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SLE AND TRIANGLES

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Abstract
By analogy with Carleson's observation on Cardy's formula describing crossing probabilities for the scaling limit of critical percolation, we exhibit "privileged geometries" for Stochastic Loewner Evolutions with various parameters, for which certain hitting distributions are uniformly distributed. We then examine consequences for limiting probabilities of events concerning various critical plane discrete models.

1 Introduction

It had been conjectured that many critical two-dimensional models from statistical physics are conformally invariant in the scaling limit; for instance, percolation, Ising/Potts models, FK percolation or dimers. The Stochastic Loewner Evolution (SLE) introduced by Oded Schramm in [Sch00] is a one-parameter family of random paths in simply connected planar domains. These processes are the only possible candidates for conformally invariant continuous limits of the aforementioned discrete models. See [RohSch01] for a discussion of explicit conjectures.

Cardy [Ca92] used conformal field theory techniques to predict an explicit formula (involving a hypergeometric function) that should describe crossing probabilities of conformal rectangles for critical percolation as a function of the aspect ratio of the rectangle. Carleson pointed out that Cardy's formula could be expressed in a much simpler way by choosing another geometric setup, specifically by mapping the rectangle onto an equilateral triangle ABC. The formula can then be simply described by saying that the probability of a crossing (in the triangle) between AC and BX for $X \in [BC]$ is BX/BC. Smirnov [Smi01] proved rigorously Cardy's formula for critical site percolation on the triangular lattice and his proof uses the global geometry of the equilateral triangle (more than the local geometry of the triangular lattice).

In the present paper, we show that each SLE_κ is in some sense naturally associated to some geometrical normalization in that the formulas corresponding to Cardy's formula can again be expressed in a simple way. Combining this with the conjectures on continuous limits of various discrete models, this yields precise simple conjectures on some asymptotics for these models in particular geometric setups. Just as percolation may be associated with equilateral

triangles, it turns that, for instance, the critical 2d Ising model (and the FK percolation with parameter $q = 2$) seems to be associated with right-angled isosceles triangles (because $\mathrm{SLE}_{16/3}$ hitting probabilities in such triangles are "uniform"). Other isosceles triangles correspond to FK percolation with different values of the q parameter. In particular, $q = 3$ corresponds to isosceles triangle with angle $2\pi/3$. Similarly, double dimer-models or $q = 4$ Potts models (conjectured to correspond to $\kappa = 4$) seem to be best expressed in strips (i.e., domains like $\mathbb{R} \times [0, 1]$), and half-strips (i.e., $[0, \infty) \times [0, 1]$) are a favorable geometry for uniform spanning trees.
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2 Chordal SLE

We first briefly recall the definition of chordal SLE in the upper half-plane $\mathbb{H}$ going from 0 to ∞ (see for instance [LawSchWer01, RohSch01] for more details). For any $z \in \mathbb{H}$, $t \geq 0$, define $g_t(z)$ by $g_0(z) = z$ and

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}$$

where $(W_t/\sqrt{\kappa}, t \geq 0)$ is a standard Brownian motion on $\mathbb{R}$, starting from 0. Let τ_z be the first time of explosion of this ODE. Define the hull K_t as

$$K_t = \overline{\{z \in \mathbb{H} \ : \ \tau_z < t\}}$$

The family $(K_t)_{t\geq 0}$ is an increasing family of compact sets in $\overline{\mathbb{H}}$; furthermore, g_t is a conformal equivalence of $\mathbb{H}\backslash K_t$ onto $\mathbb{H}$. It has been proved ([RohSch01], see [LawSchWer02] for the case $\kappa = 8$) that there exists a continuous process $(\gamma_t)_{t\geq 0}$ with values in $\overline{\mathbb{H}}$ such that $\mathbb{H}\backslash K_t$ is the unbounded connected component of $\mathbb{H} \setminus \gamma_{[0,t]}$, a.s. This process is the trace of the SLE and it can recovered from g_t (and therefore from W_t) by

$$\gamma_t = \lim_{z \in \mathbb{H} \to W_t} g_t^{-1}(z)$$

For any simply connected domain D with two boundary points (or prime ends) a and b, chordal SLE_κ in D from a to b is defined as $K_t^{(D,a,b)} = h^{-1}(K_t^{(\mathbb{H},0,\infty)})$, where $K_t^{(\mathbb{H},0,\infty)}$ is as above, and h is a conformal equivalence of (D, a, b) onto $(\mathbb{H}, 0, \infty)$. This definition is unambiguous up to a linear time change thanks to the scaling property of SLE in the upper half-plane (inherited from the scaling property of the driving process W_t).

3 A normalization of SLE

The construction of SLE relies on the conformal equivalence g_t of $\mathbb{H}\backslash K_t$ onto $\mathbb{H}$. As $\mathbb{H}$ has non-trivial conformal automorphisms, one can choose other conformal mappings. The original g_t is natural as all points of the real line seen from infinity play the same role (hence the driving process (W_t) is a Brownian motion). Other normalizations, such as the one used in [LawSchWer01] may prove useful for different points of view.
A by-product of Smirnov's results ([Smi01]) is the following: let $\kappa = 6$, and F be the conformal mapping of $(\mathbb{H}, 0, 1, \infty)$ onto an equilateral triangle (T, a, b, c). Let h_t be the conformal automorphism of (T, a, b, c) such that $h_t(F(W_t)) = a$, $h_t(F(g_t(1))) = b$, $h_t(c) = c$. Then, for any

$z \in \mathbb{H}$, $h_t(F(g_t(z)))$ is a local martingale. Our goal in this section is to find similar functions F for other values of κ.
Recall the definitions and notations of section 2. For $t < \tau_1$, consider the conformal mapping of $\mathbb{H}\backslash K_t$ onto $\mathbb{H}$ defined as:

$$\tilde{g}_t(z) = \frac{g_t(z) - W_t}{g_t(1) - W_t}$$

so that $\tilde{g}_t(\infty) = \infty$, $\tilde{g}_t(1) = 1$ and $\tilde{g}_t(\gamma_t) = 0$, where γ_t is the SLE trace.
Notice that if F is an holomorphic map $D \to \mathbb{C}$ and $(Y_t)_{t\geq 0}$ is a D-valued semimartingale, then (the bivariate real version of) Itô's formula yields:

$$dF(Y_t) = \frac{dF}{dz} dY_t + \frac{1}{2}\frac{d^2F}{dz^2} d\langle Y_t\rangle$$

where the quadratic covariation $\langle .,.\rangle$ for real semimartingales is extended in a $\mathbb{C}$-bilinear fashion to complex semimartingales:

$$\langle Y_1, Y_2\rangle = (\langle \Re Y_1, \Re Y_2\rangle - \langle \Im Y_1, \Im Y_2\rangle) + i(\langle \Re Y_1, \Im Y_2\rangle + \langle \Im Y_1, \Re Y_2\rangle)$$

so that $d\langle C_t\rangle = 0$ for an isotropic complex Brownian motion (C_t). The setup here is slightly different from conformal martingales as described in [RevYor94].
In the present case, one gets:

$$d\tilde{g}_t(z) = \left[\frac{2}{\tilde{g}_t(z)} - 2\tilde{g}_t(z) + \kappa(\tilde{g}_t(z) - 1)\right] \frac{dt}{(g_t(1) - W_t)^2} + (\tilde{g}_t(z) - 1)\frac{dW_t}{g_t(1) - W_t}$$

For notational convenience, define $w_t = \tilde{g}_t(z)$. After performing the time change

$$u(t) = \int_0^t \frac{ds}{(g_s(1) - W_s)^2}$$

one gets the autonomous SDE:

$$dw_u = (w_u - 1)\left[\kappa - \frac{2}{w_u}(1 + w_u)\right] du + (w_u - 1) d\tilde{W}_u$$

where $(\tilde{W}_u/\sqrt{\kappa})_{u\geq 0}$ is a standard Brownian motion.
Let us take a closer look at the time change. Let $Y_t = g_t(1) - W_t$; then, $dY_t = -dW_t + 2dt/Y_t$, so that $(Y_t/\sqrt{\kappa})_{t\geq 0}$ is a Bessel process of dimension $(1+4/\kappa)$. For $\kappa \leq 4$, this dimension is not smaller than 2, so that Y almost surely never vanishes (see e.g. [RevYor94]); moreover, a.s.,

$$\int_0^\infty \frac{dt}{Y_t^2} = \infty$$

Indeed, let $T_n = \inf\{t > 0 \;:\; Y_t = 2^n\}$. Then, the positive random variables $(\int_{T_n}^{T_{n+1}} dt/Y_t^2, n \geq 1)$ are i.i.d. (using the Markov and scaling properties of Bessel processes). Hence:

$$\int_0^\infty \frac{dt}{Y_t^2} \geq \sum_{n=1}^\infty \int_{T_n}^{T_{n+1}} \frac{dt}{Y_t^2} = \infty \quad \text{a.s.}$$

So the time change is a.s. a bijection from $\mathbb{R}_+$ onto $\mathbb{R}_+$ if $\kappa \leq 4$.

When $\kappa > 4$, the dimension of the Bessel process Y is smaller than 2, so that $\tau_1 < \infty$ almost surely. In this case, using a similar argument with the stopping times T_n for $n < 0$, one sees that

$$\int_0^{\tau_1} \frac{dt}{Y_t^2} = \infty$$

Hence, if $\kappa > 4$, the time change is a.s. a bijection $[0, \tau_1) \to \mathbb{R}_+$.
We conclude that for all $\kappa > 0$, the stochastic flow $(\tilde{g}_u)_{u \geq 0}$ does almost surely not explode in finite time.
We now look for holomorphic functions F such that $(F(w_u))_{u \geq 0}$ are local martingales. As before, one gets:

$$dF(w_u) = \left[F'(w_u)(\kappa - \frac{2}{w_u}(1 + w_u)) + \frac{\kappa}{2} F''(w_u)(w_u - 1) \right] (w_u - 1)du + F'(w_u)(w_u - 1)d\tilde{W}_u$$

Hence we have to find holomorphic functions defined on $\mathbb{H}$ satisfying the following equation:

$$F'(w) \left[\kappa - \frac{2}{w}(1 + w) \right] + F''(w) \frac{\kappa}{2}(w - 1) = 0$$

The solutions are such that

$$F'(w) \propto w^{\alpha - 1}(w - 1)^{\beta - 1},$$

where

$$\begin{cases} \alpha &= 1 - \frac{4}{\kappa} \\ \beta &= \frac{8}{\kappa} - 1 \end{cases}$$

For $\kappa = 4$, $F(w) = \log(w)$ is a solution.

4 Privileged geometries

In this section we attempt to identify the holomorphic map F depending on the value of the κ parameter.

- **Case** $4 < \kappa < 8$

  Using the Schwarz-Christoffel formula [Ahl79], one can identify F as the conformal equivalence of $(\mathbb{H}, 0, 1, \infty)$ onto an isosceles triangle (T_κ, a, b, c) with angles $\hat{a} = \hat{c} = \alpha\pi = (1 - \frac{4}{\kappa})\pi$ and $\hat{b} = \beta\pi = (\frac{8}{\kappa} - 1)\pi$. Special triangles turn out to correspond to special values of κ. Thus, for $\kappa = 6$, one gets an equilateral triangle, as was foreseeable from Smirnov's work ([Smi01]). For $\kappa = \frac{16}{3}$, a value conjectured to correspond to FK percolation with $q = 2$ and to the Ising model, one gets an isorectangle triangle.

  Since $F(\mathbb{H})$ is bounded, the local martingales $F(\tilde{g}_{t \wedge \tau_1}(z))$ are bounded (complex-valued) martingales, so that one can apply the optional stopping theorem. We therefore study what happens at the stopping time $\tau_{1,z} = \tau_1 \wedge \tau_z$. There are three possible cases, each having positive probability: $\tau_1 < \tau_z$, $\tau_1 = \tau_z$ and $\tau_1 > \tau_z$. Clearly, $\lim_{t \nearrow \tau_z}(g_t(z) - W_t) = 0$, and on the other hand $(g_t(z) - W_t)$ is bounded away from zero if t stays bounded away from τ_z. Recall that

$$\tilde{g}_t(z) = \frac{g_t(z) - W_t}{g_t(1) - W_t}$$

So, as $t \nearrow \tau_{1,z}$, $\tilde{g}_t(z) \to \infty$ if $\tau_1 < \tau_z$ and $\tilde{g}_t(z) \to 0$ if $\tau_z < \tau_1$. In the case $\tau_1 = \tau_z = \tau$, the points 1 and z are disconnected at the same moment, with $\gamma_\tau \in \partial\mathbb{H}$. As $t \nearrow \tau$, the harmonic measure of $(-\infty, 0)$ seen from z tends to 0; indeed, to reach $(-\infty, 0)$, a Brownian motion starting from z has to go through the straits $[\gamma_t, \gamma_\tau]$ the width of which tends to zero. At the same time, the harmonic measures of $(0,1)$ and $(1,\infty)$ seen from z stay bounded away from 0. This implies that $\tilde{g}_t(z)$ tends to 1, as is easily seen by mapping $\mathbb{H}$ to strips.

Now one can apply the optional stopping theorem to the martingales $F(\tilde{g}_{t\wedge\tau_{1,z}}(z))$. The mapping F has a continuous extension to $\overline{\mathbb{H}}$, hence:

$$F(z) = F(0)\mathbb{P}(\tau_z < \tau_1) + F(1)\mathbb{P}(\tau_z = \tau_1) + F(\infty)\mathbb{P}(\tau_z > \tau_1)$$

Thus:

Proposition 1.
The barycentric coordinates of $w = F(z)$ in the triangle T_κ are the probabilities of the events $\tau_z < \tau_1$, $\tau_z = \tau_1$, $\tau_z > \tau_1$.

Define $T^0 = \{w \in T_\kappa \;:\; \tau_z < \tau_1\}$, $T^1 = \{w \in T_\kappa \;:\; \tau_z = \tau_1\}$, $T^\infty = \{w \in T_\kappa \;:\; \tau_z > \tau_1\}$, which is a random partition of T_κ. These three sets are a.s. borelian; indeed, $T^\infty = F(\mathbb{H}\backslash K_{\tau_1})$ is a.s. open, and $T^0 = \bigcup_{t<\tau_1} K_t$ is a.s. an F_σ borelian. The integral of the above formula with respect to the Lebesgue measure on T_κ yields:

Corollary 1.
The following relation holds:

$$\mathbb{E}(\mathcal{A}(T^0)) = \mathbb{E}(\mathcal{A}(T^1)) = \mathbb{E}(\mathcal{A}(T^\infty)) = \frac{\mathcal{A}(T_\kappa)}{3}$$

where $\mathcal{A}$ designates the area.

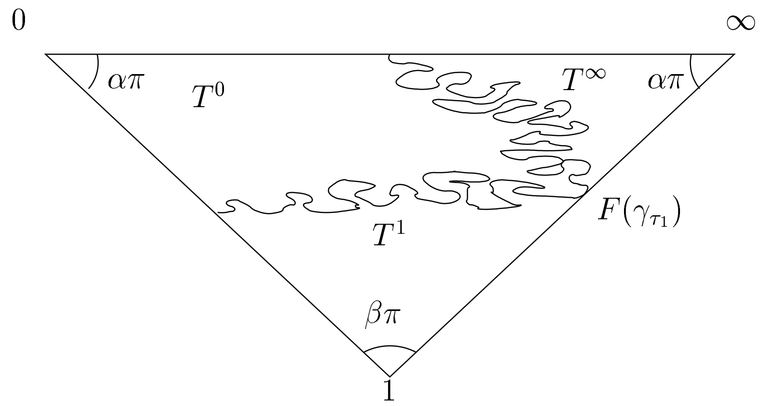

Figure 1: The random partition

Another easy consequence is a Cardy's formula for SLE.

Corollary 2 (Cardy's Formula).
Let γ be the trace of a chordal SLE_κ *going from a to c in the isosceles triangle T_κ, $4 < \kappa < 8$. Let τ be the first time γ hits (b, c). Then γ_τ has uniform distribution on (b, c).*

One can translate this result on the usual half-plane setup.

Corollary 3. *Let γ be the trace of a chordal* SLE_κ *going from 0 to ∞ in the half-plane, and γ_{τ_1} be the first hit of the half-line $[1, \infty)$ by γ. Then, if $4 < \kappa < 8$, the law of $1/\gamma_{\tau_1}$ is that of the beta distribution $B(1 - \frac{4}{\kappa}, \frac{8}{\kappa} - 1)$.*

It is easy to see that, the law of γ_{τ_1} converges weakly to δ_1 when $\kappa \nearrow 8$. This is not surprising as for $\kappa \geq 8$, the SLE trace γ is a.s. a Peano curve, and $\gamma_{\tau_1} = 1$ a.s.

- **Case** $\kappa = 4$

In this case, $F(w) = \log(w)$ is a solution. One can choose a determination of the logarithm such that $\Im(\log(\mathbb{H})) = (0, \pi)$. Then $\Im(\log(\tilde{g}_t(z))) = \arg(\tilde{g}_t(z))$ is a bounded local martingale. Let $\mathbb{H}_r$ (resp. $\mathbb{H}_l$) be the points in $\mathbb{H}$ left on the right (resp. on the left) by the SLE trace (a precise definition is to be found in [Sch01]). If $z \in \mathbb{H}_l$, the harmonic measure of $g_t^{-1}((W_t, \infty))$ seen from z in $\mathbb{H}\backslash\gamma_{[0,t]}$ tends to 0 as $t \to \infty = \tau_z$. This implies that the argument of $\tilde{g}_t(z)$ tends to π. For $z \in \mathbb{H}_r$, an argument similar to the case $4 < \kappa < 8$ shows that $\tilde{g}_t(z) \to 1$. Hence, applying the optional stopping theorem to the bounded martingale $\arg(\tilde{g}_t(z))$, one gets:

$$\arg(z) = 0 \times \mathbb{P}(z \in \mathbb{H}_r) + \pi\mathbb{P}(z \in \mathbb{H}_l)$$

or $\mathbb{P}(z \in \mathbb{H}_l) = \arg(z)/\pi$, in accordance with [Sch01].

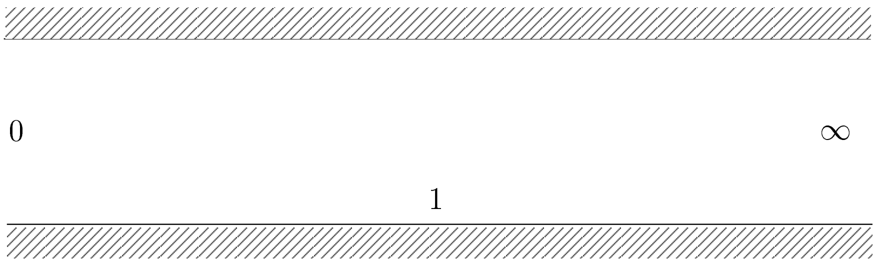

Figure 2: $F(\mathbb{H})$, case $\kappa = 4$: slit

- **Case** $\kappa = 8$

Let $F(z) = \int w^{-\frac{1}{2}}(w-1)^{-1}dw$; F maps $(\mathbb{H}, 0, 1, \infty)$ onto a half-strip (D, a, ∞, b). One may choose F so that $F(\mathbb{H}) = \{z \;:\; 0 < \Re z < 1, \Im z > 0\}$. Then $F(\infty) = 0$ and $F(0) = 1$. Moreover, $\Re F(\tilde{g}_t(z))$ is a bounded martingale. In the case $\kappa \geq 8$, it is known that $\tau_1 < \infty$, $\tau_z < \infty$, and $\tau_1 \neq \tau_z$ a.s. if $z \neq 1$ (see [RohSch01]). Hence, if $\tau = \tau_1 \wedge \tau_z$, $\tilde{g}_\tau(z)$ equals 0 or ∞, depending on whether $\tau_z < \tau_1$ or $\tau_z > \tau_1$. Applying the optional stopping theorem to the bounded martingale $\Re F(\tilde{g}_t(z))$, one gets:

$$\mathbb{P}(\tau_z < \tau_1) = \Re F(z)$$

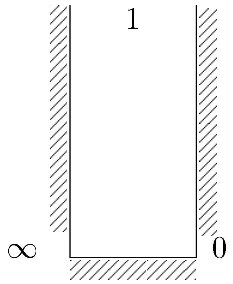

Figure 3: $F(\mathbb{H})$, case $\kappa = 8$: half-strip

- **Case** $\kappa > 8$

  In this case, one can choose F so that it maps $(\mathbb{H}, 0, 1, \infty)$ onto $(D, 1, \infty, 0)$ where

$$D = \left\{ z \ : \ \Im z > 0, 0 < \arg(z) < \left(1 - \frac{4}{\kappa}\right)\pi, \frac{4}{\kappa}\pi < \arg(z-1) < \pi \right\}$$

  Then $F(\mathbb{H})$ is not bounded in any direction, preventing us from using the optional stopping theorem.

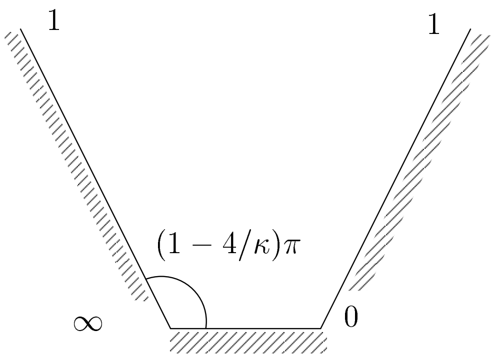

Figure 4: $F(\mathbb{H})$, case $\kappa > 8$

- **Case** $\kappa < 4$

  If $\kappa \geq 8/3$, one can choose F so that it maps $(\mathbb{H}, 0, 1, \infty)$ onto $(D, \infty, 0, \infty)$, where

$$D = \left\{ z \ : \ \Im z < 1, -\left(\frac{4}{\kappa} - 1\right)\pi < \arg(z) < \frac{4}{\kappa}\pi \right\}$$

  For $\kappa = 8/3$, one gets a slit half-plane. For $\kappa < 8/3$, the map F ceases to be univalent.

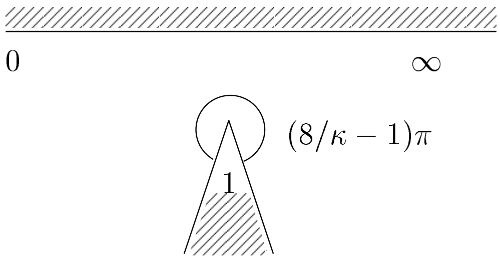

Figure 5: $F(\mathbb{H})$, case $\frac{8}{3} \leq \kappa \leq 4$

5 Radial SLE

Let D be the unit disk. Radial SLE in D starting from 1 is defined by $g_0(z) = z$, $z \in D$ and the ODEs:

$$\partial_t g_t(z) = -g_t(z)\frac{g_t(z) + \xi(t)}{g_t(z) - \xi(t)}$$

where $\xi(t) = \exp(iW_t)$ and $W_t/\sqrt{\kappa}$ is a real standard Brownian motion. The hulls (K_t) and the trace (γ_t) are defined as in the chordal case ([RohSch01]). Define $\tilde{g}_t(z) = g_t(z)\xi_t^{-1}$, so that $\tilde{g}_t(0) = 0$, $\tilde{g}(\gamma_t) = 1$, where (γ_t) is the SLE trace. One may compute:

$$d\tilde{g}_t(z) = -\tilde{g}_t(z)\frac{\tilde{g}_t(z) + 1}{\tilde{g}_t(z) - 1}dt + \tilde{g}_t(z)(-idW_t - \frac{1}{2}\kappa dt)$$

The above SDE is autonomous. As before, one looks for holomorphic functions F such that $(F(\tilde{g}_t(z)))_{t\geq 0}$ are local martingales. A sufficient condition is:

$$F'(z)\left(-z\frac{z+1}{z-1} - \frac{\kappa}{2}z\right) - \frac{\kappa}{2}F''(z)z^2 = 0$$

i.e.,

$$\frac{F''(z)}{F'(z)} = \left(\frac{2}{\kappa} - 1\right)\frac{1}{z} - \frac{4}{\kappa}\frac{1}{z-1}.$$

Meromorphic solutions of this equation defined on D exist for $\kappa = 2/n$, $n \in \mathbb{N}^*$. For $\kappa = 2$, $F(z) = (z-1)^{-1}$ is an (unbounded) solution.

6 Related conjectures

In this section we formulate various conjectures pertaining to continuous limits of discrete critical models using the privileged geometries for SLE described above.

6.1 FK percolation in isosceles triangles

For a survey of FK percolation, also called random-cluster model, see [Gri97]. We build on a conjecture stated in [RohSch01] (Conjecture 9.7), according to which the discrete exploration process for critical FK percolation with parameter q converges weakly to the trace of SLE_κ for $q \in (0,4)$, where the following relation holds:

$$\kappa = \frac{4\pi}{\cos^{-1}(-\sqrt{q}/2)}$$

Then the associated isosceles triangle T_κ has angles $\hat{a} = \hat{c} = \cos^{-1}(\sqrt{q}/2)$, $\hat{b} = \pi - 2\hat{a}$. Let Γ_n be a discrete approximation of the triangle T_κ on the square lattice with mesh $\frac{1}{n}$; all vertices on the edges $(a,b]$ and $[b,c)$ are identified. Let $\Gamma_n^\dagger$ be the dual graph. The discrete exploration process β runs between the opened connected component of $(a,b] \cup [b,c)$ in Γ_n and the closed connected component of (a,c) in $\Gamma_n^\dagger$.

Conjecture 1. *Cardy's Formula*
Let τ be the first time β hits (b,c). Then, as n tends to infinity (i.e. as the mesh tends to zero), the law of β_τ converges weakly towards the uniform law on (b,c).

Kenyon [Ken02] has proposed an FK percolation model for any isoradial lattice, in particular for any rectangular lattice. Let κ, q and α be as above, i.e. $4 < \kappa < 8$, $\frac{4\pi}{\kappa} = \cos^{-1}(-\sqrt{q}/2)$ and $\alpha = 1 - \frac{4}{\kappa}$. Consider the rectangular lattice $\mathbb{Z}\cos\alpha\pi + i\mathbb{Z}\sin\alpha\pi$. Then isosceles triangles homothetic to T_κ naturally fit in the lattice (see figure 6). Let $\Gamma = (V,E)$ be the finite graph resulting from the restriction of the lattice to a (large) T_κ triangle, with appropriate boundary conditions. A configuration $\omega \in \{0,1\}^E$ of open edges has probability:

$$p_\Gamma(\omega) \propto q^{k(\omega)} \nu_h^{e_h(\omega)} \nu_v^{e_v(\omega)}$$

where $k(\omega)$ is the number of connected components in the configuration, and e_h (resp. e_v) is the number of open horizontal (resp. vertical) edges. The weights ν_h, ν_v are given by the formulas:

$$\nu_v = \sqrt{q}\frac{\sin(2\alpha^2\pi)}{\sin(\alpha(1-2\alpha)\pi)}$$
$$\nu_h = \frac{q}{\nu_v}$$

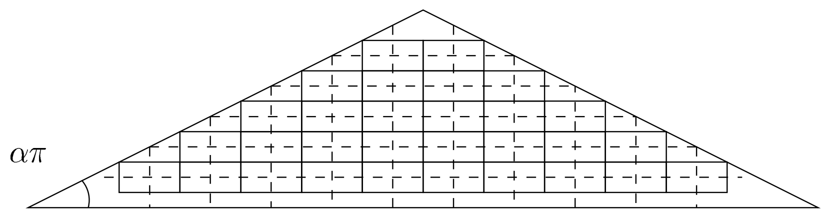

Figure 6: Rectangle lattice, dual graph and associated isosceles triangle

For this model, one may conjecture Cardy's formula as stated above. Note that for $q = 2$, $\kappa = \frac{16}{3}$, one retrieves the usual critical FK percolation on the square lattice.
Let us now focus on the integral values of the q parameter. It is known that for these values there exists a stochastic coupling between FK percolation and the Potts model (with parameter q) (see [Gri97]).

- $q = 1$
  In this case FK percolation is simply percolation, $\kappa = 6$, and the privileged geometry is the equilateral triangle. This corresponds to Carleson's observation on Cardy's formula.

- $q = 2$

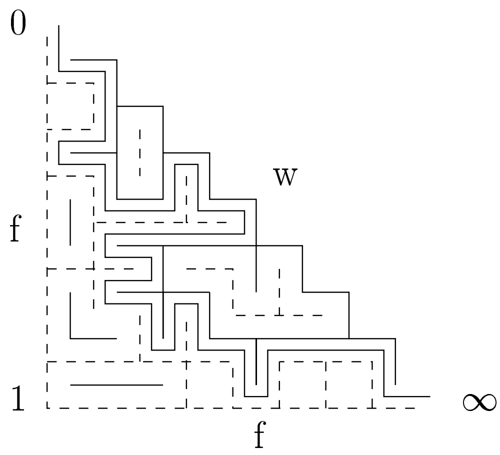

Figure 7: Discrete exploration process for FK percolation ($q = 2$, $\kappa = \frac{16}{3}$)

  Here $\kappa = \frac{16}{3}$, and T_κ is an isorectangle triangle. As there is a stochastic coupling between FK percolation with parameter $q = 2$ and the Ising model (Potts model with $q = 2$), this suggests that the isorectangle triangle may be of some significance for the Ising model.

- $q = 3$
  The corresponding geometry is the isosceles triangle $T_{\frac{24}{5}}$, which has angles $\hat{a} = \hat{c} = \frac{\pi}{6}$, $\hat{b} = \frac{2\pi}{3}$. The possible relationship with the $q = 3$ Potts model is not clear, as this model is not naturally associated with an exploration process.

6.2 UST in half-strips

It is proved in [LawSchWer02] that the scaling limit of the uniform spanning trees (UST) Peano curve is the SLE_8 chordal path. Let $R_{n,L}$ be the square lattice $[0, n] \times [0, nL]$, with the following boundaries conditions: the two horizontal arcs as well as the top one are wired, and the bottom one is free. In fact, as we will consider the limit as L goes to infinity, one may as well consider that the top arc is free, which makes the following lemma neater. We consider the uniform spanning tree in $R_{n,L}$. Let w be a point of the half-strip $\{z \;:\; 0 < \Re z < 1, \Im z > 0\}$, and w_n an integral approximation of nw. Let $a \in [0, n]$ be the unique triple point of the minimal subtree T containing $(0, 0)$, $(n, 0)$ and w_n, and let b be the other triple point of the minimal subtree containing $(0, 0)$, $(n, 0)$, w_n and $(0, nL)$. One can formulate the following easy consequence of the identification of the scaling limit of the UST:

Lemma 1. *The following limits hold:*

$$\lim_{n\to\infty}\lim_{L\to\infty}\mathbb{P}_{R_{n,L}}(b \text{ belongs to the oriented arc } [0,a]\cup[a,w_n] \textit{ in } T) = \Re w$$
$$\lim_{L\to\infty}\lim_{n\to\infty}\mathbb{P}_{R_{n,L}}(b \text{ belongs to the oriented arc } [0,a]\cup[a,w_n] \textit{ in } T) = \Re w$$

Let us clarify the alternative (up to events of negligible probability): either b belongs to the (oriented) arc $[0,a]\cup[a,w_n]$, or to the (oriented) arc $[w_n;a]\cup[a,1]$. Recall that we have computed $\mathbb{P}(\tau_{F^{-1}(w)} > \tau_1) = \Re w$ for a chordal SLE_8 going from 0 to 1 in the half-strip (in accordance with earlier conventions, subscripts refer to points in the half-plane, not in the half-strip). As this path is identified as the scaling limit of the UST Peano curve (start from 0 and go to 1 with the UST rooted on the bottom always on your right-hand), the event $\{\tau_1 < \tau_{F^{-1}(w)}\}$ appears as a scaling limit of an event involving only the subtree T. If one removes the arc joining a to iLn, w_n is either on the left connected component or on the right one depending on whether w_n is "visited" by the exploration process before or after the top arc, up to events of negligible probaility.

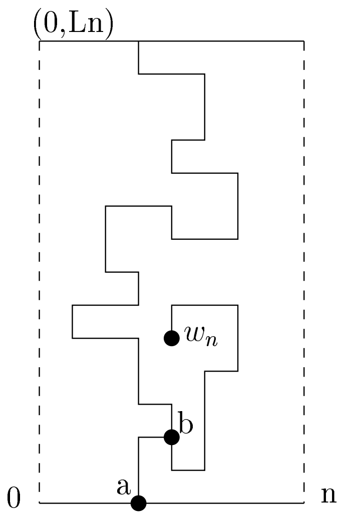

Figure 8: The alternative

In fact, one can prove the lemma without using the continuous limit for UST. Indeed, let $w_n^\dagger$ be a point on the dual grid standing at distance $\frac{\sqrt{2}}{2}$ from w_n. Then, as n tends to infinity,

$$\mathbb{P}_{R_{n,L}}(b \text{ belongs to the arc } [0,a]\cup[a,w_n] \text{ in } T)$$
$$-\mathbb{P}_{R^\dagger_{n,L}}(w_n^\dagger \text{ is connected to the right-hand boundary in the dual tree}) \to 0$$

According to Wilson's algorithm [Wil96], the minimal subtree in the dual tree connecting $w_n^\dagger$ to the boundary has the law of a loop-erased random walk (LERW) stopped at its first hit of the boundary. The probability of hitting the right-hand boundary or the left-hand boundary for a LERW equals the corresponding probability for a simple random walk. The continuous limit for a simple random walk with these boundary conditions is a Brownian motion reflected on the bottom of the half-strip; as the harmonic measure of the right-hand boundary of the whole slit $\{0 < \Re z < 1\}$ seen from $w_n^\dagger$ is $\Re w + o(1)$, this proves the lemma.

6.3 Double domino tilings in plane strips

For an early discussion of the double domino tiling model, see [RagHenAro97]. It is conjectured that the scaling limit of the path arising in this model is the SLE_4 trace (see [RohSch01], Problem 9.8). Building on Kenyon's work [Ken97, Ken00], we show that the continuous limit of a particular discrete event is compatible with the SLE_4 conjecture.

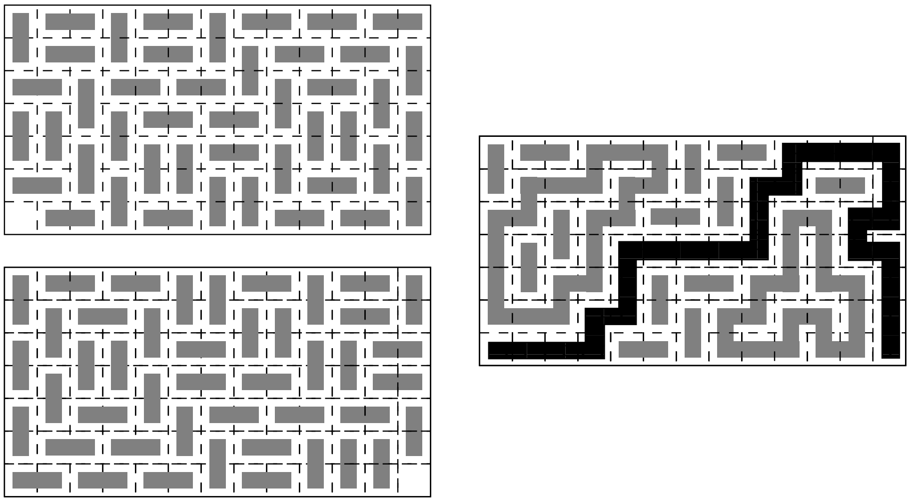

Figure 9: Double domino tilings and associated path

Consider the rectangle $R_{n,L} = [-nL, nL+1] \times [0, 2n+1]$ (it is important that the rectangle have odd length and width). Remove a unit square at the corner $(-nL, 0)$ or $(nL+1, 0)$ to get two Temperleyan polyominos (for general background on domino tilings, see [Ken00]). Let γ be the random path going from $(-nL, 0)$ to $(nL+1, 0)$, arising from the superposed uniform domino tilings on the two polyominos. Let w be a point of the strip $\{z \ : \ 0 < \Im z < 1\}$, and w_n an integral approximation of $2nw$ in $R_{n,L}$.

Proposition 2.
The following limit holds:

$$\lim_{L\to\infty} \lim_{n\to\infty} \mathbb{P}_{R_{n,L}}(w_n \text{ lies above } \gamma) = \Im z$$

Proof. We use a similar argument to the one given in [Ken97], 4.7. Let R_1, R_2 be the two polyominos, and h_1, h_2 the height functions associated with the two polyominos (these random integer-valued functions are defined up to a constant). It is easily seen that one may choose h_1, h_2 so that $h = h_1 - h_2 = 0$ on the bottom side, and $h = 4$ on the three other sides. Let x be an inner lattice point. Then:

$$\mathbb{E}(h(x)) = 4\mathbb{P}(x \ \text{ lies above } \gamma)$$

Indeed, condition on the union of the two dominos tilings. This union consists of the path γ, doubled dominos and disjoint cycles. Then x is separated from the bottom side by a certain number of closed cycles, and possibly γ. Conditionally on the union, each closed cycle accounts

for ± 4 with equal probability in $h(x)$. Moreover, crossing γ from below increases h by 4. This yields the formula.
As n goes to infinity, the average height functions converge to harmonic functions ([Ken00], Theorem 23). Then take the limit as L goes to infinity to conclude (one may map any finite rectangle R_L to the whole slit, fixing a given point x; the boundary conditions converge to the appropriate conditions, one concludes with Poisson's formula). $\square$

7 SLE(κ, ρ) processes and general triangles

In this section we quickly discuss how any triangle may be associated with a certain SLE process, in the same way as isoscele triangles were associated with SLE_κ processes.

7.1 SLE(κ, ρ) processes

Let us briefly describe SLE(κ, ρ) processes, defined in [LSW02b]. Let $(W_t, O_t)_{t\geq 0}$ be a two-dimensional semimartingale satisfying the following SDEs:

$$\left\{ \begin{array}{l} dW_t = \sqrt{\kappa} dB_t + \frac{\rho}{W_t - O_t} dt \\ dO_t = \frac{2}{O_t - W_t} dt \end{array} \right. \tag{1}$$

where B is a standard Brownian motion, as well as the inequality $W_t \leq O_t$ valid for all positive times (the convention here differs from the one in [LSW02b]). This process is well defined for $\kappa > 0$, $\rho > -2$. Indeed, one may consider $Z_t = O_t - W_t$. The process $(Z_t/\sqrt{\kappa})_{t\geq 0}$ is a Bessel process in dimension $d = 1 + 2\frac{\rho+2}{\kappa}$. Such processes are well defined semimartingales if $d > 1$, or $\rho > -2$ (see for instance [RevYor94]). Then $O_t = 2\int_0^t \frac{du}{Z_u}$ and $W_t = O_t - Z_t$.
Hence one may define a SLE(κ, ρ) as a stochastic Loewner chain the driving process of which has the law of the process (W_t) defined above. The starting point (or rather state) of the process is a couple (w, o) with $w \leq o$, usually set to $(0, 0^+)$. Then O_t represents the image under the conformal mapping g_t of the rightmost point of $\partial K_t \cup O_0$. Obviously, for $\rho = 0$, one recovers a standard SLE(κ) process.

Proposition 3. *Let (W_t, O_t) be the driving process of a* SLE(κ, ρ) *process starting from $(0, 1)$, and (g_t) be the associated conformal equivalences. Let $z \in \mathbb{H}$. Then if F is any analytic function on $\mathbb{H}$, the complex-valued semimartingale*

$$t \mapsto F\left(\frac{g_t(z) - W_t}{O_t - W_t}\right)$$

is a local martingale if and only if:

$$F'(z) \propto z^{-\frac{4}{\kappa}}(1-z)^{2\frac{\rho-\kappa+4}{\kappa}}.$$

The proof is routine and is omitted. Once again, the conformal mapping F may be identified using the Schwarz-Christoffel formula.

7.2 A particular case

In [Sch01], Schramm derives expressions of the form

$$\mathbb{P}(z \in \mathbb{H} \text{ lies to the left of } \gamma) = F_\kappa(\arg z)$$

where γ is the trace of a SLE(κ) process, for $\kappa \leq 4$. The function F_κ involves hypergeometric functions, and $F_\kappa(x) \propto x$ iff $\kappa = 4$ (in this case $F \circ \arg$ is a harmonic function). Now it is easily seen that for any $\kappa > 0$, $\rho > -2$, if δ designates the right boundary of a SLE(κ, ρ) process starting from $(0, 0^+)$, then a simple consequence of scaling is the existence of a function $F_{\kappa,\rho}$ such that:

$$\mathbb{P}(z \in \mathbb{H} \text{ lies to the left of } \delta) = F_{\kappa,\rho}(\arg z)$$

Moreover, this function is not identically zero if $\rho \geq \kappa/2 - 2$. This motivates the following result:

Proposition 4. *Let $\kappa > 4$, $\rho = \frac{\kappa}{2} - 2$. Then:*

$$\mathbb{P}(z \in \mathbb{H} \text{ lies to the left of } \delta) = \arg z/\pi$$

Proof. Lying to the left of the right boundary of the hull is the same thing as being absorbed if $\kappa > 4$. Let (W_t, O_t) be the driving mechanism of the SLE($\kappa, \frac{\kappa}{2} - 2$), and let $z_t = g_t(z)$. Suppose for now that the starting state of the SLE is $(W_0, O_0) = (0, 1)$. Let $h : \mathbb{H} \to \mathbb{C}$ be a holomorphic function. We have seen that a necessary and sufficient condition for $h(\frac{z_t - W_t}{O_t - W_t})$ to be a ($\mathbb{C}$-valued) local martingale is the holomorphic differential equation:

$$\frac{h''(z)}{h'(z)} = -\frac{4}{\kappa}\frac{1}{z} - 2\frac{\rho - \kappa + 4}{\kappa}\frac{1}{1 - z}$$

or $h(z) \propto z^{-\frac{4}{\kappa}}(1 - z)^{2\frac{\rho - \kappa + 4}{\kappa}}$. In the case $\rho = \kappa/2 - 2$, using the Schwarz-Christoffel formula (see [Ahl79]), one sees that h is (up to a constant factor) the conformal equivalence between $(\mathbb{H}, 0, 1, \infty)$ and $(D, 0, 1, \infty)$, where D is the degenerate triangle defined by:

$$D = \{z \in \mathbb{H} \ : \ \arg(z) \leq \pi(1 - 4/\kappa), \arg(z - 1) \geq \pi(1 - 4/\kappa)\}$$

Let $\varphi(z) = \Re z - \text{cotan}(\pi(1 - 4/\kappa))\Im z$. Then the image of D under this $\mathbb{R}$-linear form is $[0, 1]$. Hence $\varphi \circ h(\frac{z_t - W_t}{O_t - W_t})$ is a bounded martingale. Moreover, standard convergence arguments imply that $\frac{z_t - W_t}{O_t - W_t}$ goes to 0 in finite time if z is absorbed and to 1 in infinite time in the other case. A straightforward application of the optional stopping theorem yields:

$$\mathbb{P}(z \in \mathbb{H} \text{ lies to the right of } \delta) = \varphi\left(\int_0^z w^{-\frac{4}{\kappa}}(1 - w)^{\frac{4}{\kappa} - 1} dw\right) / B(1 - 4/\kappa, 4/\kappa)$$

Taking the asymptotics of this formula when $z = r \exp^{i\theta}$ goes to infinity with constant argument (making use of $B(1 - x, x) = \pi/\sin(\pi x)$), one finds that for a SLE($\kappa, \frac{\kappa}{2} - 2$) starting from $(0, 0^+)$:

$$\mathbb{P}(z \in \mathbb{H} \text{ lies to the right of } \delta) = 1 - \arg z/\pi$$

□

In other words, $F_{\kappa,\kappa/2-2} = F_4$ for all $\kappa \geq 4$. This raises several questions, such as whether this still holds for $\kappa < 4$, or whether in full generality $F_{\kappa,\rho} = F_{2\kappa/(\rho+2)}$, this last conjecture being based on the dimension of the Bessel process $(O_t - W_t)$, where (W_t, O_t) designates the driving process of a SLE(κ, ρ) process.

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