corresponds to the notion of strong solution given above for the equation (1.1).

In this paper we introduce a weaker notion of solution which, as in the classical case above, removes the restriction that U_t should map $\mathcal{D} \odot \mathcal{E}$ into $\operatorname{Dom} F_0^0 \otimes 1$, the domain of the time coefficient. Instead we demand this behaviour from the smeared operator $\int_0^t U_s\,ds$, so that a *mild solution* is a process U such that

$$\bigcup_{t>0} U_t(\mathcal{D} \odot \mathcal{E}) \subset \bigcap_{\alpha+\beta>0} \operatorname{Dom} F_\beta^\alpha \otimes 1, \tag{1.4}$$
$$\bigcup_{t>0} \int_0^t U_s\,ds(\mathcal{D} \odot \mathcal{E}) \subset \operatorname{Dom} F_0^0 \otimes 1,$$

and

$$U_t = 1 + (F_0^0 \otimes 1) \int_0^t U_s\,ds + \sum_{\alpha+\beta>0}^{d} \int_0^t (F_\beta^\alpha \otimes 1) U_s\,d\Lambda_\alpha^\beta(s).$$

We show how this relates to the classical notion of mild solution in Proposition 2.1. Moreover in Section 3 we show that this distinction between strong and mild solutions is nontrivial by exhibiting a class of matrices F for which it is possible to construct a mild unitary solution of (R), but for which no strong solution can exist.

The other main result of this paper (Theorem 2.3) is a general method for constructing mild solutions of (R). This is a modification of the method developed in [FW] for obtaining strong solutions, and both rely on the introduction of a positive self-adjoint operator C that behaves well with respect to the F_β^α. In particular in [FW] it was necessary to assume that $\operatorname{Dom} C^{1/2}$ is contained in $\bigcap_{\alpha,\beta\geq 0} \operatorname{Dom} F_\beta^\alpha$ in order to prove that (1.3) holds for $\mathcal{D} = \operatorname{Dom} C^{1/2}$. Here, since we need only prove that (1.4) holds, it suffices to assume that $\operatorname{Dom} C^{1/2}$ is contained in $\bigcap_{\alpha+\beta>0} \operatorname{Dom} F_\beta^\alpha$. That this is a significant weakening of the conditions imposed on C can be deduced (at least formally) by an application of the quantum Itô formula: for U to be a solution consisting of contractions it is necessary that F_0^0 have the same order of unboundedness as $(F_0^i)^* F_0^i$ and $F_j^0 (F_j^0)^*$ for $i, j = 1, \ldots d$, and that the conservation coefficients F_j^i be bounded (compare this with the classical equation (1.1), where again the Itô formula can be used to deduce that that time coefficient A is of the same order as B^*B.) For example, when realising diffusion processes in the quantum setting F_0^0 is taken to be a second-order differential operator and F_0^i, F_j^0 first-order. Thus a natural candidate for C when constructing a strong solution turns out to be $\partial^4 + 1$ (where ∂ denotes the differentiation operator on $L^2(\mathbb{R})$), but if we only require a mild solution then we may replace this by $\partial^2 + 1$, which is the same reference operator used to study the conservativity of the quantum Markov semigroup associated to the process U ([F1],[ChF],[F2]).

As a final remark it should be noted that we have chosen to work with the HP calculus, in part because the majority of results on QSDEs have been obtained in this setting. Other calculi have been developed, for example in the boson Fock space case the recent reformulation by Attal and Lindsay ([AtL]) identifies maximal domains for quantum stochastic integrals of any densely defined process, so in particular if the domain of the process contains the exponential vectors then its AL integral is an extension of the HP one. However the idea of mild solutions should also play a role in the future study of QSDEs using these alternative calculi.

1.1 Tensor product conventions

We maintain the conventions concerning tensor products used in [FW], namely that the symbol $\odot$ is used to denote algebraic tensor products, whereas $\otimes$ is reserved for the Hilbert space tensor product of Hilbert spaces and their vectors. Moreover, if S and T are closable operators on Hilbert spaces H and K respectively then $S \otimes T$ will denote the closure of the operator $S \odot T$ whose domain is the inner product space $\operatorname{Dom} S \odot \operatorname{Dom} T$. So in particular if $S \in \mathcal{B}(\mathsf{H})$ and $T \in \mathcal{B}(\mathsf{K})$, then $S \otimes T$ is the unique continuous extension of $S \odot T$ from $\mathsf{H} \odot \mathsf{K}$ to the Hilbert space $\mathsf{H} \otimes \mathsf{K}$. At times we will follow the trends prevelant in the literature and identify *bounded* operators with their ampliations whenever this causes no confusion.

2 Mild solutions of the right HP equation

2.1 Quantum stochastic calculus

Fix a Hilbert space $\mathfrak{h}$, called the *initial space*, and a number $d \geq 1$, the number of dimensions of quantum noise. Let $\mathcal{H} = \mathfrak{h} \otimes \mathcal{F}$, the Hilbert space tensor product of the initial space and $\mathcal{F} = \Gamma(L^2(\mathbb{R}_+; \mathbb{C}^d))$, the symmetric Fock space over $L^2(\mathbb{R}_+; \mathbb{C}^d)$. Put

$$\mathbb{M} = L^2(\mathbb{R}_+; \mathbb{C}^d) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{C}^d) \quad \text{and} \quad \mathcal{E} = \operatorname{Lin}\{\varepsilon(f) : f \in \mathbb{M}\},$$

where $\varepsilon(f) = ((n!)^{-1/2} f^{\otimes n})$ is the exponential vector associated to the test function f. Since the subspace $\mathbb{M}$ is dense in $L^2(\mathbb{R}_+; \mathbb{C}^d)$, it follows that $\mathcal{E}$ is a dense subspace of $\mathcal{F}$. The elementary tensor $u \otimes \varepsilon(f)$ will (usually) be abbreviated to $u\varepsilon(f)$ below.
A crucial ingredient of the HP quantum stochastic calculus is that all of the processes considered are adapted, a property that is defined through the continuous tensor product factorisation property of Fock space: for each $t > 0$ let

$$\mathcal{F}_t = \Gamma(L^2([0,t[; \mathbb{C}^d)), \quad \mathcal{F}^t = \Gamma(L^2([t,\infty[; \mathbb{C}^d)).$$

Then $\mathcal{F} = \mathcal{F}_t \otimes \mathcal{F}^t$ via continuous linear extension of the isometric map $\varepsilon(f) \mapsto \varepsilon(f|_{[0,t[}) \otimes \varepsilon(f|_{[t,\infty[})$; the spaces $\mathcal{F}_t$ and $\mathcal{F}^t$ are viewed as subspaces of $\mathcal{F}$ by tensoring with the appropriate vacuum vector $\varepsilon(0)$.
Let $\mathcal{D}$ be a dense subspace of $\mathfrak{h}$. An *operator process on* $\mathcal{D}$ is a family $X = (X_t)_{t \geq 0}$ of operators on $\mathcal{H}$ satisfying:

(i) $\mathcal{D} \odot \mathcal{E} \subset \bigcap_{t \geq 0} \operatorname{Dom} X_t$,

(ii) $t \mapsto \langle u\varepsilon(f), X_t v\varepsilon(g)\rangle$ is measurable,

(iii) $X_t v\varepsilon(g|_{[0,t[}) \in \mathfrak{h} \otimes \mathcal{F}_t$, and $X_t v\varepsilon(g) = [X_t v\varepsilon(g|_{[0,t[})] \otimes \varepsilon(g|_{[t,\infty[})$,

for all $u \in \mathfrak{h}, v \in \mathcal{D}, f, g \in \mathbb{M}$ and $t > 0$ — condition (iii) is the adaptedness condition. Any process that satisfies the further condition

(iv) $t \mapsto X_t v\varepsilon(g)$ is strongly measurable and $\int_0^t \|X_s v\varepsilon(g)\|^2 \, ds < \infty \; \forall t > 0$,

is called *stochastically integrable on* $\mathcal{D}$.
The stochastic integrals $\int_0^t X_s \, d\Lambda^\alpha_\beta(s)$ are defined for any stochastically integrable process X in [HuP], where $[\Lambda^\alpha_\beta]^d_{\alpha,\beta=0}$ are the fundamental noise processes defined with respect to the

standard basis of $\mathbb{C}^d$. The integral has domain $\mathcal{D} \odot \mathcal{E}$, and for all $u \in \mathfrak{h}, v \in \mathcal{D}, f, g \in \mathbb{M}$ and $t > 0$

$$\langle u\varepsilon(f), \int_0^t X_s \, d\Lambda^\alpha_\beta(s) \, v\varepsilon(g)\rangle = \int_0^t f_\beta(s) g^\alpha(s) \langle u\varepsilon(f), X_s v\varepsilon(g)\rangle \, ds. \tag{2.1}$$

Here $f^1, \ldots, f^d$ are the components of the $\mathbb{C}^d$-valued function f, and by convention $f^0 \equiv 1$ and $f_\alpha = \overline{f^\alpha}$. If Y is another process that is stochastically integrable on some subspace $\mathcal{D}'$, then, putting $I^X_t = \int_0^t X_s \, d\Lambda^\alpha_\beta(s)$ and $I^Y_t = \int_0^t Y_s \, d\Lambda^\mu_\nu(s)$,

$$\begin{aligned}\langle I^X_t u\varepsilon(f), I^Y_t v\varepsilon(g)\rangle = \int_0^t \Big\{ & f_\nu(s) g^\mu(s) \langle I^X_s u\varepsilon(f), Y_s v\varepsilon(g)\rangle \\ & + f_\alpha(s) g^\beta(s) \langle X_s u\varepsilon(f), I^Y_s v\varepsilon(g)\rangle + \widehat{\delta}^\beta_\nu f_\alpha(s) g^\mu(s) \langle X_s u\varepsilon(f), Y_s v\varepsilon(g)\rangle \Big\} \, ds \end{aligned} \tag{2.2}$$

for $u \in \mathcal{D}, v \in \mathcal{D}', f, g \in \mathbb{M}$, and where $\widehat{\delta} \in \mathrm{M}_{d+1}(\mathbb{C})$ is the Evans delta matrix defined by

$$\widehat{\delta}^\alpha_\beta = \begin{cases} 1, & 1 \le \alpha = \beta \le d \\ 0, & \text{otherwise} \end{cases}.$$

Finally we have the estimate

$$\|I^X_t u\varepsilon(f)\|^2 \le 2 \exp(\nu_f(t)) \int_0^t \|X_s u\varepsilon(f)\|^2 \, d\nu_f(s) \tag{2.3}$$

for all $u \in \mathcal{D}, f \in \mathbb{M}$ and $t > 0$, where $\nu_f(t) = \int_0^t (1 + \|f(s)\|^2) \, ds$. This implies in particular that the map $t \mapsto \int_0^t X_s \, d\Lambda^\alpha_\beta(s)\xi$ is continuous for all $\xi \in \mathcal{D} \odot \mathcal{E}$.

2.2 The right and left equations; notions of solution

As stated in the introduction our main concern in this paper is the *right HP equation* (R) determined by $F = [F^\alpha_\beta]$, a matrix of operators on $\mathfrak{h}$, although we shall encounter the *left equation*:

$$dV_t = \sum_{\alpha,\beta=0}^{d} V_t G^\alpha_\beta \, d\Lambda^\beta_\alpha(t), \quad V_0 = 1, \tag{L}$$

and in either case we shall only be concerned with contraction process solutions, that is processes U or V such that $\|U_t\| \le 1$ or $\|V_t\| \le 1$ for all t. If each F^α_β is densely defined then F^* will denote the *adjoint matrix* $[(F^\beta_\alpha)^*]$. Associated to any such matrix F of operators is the following subspace of $\mathfrak{h}$:

$$\mathrm{Dom}\,[F] := \bigcap_{\alpha,\beta \ge 0} \mathrm{Dom}\, F^\alpha_\beta.$$

Given a dense subspace $\mathcal{D} \subset \mathfrak{h}$ and a matrix of operators G, a contraction process V is a *strong solution to* (L) *on* $\mathcal{D}$ *for the operator matrix* G if

(Li) $\mathcal{D} \subset \mathrm{Dom}\,[G]$ and each process $(V_t G^\alpha_\beta)_{t \ge 0}$ is stochastically integrable on $\mathcal{D}$;

(Lii) $V_t = 1 + \sum_{\alpha,\beta \ge 0} \int_0^t V_s G^\alpha_\beta \, d\Lambda^\beta_\alpha(s)$ on $\mathcal{D} \odot \mathcal{E}$.

Because V is assumed to be a contraction process the matrix of processes $[V_t G^\alpha_\beta]$ is well-defined on $\mathcal{D}$, and so to check that (Li) holds it is sufficient to check that the maps $t \mapsto V_t G^\alpha_\beta \xi$ are strongly *measurable* for all $\alpha, \beta \geq 0$ and $\xi \in \mathcal{D} \odot \mathcal{E}$. However, note that as soon as (Lii) is shown to hold we know more, since then $t \mapsto V_t \eta$ is strongly continuous for all $\eta \in \mathfrak{h} \otimes \mathcal{F}$.
For the right equation (R) the situation is more delicate, as noted in [FW], since we must now pay attention to the image of U. If T is a closable operator on $\mathfrak{h}$ then $T \odot 1$, the algebraic tensor ampliation with the identity operator on $\mathcal{F}$, is again closable — see Section 1 of [FW] for a discussion of these matters. We will always assume that each component F^α_β of the stochastic generator in (R) is closable, and denote by $F \otimes 1$ the matrix $[F^\alpha_\beta \otimes 1]$ of closed operators on $\mathcal{H}$. With $\mathcal{D}$ as above, a contraction process U is a *strong solution of* (R) *on* $\mathcal{D}$ *for the operator matrix* F if:

(Ri) $\bigcup_{t>0} U_t(\mathcal{D} \odot \mathcal{E}) \subset \operatorname{Dom}[F \otimes 1]$, and each of the processes $(F^\alpha_\beta \otimes 1)U_t$ is stochastically integrable on $\mathcal{D}$;

(Rii) $U_t = 1 + \sum_{\alpha,\beta \geq 0} \int_0^t (F^\alpha_\beta \otimes 1) U_s \, d\Lambda^\beta_\alpha(s)$ on $\mathcal{D} \odot \mathcal{E}$.

The new notion of solution that we are introducing in this paper is the following: the process U is a *mild solution of* (R) *on* $\mathcal{D}$ *for the operator matrix* F if:

(Mi) $\bigcup_{t>0} U_t(\mathcal{D} \odot \mathcal{E}) \subset \bigcap_{\alpha+\beta>0} \operatorname{Dom} F^\alpha_\beta \otimes 1$, and each of the processes $(F^\alpha_\beta \otimes 1)U_t$ is stochastically integrable on $\mathcal{D}$;

(Mii) The map $t \mapsto U_t \xi$ is strongly measurable for all $\xi \in \mathcal{H}$, and $\int_0^t U_s \, ds (\mathcal{D} \odot \mathcal{E}) \subset \operatorname{Dom} F^0_0 \otimes 1$ for all $t > 0$;

(Miii) $U_t = 1 + (F^0_0 \otimes 1) \int_0^t U_s \, ds + \sum_{\alpha+\beta>0} \int_0^t (F^\alpha_\beta \otimes 1) U_s \, d\Lambda^\beta_\alpha(s)$ on $\mathcal{D} \odot \mathcal{E}$.

Note. The operator $\int_0^t U_s \, ds$ in the HP quantum stochastic calculus is defined by $\int_0^t U_s \, ds \xi := \int_0^t U_s \xi \, ds$ — the Bochner integral of a *vector*-valued function rather than the integral of an *operator*-valued function.

It is easy to see that any strong solution is also a mild solution. Also (2.3) implies that any strong solution must consist of a strongly continuous family of operators, but the presence of the term $(F^0_0 \otimes 1) \int_0^t U_s \, ds$ seems at first glance only to imply that a mild solution is weakly continuous. The next result allows us to improve on this by giving an alternative characterisation of mild solutions, once we make the reasonable assumption that F^0_0 is the generator of a strongly continuous contraction semigroup on $\mathfrak{h}$. We also justify our terminology since (2.4) below contains stochastic convolution terms analogous to those appearing in (1.2).

Proposition 2.1 *Let U be a contraction process, $F = [F^\alpha_\beta]$ a matrix of closable operators, and $\mathcal{D} \subset \mathfrak{h}$ a dense subspace. Suppose further that F^0_0 is the generator of a strongly continuous one-parameter semigroup of contractions $(P_t)_{t \geq 0}$ on $\mathfrak{h}$. The following are equivalent*:

(i) *U is a mild solution of* (R) *on $\mathcal{D}$ for this F.*

(ii) (Mi) *holds, and the integral identity*

$$U_t = P_t + \sum_{\alpha+\beta>0} \int_0^t P_{t-s} (F^\alpha_\beta \otimes 1) U_s \, d\Lambda^\beta_\alpha(s) \tag{2.4}$$

holds on $\mathcal{D} \odot \mathcal{E}$.

Moreover if $\mathrm{Dom}\,[F^*]$ *is a core for* $(F_0^0)^*$ *and either of the above hold then* U *is the unique mild solution on* $\mathcal{D}$ *for this* F.

Proof. Set $K = F_0^0$ throughout the proof.
(i $\Rightarrow$ ii): Fix $t \geq 0, u \in \mathrm{Dom}\,(K^*)^2, v \in \mathcal{D}$ and $f, g \in \mathbb{M}$. Then for all $s \in [0,t]$, (2.1) implies

$$\begin{aligned}\langle P_{t-s}^* u\varepsilon(f), U_s v\varepsilon(g)\rangle &= \langle P_{t-s}^* u\varepsilon(f), v\varepsilon(g)\rangle + \langle K^* P_{t-s}^* u\varepsilon(f), \int_0^s U_r v\varepsilon(g)\,dr\rangle \\ &\quad + \sum_{\alpha+\beta>0} \langle P_{t-s}^* u\varepsilon(f), \int_0^s f_\alpha(r) g^\beta(r)(F_\beta^\alpha \otimes 1) U_r v\varepsilon(g)\,dr\rangle,\end{aligned} \tag{2.5}$$

and since $P_{t-s}^* u\varepsilon(f) = \int_0^{t-s} K^* P_r^* u\varepsilon(f)\,dr$ it is straightforward to check that the map $s \mapsto \langle P_{t-s}^* u\varepsilon(f), U_s v\varepsilon(g)\rangle$ is absolutely continuous. Moreover, since each of the Bochner integrals that appears in (2.5) is a.e. differentiable, we have

$$\begin{aligned}\frac{d}{ds}\langle P_{t-s}^* u\varepsilon(f), U_s v\varepsilon(g)\rangle &= -\langle K^* P_{t-s}^* u\varepsilon(f), v\varepsilon(g)\rangle + \langle K^* P_{t-s}^* u\varepsilon(f), U_s v\varepsilon(g)\rangle \\ &\quad - \langle (K^*)^2 P_{t-s}^* u\varepsilon(f), \int_0^s U_r v\varepsilon(g)\,dr\rangle \\ &\quad + \sum_{\alpha+\beta>0} f_\alpha(s) g^\beta(s) \langle P_{t-s}^* u\varepsilon(f), (F_\beta^\alpha \otimes 1) U_s v\varepsilon(g)\rangle \\ &\quad - \sum_{\alpha+\beta>0} \langle K^* P_{t-s}^* u\varepsilon(f), \int_0^s f_\alpha(r) g^\beta(r)(F_\beta^\alpha \otimes 1) U_r v\varepsilon(g)\,dr\rangle\end{aligned}$$

for a.a. $s \in [0,t]$. Now the identity (2.5), with u replaced by $-K^* u$, appears on the right hand side of the above, and so cancellations give

$$\frac{d}{ds}\langle P_{t-s}^* u\varepsilon(f), U_s v\varepsilon(g)\rangle = \sum_{\alpha+\beta>0} f_\alpha(s) g^\beta(s) \langle u\varepsilon(f), P_{t-s}(F_\beta^\alpha \otimes 1) U_s v\varepsilon(g)\rangle$$

for a.a. $s \in [0,t]$. The processes $(P_{t-s}(F_\beta^\alpha \otimes 1) U_s)_{0 \leq s \leq t}$ are clearly stochastically integrable for all $\alpha + \beta > 0$, and so integrating over $[0,t]$ and applying (2.1) gives

$$\langle u\varepsilon(f), (U_t - P_t) v\varepsilon(g)\rangle = \sum_{\alpha+\beta>0} \langle u\varepsilon(f), \int_0^t P_{t-s}(F_\beta^\alpha \otimes 1) U_s\, d\Lambda_\alpha^\beta(s)\, v\varepsilon(g)\rangle,$$

as required.

(ii $\Rightarrow$ i): Let $u \in \mathrm{Dom}\,K^*, v \in \mathcal{D}$ and $f, g \in \mathbb{M}$. Then (2.1) applied to (2.4) gives

$$\begin{aligned}\langle u\varepsilon(f), U_t v\varepsilon(g)\rangle &= \langle P_t^* u\varepsilon(f), v\varepsilon(g)\rangle \\ &\quad + \sum_{\alpha+\beta>0} \int_0^t f_\alpha(s) g^\beta(s) \langle P_{t-s}^* u\varepsilon(f), (F_\beta^\alpha \otimes 1) U_s v\varepsilon(g)\rangle\,ds.\end{aligned} \tag{2.6}$$

Again the function $t \mapsto \langle u\varepsilon(f), U_t v\varepsilon(g)\rangle$ is absolutely continuous, and so

$$\begin{aligned}\frac{d}{dt}\langle u\varepsilon(f), U_t v\varepsilon(g)\rangle &= \langle K^* u\varepsilon(f), U_t v\varepsilon(g)\rangle \\ &\quad + \sum_{\alpha+\beta>0} f_\alpha(t) g^\beta(t) \langle u\varepsilon(f), (F_\beta^\alpha \otimes 1) U_t v\varepsilon(g)\rangle\end{aligned}$$

for a.a. t, since on differentiating (2.6) appears with u replaced by K^*u. Integrating this over $[0,t]$, and using (2.1) and the stochastic integrability assumptions on the non-time coefficients, yields

$$\langle u\varepsilon(f), (U_t - 1)v\varepsilon(g)\rangle = \langle (K^* \odot 1)u\varepsilon(f), \int_0^t U_s\, ds\, v\varepsilon(g)\rangle$$
$$+ \sum_{\alpha+\beta>0} \langle u\varepsilon(f), \int_0^t (F^\alpha_\beta \otimes 1)U_s\, d\Lambda^\beta_\alpha(s)\, v\varepsilon(g)\rangle,$$

and since $\mathrm{Dom}\, K^* \odot \mathcal{E}$ is a core for $K^* \otimes 1 = (K \otimes 1)^*$, we see that $\int_0^t U_s\, ds\, v\varepsilon(g) \in \mathrm{Dom}\, K \otimes 1$ and so U is a mild solution as required.

Finally, for the uniqueness part, if U is a mild solution to (R) on $\mathcal{D}$ for this F then it is easy to check that the adjoint process U^* is a weak solution of the adjoint left equation $dU^*_t = U^*_t (F^\beta_\alpha)^*\, d\Lambda^\beta_\alpha(t)$ on $\mathrm{Dom}\, [F^*]$. That is, the matrix elements $\langle u\varepsilon(f), U^*_t v\varepsilon(g)\rangle$ satisfy the same integral identity satisfied by any strong solution to this equation, but we do not demand that $t \mapsto U^*_t (F^\alpha_\beta)^* u\varepsilon(f)$ is strongly measurable, and hence stochastically integrable. However, if $\mathrm{Dom}\, [F^*]$ is a core for $K^* = (F^0_0)^*$ then there is at most one weak solution by Mohari's uniqueness result for the left HP equation ([Moh], Proposition 3.6; see also the remark after Proposition 2.2 of [FW]), which thus guarantees the uniqueness of the mild solution to (R). □

Remark. For the proof (ii ⇒ i) we need to know that $\int_0^t U_s\, ds$ is well-defined, i.e. that the map $t \mapsto U_t\xi$ is strongly measurable for all $\xi \in \mathfrak{h} \otimes \mathcal{F}$. But this is immediate from (2.4) — indeed this identity together with (2.3) can be used to show that $(U_t)_{t\geq 0}$ is *strongly* continuous.

2.3 Existence results

We now establish two existence results for mild solutions of (R). For the first we make the very strong assumption that the only unbounded term in F is the time coefficient F^0_0 (as happens in our example in Section 3), and so condition (Mi) becomes a triviality to verify. However the most interesting examples from a probabilistic or physical point of view do not satisfy this assumption — the creation and annihilation coefficients will typically be unbounded — and so Theorem 2.3 below shows how to adapt Theorem 2.3 of [FW] in order to be able to check that (Mi) holds in these cases.
The statements of both results involve form inequalities: given any operator matrix $G = [G^\alpha_\beta]$ and $S \in \mathcal{B}(\mathfrak{h})$, let $\theta_G(S)$ denote the form defined by

$$\theta_G(S)((u^\gamma),(v^\gamma)) = \sum_{\alpha,\beta=0}^{d} \Big\{ \langle u^\alpha, SG^\alpha_\beta v^\beta\rangle + \langle G^\beta_\alpha u^\alpha, Sv^\beta\rangle + \sum_{i=1}^{d} \langle G^i_\alpha u^\alpha, SG^i_\beta v^\beta\rangle \Big\},$$

with domain $\bigoplus_{\gamma=0}^d \mathrm{Dom}\, [G]$. Also, let $\iota(S)$ denote the identity form defined by

$$\iota(S)((u^\gamma),(v^\gamma)) = \sum_{\alpha\geq 0} \langle u^\alpha, Sv^\alpha\rangle,$$

that is $\iota(S) = S \otimes 1_{\mathbb{C}^{d+1}} \in \mathcal{B}(\mathfrak{h}^{d+1})$.

Theorem 2.2 *Assume that the initial space $\mathfrak{h}$ is separable, and let F be an operator matrix satisfying the following*:

(i) F_0^0 *is the generator of a strongly continuous one-parameter semigroup of contractions.*

(ii) $F_\beta^\alpha \in \mathcal{B}(\mathfrak{h})$ *whenever* $\alpha + \beta > 0$.

(iii) $\theta_{F^*}(1) \leq 0$ *on* $\bigoplus_{\gamma=0}^d \mathrm{Dom}\,[F^*]$.

Then there is a contraction process U that satisfies (R) *mildly on* $\mathfrak{h}$.

Proof. Since $\theta_{F^*}(1) \leq 0$ on $\bigoplus_{\gamma=0}^d \mathrm{Dom}\,[F^*]$ and $\mathfrak{h}$ is separable we can apply Theorem 3.6 of [F1] to show the existence of a contraction process U^* that is a strong solution to (L) on $\mathrm{Dom}\,(F_0^0)^*$ for the operator matrix F^*. Then (2.1) gives

$$\langle u\varepsilon(f), (U_t - 1)v\varepsilon(g)\rangle = \sum_{\alpha,\beta \geq 0} \int_0^t f_\alpha(s) g^\beta(s) \langle (F_\beta^\alpha)^* u\varepsilon(f), U_s v\varepsilon(g)\rangle\, ds$$

for all $u \in \mathrm{Dom}\,(F_0^0)^*$, $v \in \mathfrak{h}$ and $f, g \in \mathbb{M}$. Now $F_\beta^\alpha \otimes 1 \in \mathcal{B}(\mathcal{H})$ whenever $\alpha + \beta > 0$, so we have

$$\langle u\varepsilon(f), \Big[U_t - 1 - \sum_{\alpha+\beta>0} \int_0^t (F_\beta^\alpha \otimes 1) U_s\, d\Lambda_\alpha^\beta(s)\Big] v\varepsilon(g)\rangle = \langle (F_0^0)^* u\varepsilon(f), \int_0^t U_s\, ds\, v\varepsilon(g)\rangle,$$

and the result follows since $\mathrm{Dom}\,(F_0^0)^* \odot \mathcal{E}$ is a core for $(F_0^0 \otimes 1)^*$. □

In order to be able to deal with unbounded coefficients in the next theorem we follow the ideas of [FW] and introduce a positive self-adjoint operator C with which we can gain some control. Both the next result and Theorem 2.3 of [FW] make the same basic assumptions, namely that we hypothesise the existence of a family of (continuous) bounded maps $(f_\epsilon : [0,\infty[\to [0,\infty[)_{\epsilon>0}$ such that $f_\epsilon(x) \uparrow x$ as $\epsilon \downarrow 0$ for all $x \in [0,\infty[$, and which satisfy

$$\theta_F(f_\epsilon(C)) \leq b_1 \iota(f_\epsilon(C)) + b_2 1, \text{ and}$$
$$(F_\beta^\alpha)^* f_\epsilon(C)^{1/2} \text{ is bounded } \forall \alpha, \beta \geq 0,$$

where b_1, b_2 are positive constants that do not depend on ϵ. In [FW] we must also check that (Ri) holds, and so we also demand that $\mathrm{Dom}\, C^{1/2} \subset \mathrm{Dom}\,[\overline{F}]$, forcing C to be "as unbounded as" $(F_0^0)^2$. An appropriate choice for f_ϵ in this case is $f_\epsilon(x) = x(1+\epsilon x)^{-2}$. However we are now only looking for mild solutions, and so to satisfy (Mi) it is enough to assume $\mathrm{Dom}\, C^{1/2} \subset \bigcap_{\alpha+\beta>0} \mathrm{Dom}\, \overline{F_\beta^\alpha}$. Thus C will be of the same order as F_0^0, and so if $(F_\beta^\alpha)^* f_\epsilon(C)^{1/2}$ is to be bounded for all $\alpha, \beta \geq 0$, in particular for $\alpha = \beta = 0$, then we must use higher powers of the resolvent; a reasonable choice is to set

$$C^\epsilon := C(1+\epsilon C)^{-4} \quad \forall \epsilon > 0.$$

Theorem 2.3 *Let U be a contraction process and F a matrix of closable operators with F_0^0 the generator of a strongly continuous one-parameter semigroup of contractions. Suppose that C is a positive self-adjoint operator on $\mathfrak{h}$, and $\delta > 0$ and $b_1, b_2 \geq 0$ are constants such that the following hold*:

(i) *There is a dense subspace* $\mathcal{D} \subset \mathfrak{h}$ *that is a core for* $(F_0^0)^*$, *and the adjoint process* U^* *is a strong solution to* (L) *on* $\mathcal{D}$ *for the operator matrix* F^*.

(ii) *For each* $0 < \epsilon < \delta$ *there is a dense subspace* $\mathcal{D}_\epsilon \subset \mathcal{D}$ *such that* $(C^\epsilon)^{1/2}(\mathcal{D}_\epsilon) \subset \mathcal{D}$ *and* $(F_\beta^\alpha)^*(C^\epsilon)^{1/2}|_{\mathcal{D}_\epsilon}$ *is bounded for all* $\alpha, \beta \geq 0$.

(iii) $\operatorname{Dom} C^{1/2} \subset \bigcap_{\alpha+\beta>0} \operatorname{Dom} \overline{F_\beta^\alpha}$.

(iv) $\operatorname{Dom}[F]$ *is dense in* $\mathfrak{h}$, *and for all* $0 < \epsilon < \delta$ *the form* $\theta_F(C^\epsilon)$ *satisfies the inequality*

$$\theta_F(C^\epsilon) \leq b_1 \iota(C^\epsilon) + b_2 1$$

on $\operatorname{Dom}[F]$.

Then U *is a mild solution to the right equation* (R) *on* $\operatorname{Dom} C^{1/2}$ *for the operator matrix* F.

Remark. Since F_0^0 is the generator of a strongly continuous one-parameter semigroup of contractions on $\mathfrak{h}$, the fact that $\mathcal{D}$ is assumed to be a core for $(F_0^0)^*$ implies that there is at most one (strong) solution U^* to (L) on $\mathcal{D}$ for this F^* by the result of Mohari ([Moh], cf. the proof of our Proposition 2.1). The uniqueness of U^* implies that it is a Markovian cocycle and hence U^* and U are both strongly continuous.

Proof. To prove this result it is possible to recycle almost all of the argument used in the proof of Theorem 2.3 of [FW]. In particular the form $\theta_F(C^\epsilon)$ is bounded, and so the inequality in (iv) holds on all of $\mathfrak{h}$. Also the integral identity

$$U_t^*(C^\epsilon)^{1/2} = (C^\epsilon)^{1/2} + \sum_{\alpha,\beta\geq 0} \int_0^t U_s^*(F_\alpha^\beta)^*(C^\epsilon)^{1/2}\, d\Lambda_\alpha^\beta(s)$$

holds on $\mathfrak{h} \odot \mathcal{E}$, and since all of the terms appearing above are bounded we may take the adjoint of this expression and apply the quantum Itô formula (2.2), the Gronwall Lemma and the Spectral Theorem to conclude that

$$U_t(\operatorname{Dom} C^{1/2} \odot \mathcal{E}) \subset \operatorname{Dom} C^{1/2} \otimes 1.$$

Then by (iii) we have $U_t(\operatorname{Dom} C^{1/2} \odot \mathcal{E}) \subset \bigcap_{\alpha+\beta>0} \operatorname{Dom} F_\beta^\alpha \otimes 1$, and it is straightforward to show that the processes $\{(F_\beta^\alpha \otimes 1)U_t\}$ are stochastically integrable for $\alpha + \beta > 0$. So for all $u \in \mathcal{D}$, $v \in \operatorname{Dom} C^{1/2}$ and $f, g \in \mathbb{M}$

$$\langle u\varepsilon(f), (U_t - 1)v\varepsilon(g)\rangle = \sum_{\alpha,\beta\geq 0} \int_0^t f_\alpha(s) g^\beta(s) \langle (F_\beta^\alpha)^* u\varepsilon(f), U_s v\varepsilon(g)\rangle\, ds$$

since U^* is a (strong) solution of (L) for F^*, and so by what we have shown so far

$$\langle u\varepsilon(f), \Big[U_t - 1 - \sum_{\alpha+\beta>0} \int_0^t (F_\beta^\alpha \otimes 1) U_s\, d\Lambda_\alpha^\beta(s)\Big] v\varepsilon(g)\rangle = \langle (F_0^0)^* u\varepsilon(f), \int_0^t U_s\, ds\, v\varepsilon(g)\rangle.$$

The result follows once more because $\mathcal{D} \odot \mathcal{E}$ is a core for $(F_0^0 \otimes 1)^*$. □

3 A mild solution that cannot be a strong solution

We now show that for a certain class of operator matrices F one can construct a unitary mild solution of (R) on $\mathfrak{h}$, but that there is *no strong solution* for *any* choice of domain $\mathcal{D} \subset \mathfrak{h}$. To achieve this we need the following general result from semigroup theory:

Lemma 3.1 *Let $(T_r)_{r\geq 0}$ be a strongly continuous one-parameter semigroup on a Hilbert space H, and denote its generator by Z. Let $u \in \mathsf{H}$, then*

$$u \in \operatorname{Dom} Z \iff \sup_{r\in]0,1]} \|r^{-1}(T_r - 1)u\| < \infty.$$

Proof. One implication is obvious, so assume that $\sup_{r\in]0,1]} \|r^{-1}(T_r - 1)u\| < \infty$. Since T is strongly continuous the closed subspace $\mathsf{H}_0 = \overline{\operatorname{Lin}}\,\{T_r u : r \geq 0\}$ of H is separable. Let $(e_k)_{k\geq 1}$ be a basis of H_0, and $(r_n)_{n\geq 1}$ a sequence in $]0, \infty[$ with $\lim_n r_n = 0$. Then for each k the sequence $\{\langle e_k, r_n^{-1}(T_{r_n} - 1)u\rangle\}$ is bounded. A diagonalisation argument allows us to find a subsequence $(s_m)_{m\geq 1}$ of $(r_n)_{n\geq 1}$ and numbers $v_k \in \mathbb{C}$ such that

$$\lim_{m\to\infty} \langle e_k, s_m^{-1}(T_{s_m} - 1)u\rangle = v_k \quad \forall k \geq 1.$$

Fatou's Lemma implies

$$\sum_{k\geq 1} |v_k|^2 \leq \liminf_{m\to\infty} \sum_{k\geq 1} |\langle e_k, s_m^{-1}(T_{s_m} - 1)u\rangle|^2 = \liminf_{m\to\infty} \|s_m^{-1}(T_{s_m} - 1)u\|^2 < \infty$$

and so the series $\sum_{k\geq 1} v_k e_k$ converges to an element $v \in \mathsf{H}_0$, satisfying

$$\lim_{m\to\infty} \langle w, s_m^{-1}(T_{s_m} - 1)u\rangle = \langle w, v\rangle \quad \forall w \in \mathsf{H}.$$

It then follows by Theorem 1.24 of [Dav] that $u \in \operatorname{Dom} Z$, with $Zu = v$. □

Let $\mathfrak{h}$ be any separable Hilbert space, H an unbounded self-adjoint operator on $\mathfrak{h}$ and choose $u_0 \in \mathfrak{h}$ outside the domain of H. Let $d = 1$, so that we are working with only one dimension of quantum noise, and define an operator matrix F by

$$F_0^0 = iH - \frac{1}{2}E, \quad F_0^1 = E = -F_1^0, \quad F_1^1 = 0,$$

where E is the orthogonal projection onto the one-dimensional subspace spanned by u_0. Written as matrices we have

$$F = \begin{bmatrix} iH - \frac{1}{2}E & -E \\ E & 0 \end{bmatrix}, \quad F^* = \begin{bmatrix} -iH - \frac{1}{2}E & E \\ -E & 0 \end{bmatrix},$$

with $\operatorname{Dom}[F] = \operatorname{Dom}[F^*] = \operatorname{Dom} H$. Standard results from perturbation theory for semigroups show that F_0^0 is the generator of a contraction semigroup — a perturbation of the one-parameter unitary group with Stone generator H. Indeed F_0^0 generates a strongly continuous one-parameter *group* on $\mathfrak{h}$, denoted $(P_t)_{t\in\mathbb{R}}$, with $\|P_t\| \leq 1$ whenever $t \geq 0$.
From now on we will identify the bounded operators E and P_t with their ampliations $E \otimes 1$ and $P_t \otimes 1$, recalling that the generator of the one-parameter group $(P_t \otimes 1)_{t\in\mathbb{R}}$ is $F_0^0 \otimes 1$.

It is easy to check that the form inequality $\theta_{F^*}(1) \leq 0$ holds on $\mathrm{Dom}\, H \oplus \mathrm{Dom}\, H$, and so by Theorem 2.2 there is a contraction process U that satisfies (R) mildly on $\mathfrak{h}$ for this F, that is

$$U_t = 1 + (F_0^0 \otimes 1) \int_0^t U_s\, ds + \int_0^t EU_s dA_s^\dagger - \int_0^t EU_s dA_s$$

on $\mathfrak{h} \odot \mathcal{E}$, and all of the terms in the above make sense! In fact we have that $\theta_F(1) = 0$ and $\theta_{F^*}(1) = 0$, so that F^* satisfies the formal conditions for the process U to consist of unitary operators. That this is indeed the case follows from Section 5 of [F1] and Example 3.2 in [BhS]. That U cannot be a strong solution to (R) for any domain $\mathcal{D}$ will be proved using the following lemmas.

Lemma 3.2 *The identity*

$$P_{-r}U_t = P_{t-r} + \int_0^t P_{t-s-r} EU_s\, dA_s^\dagger - \int_0^t P_{t-s-r} EU_s\, dA_s$$

holds on $\mathfrak{h} \odot \mathcal{E}$ *for all* $t \geq 0$ *and* $r \in \mathbb{R}$.

Proof. For any stochastically integrable process X on $\mathcal{D}$ and $S \in \mathcal{B}(\mathfrak{h})$ we have

$$(S \otimes 1) \int_0^t X_s\, d\Lambda_\beta^\alpha(s) = \int_0^t (S \otimes 1) X_s\, d\Lambda_\beta^\alpha(s)$$

for all $\alpha, \beta \geq 0$ and $t > 0$, and so the result is immediate from Proposition 2.1. □

Lemma 3.3 *Let* $\phi, \psi : [0, \infty[\to \mathbb{R}$ *be continuous functions and* $k \geq 0$ *a positive constant. If*

$$\phi(t) \geq \phi(u) - k \int_u^t \phi(s)\, ds + \int_u^t \psi(s)\, ds \tag{3.1}$$

for all $0 \leq u \leq t < \infty$, *then*

$$\phi(t) \geq e^{-kt}\phi(0) + k \int_0^t e^{k(u-t)} \int_u^t \psi(s)\, ds\, du + e^{-kt} \int_0^t \psi(s)\, ds \tag{3.2}$$

for all $t \geq 0$.

Proof. Fix $t > 0$, and for each $u \in [0, t]$ let $\theta(u) = \int_u^t \phi(s)\, ds$. Then (3.1) can be rewritten as

$$-\theta'(t) \geq -\theta'(u) - k\theta(u) + \int_u^t \psi(s)\, ds$$

which implies that

$$\frac{d}{du}(e^{ku}\theta(u)) \geq e^{ku}\theta'(t) + e^{ku} \int_u^t \psi(s)\, ds$$

for all $u \in [0, t]$. Integrating over this interval gives

$$-\theta(0) = -\int_0^t \phi(s)\, ds \geq k^{-1}(e^{kt} - 1)\theta'(t) + \int_0^t e^{ku} \int_u^t \psi(s)\, ds\, du,$$

and substituting this inequality into (3.1) (with $u = 0$) gives (3.2). □

Lemma 3.4 *For each $t > 0$ let I_t denote the operator*

$$P_{-t}U_t = 1 + \int_0^t P_{-s}EU_s \, dA_s^\dagger - \int_0^t P_{-s}EU_s \, dA_s$$

with domain $\mathfrak{h} \odot \mathcal{E}$, and let $u \in \mathfrak{h}$. Then

$$\langle u, u_0 \rangle \neq 0 \Longrightarrow I_t u\varepsilon(f) \notin \operatorname{Dom} F_0^0 \otimes 1 \quad \forall f \in \mathbb{M}.$$

Proof. For each $r > 0$ set $S_r = r^{-1}(P_r - 1)$, then by Lemma 3.2 we have

$$S_r I_t = S_r I_u + \int_u^t S_r P_{-s}EU_s \, dA_s^\dagger - \int_u^t S_r P_{-s}EU_s \, dA_s$$

for all $0 \leq u \leq t$. Applying the quantum Itô formula (2.2) (with initial space $\mathfrak{h} \otimes \mathcal{F}_u$) gives

$$\begin{aligned}\|S_r I_t u\varepsilon(f)\|^2 = \|S_r I_u u\varepsilon(f)\|^2 &+ 4\int_u^t \operatorname{Im} f(s) \operatorname{Im} \langle S_r I_s u\varepsilon(f), S_r P_{-s}EU_s u\varepsilon(f)\rangle \, ds \\ &+ \int_u^t \|S_r P_{-s}EU_s u\varepsilon(f)\|^2 \, ds\end{aligned}$$

Now $\operatorname{Im}\langle \xi, \eta \rangle \geq -\frac{1}{2}\|\xi\|^2 - \frac{1}{2}\|\eta\|^2$ for any $\xi, \eta \in \mathcal{H}$, and so if we fix $T \geq 0$ then

$$\begin{aligned}\|S_r I_t u\varepsilon(f)\|^2 \geq \|S_r I_u u\varepsilon(f)\|^2 &- 8\|f|_{[0,T]}\|_\infty^2 \int_u^t \|S_r I_s u\varepsilon(f)\|^2 \, ds \\ &+ \frac{1}{2}\int_u^t \|S_r P_{-s}EU_s u\varepsilon(f)\|^2 \, ds\end{aligned}$$

for all $0 \leq u \leq t \leq T$. Applying Lemma 3.3 gives

$$\|S_r I_t u\varepsilon(f)\|^2 \geq \frac{1}{2}e^{-kt} \int_0^t \|S_r P_{-s}EU_s u\varepsilon(f)\|^2 \, ds$$

where $k = 8\|f|_{[0,T]}\|_\infty^2$. But S_r and P_{-s} commute, and since P_s is a contraction for each $s \geq 0$ we have

$$\|P_{-s}\xi\| \geq \|P_s\|\|P_{-s}\xi\| \geq \|\xi\| \quad \forall \xi \in \mathcal{H}, s \in [0, t].$$

Thus

$$\begin{aligned}\|S_r I_t u\varepsilon(f)\|^2 &\geq \frac{1}{2}e^{-kt} \int_0^t \|S_r EU_s u\varepsilon(f)\|^2 \, ds \\ &= \frac{1}{2}e^{-kt}\|S_r u_0\|^2 \int_0^t \|EU_s u\varepsilon(f)\|^2 \, ds.\end{aligned}$$

Finally note that $\sup_{r \in]0,1]} \|S_r u_0\|^2 = \infty$ by Lemma 3.1 and choice of u_0, and if $\langle u, u_0 \rangle \neq 0$ then the integral on the right hand side above is strictly positive; the result follows by another application of Lemma 3.1. □

Conclusion. Note that for each $t \in \mathbb{R}$ the operator P_t defines, by restriction, a bijective map of $\operatorname{Dom} F_0^0 \otimes 1$ onto itself. Thus the unitary process U *cannot* be a strong solution of (R) for this operator matrix F and *any* dense subspace $\mathcal{D} \subset \mathfrak{h}$, since for all $u \in \mathcal{D}$ satisfying $\langle u, u_0 \rangle \neq 0$ we have by Lemma 3.4 that $U_t u\varepsilon(f) \notin \operatorname{Dom} F_0^0 \otimes 1$, and the set of such u is dense in $\mathcal{D}$. But U is a mild solution of (R) on $\mathfrak{h}$, and so the uniqueness of mild solutions (Proposition 2.1) together with the elementary fact that any strong solution to (R) is necessarily a mild solution shows that *there are no strong solutions* to (R) for this F and *any* choice of domain $\mathcal{D}$.

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