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ON A RESULT OF DAVID ALDOUS CONCERNING THE TREES IN A CONDITIONED EXCURSION

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Abstract
The law of a random tree constructed within a Brownian excursion is calculated conditional on knowing the occupation measure of the excursion. In previous work David Aldous has used random walk approximations to obtain this result. Here it is deduced from Le Gall's description of the tree in the unconditioned excursion.

1 Introduction

In [2] Aldous considers a tree constructed within the standard Brownian excursion, and shows that after conditioning on the occupation measure of the excursion, the law of the tree is that of an inhomogeneous coalescent. This description complements earlier results of the same author [1] and of Le Gall [3] on the description of the tree within the unconditioned excursion. The argument contained in [2] relies on checking a discrete version of the result (which is very simple) and then passing to a continuous limit (which is not so simple). In this note I present two direct proofs of Aldous' description, making use of Le Gall [3].

2 The tree within the excursion

This section recalls some of the definitions of [2] and [3] and then records Aldous' result.
Throughout e is a generic positive excursion with finite lifetime $\sigma(e)$, and n is the Itô measure of positive excursions of Brownian motion. We denote by $l(e)$ the continuous occupation density of an excursion e, so for any test function f,

$$\int_0^{\sigma(e)} f(e(u))\,du = \int_0^{\infty} f(z)l(e)(z)\,dz.$$

We denote the law of $l(e)$ under n by q .
Given an excursion e and $p \geq 1$ distinct times between 0 and $\sigma(e)$ we construct a tree, denoted by $T_p(e, t_1, \ldots t_p)$, according to the following well known recipe.

- The tree contains a root at height 0 and p leaves, with the height of the kth leaf being $e(t_k)$.
- The paths from the root to leaves k_1 and k_2 split at a branchpoint having height $\inf\{e(u) : u \text{ between } t_{k_1} \text{ and } t_{k_2}\}$.
- At each branchpoint left and right are labelled.

We always assume that the branchpoints are all distinct, so a tree with p leaves has $p-1$ branchpoints. For such a tree denote the height of its leaves by $x_1 \dots x_p$ and the height of its branch points by $y_1 \dots y_{p-1}$. A tree can be thought of as a 'shape' plus a set of values for the heights x_i and y_i. We consider the uniform measure, denoted by Λ_p, on the space of all such trees- for each shape of tree it is the restriction of Lebesgue measure to the appropriate subset of $\mathbb{R}^p \times \mathbb{R}^{p-1}$. Note that this measure has infinite total mass.
Given a tree, with its leaves and branchpoints labelled as above, and for $z > 0$, let

$$n(z) = \sum_i 1_{(x_i \geq z)} - \sum_i 1_{(y_i \geq z)},$$

that is the number of branches of T at height z. Then for any $l : \mathbb{R}_+ \mapsto \mathbb{R}_+$ denote by $D(T, l)$ the quantity given by

$$\frac{l(x_1)\dots l(x_p)}{l(y_1)\dots l(y_{p-1})} \exp\left\{-2\int_0^\infty \frac{n(z)^2 - n(z)}{l(z)} dz\right\},$$

provided l is strictly positive on $\{z > 0 : n(z) > 0\}$, and otherwise 0.
The result we shall prove is as follows.

Theorem 1 (Aldous [2]). *The joint distribution of $T_p(e, t_1, \dots t_p)$ and $l(e)$ under the measure*

$$1_{[0,\sigma(e)]}(t_1)\dots 1_{[0,\sigma(e)]}(t_p)\; n(de)\; dt_1 \dots dt_p$$

is given by

$$2^{(p-1)} D(T, l)\; \Lambda_p(dT)\; q(dl).$$

In fact Aldous states the result under the measure

$$1_{[0,1]}(t_1)\dots 1_{[0,1]}(t_p)\; n_{(1)}(de)\; dt_1 \dots dt_p,$$

where $n_{(1)}$ is the law of the excursion normalized to have lifetime 1, in which case it is simply necessary to replace q by $q_{(1)}$, the law of l under $n_{(1)}$, in the above statement. The two presentations are equivalent.
This result has an interesting interpretation. Suppose l is given and we want to construct a random tree T whose law is that of $T_p(e, t_1, \dots t_p)$ under the product measure

$$1_{[0,1]}(t_1)\dots 1_{[0,1]}(t_p)\; n_{(1)}(de)\; dt_1 \dots dt_p$$

conditional on knowing the occupation measure of e is l. We should take the heights of the leaves of the tree, x_1 to x_p, to be independent, each with distribution $l(x)dx$. Now think of time running from $x = \infty$ down to $x = 0$, and the heights x_i as being the birth times of particles. As time decreases let the particles cluster together according to the rule: each pair of clusters (independently of the others) coalesces into a single cluster in the time interval $[x, x - dx]$ with probability $4dx/l(x)$. Add in random left/right choices each time clusters coalesce and it is easy to associate with this proceedure a tree having branch points y_i at the times of coalescense. This tree then has the desired conditional law. Notice that to guarantee that there is an associated tree we need there to be only one cluster on reaching $x = 0$, but this will happen (with probability one) except for a set of exceptional l that are q-negligible.

3 The first proof

This is the quicker of the two arguments but makes full use of Le Gall's description of the excursion ([3], Theorem 3). According to this the distribution of $T = T_p(e, t_1, \ldots t_p)$ is $2^{(p-1)}\Lambda_p(dT)$, and moreover conditional on T the excursion e is decomposed into pieces of path which are distributed as independent stopped BMs. Readers are strongly advised to consult Le Gall's paper if they are unfamiliar with this construction. In the case $p = 1$ the tree is completely determined by the height of its single leaf $x_1 = e(t_1)$ and Le Gall's description reduces to that of Bismut (see Revuz and Yor [4], chapter XII). Conditional on $x_1 = x$ the processes $\big(e(t_1 + u); 0 \leq u \leq \sigma(e) - t_1\big)$ and $\big(e(t_1 - u); 0 \leq u \leq t_1\big)$ are independent and are each distributed as Brownian motion started from the level x and run until first hitting zero. By virtue of the Ray-Knight theorems and the additive properties of squared Bessel processes (see Revuz and Yor, [4], chapter XI) the joint distribution of $e(t_1)$ and $l(e)$ is $dx\, q^{(0,x)}(dl)$ where $q^{(0,x)}$ is the law of the unique solution of the SDE

$$Z_t = 2\int_0^t \sqrt{Z_s}d\beta_s + 4(t \wedge x), \qquad Z_0 = 0.$$

But we know from the outset that, as a consequence of the occupation times formula, the joint distribution $e(t_1)$ and $l(e)$ is just $l(x)dx\, q(dl)$, and so we see that the measure $q^{(0,x)}$ is absolutely continuous with respect to q with density given by

$$q^{(0,x)}(dl) = l(x) \cdot q(dl).$$

More generally it follows from Le Gall's description together with the Ray-Knight theorems and the additive properties of squared Bessel processes that the joint distribution of $T = T_p(e, t_1, \ldots t_p)$ and $l(e)$ is

$$2^{(p-1)}\Lambda_p(dT)\; q^T(dl),$$

where q^T is, for fixed T, the unique law of the SDE

$$Z_t = 2\int_0^t \sqrt{Z_s}d\beta_s + 4\int_0^t n(s)ds, \qquad Z_0 = 0. \tag{3.1}$$

But a well known application of Girsanov's formula (see Revuz and Yor, [4], chapters VIII and XI) shows that this law is absolutely continuous with respect to $q^{(0,x)}$ (for sufficiently small $x > 0$) and thus to q, in fact,

$$q^T(dl) = D(T, l) \cdot q(dl), \tag{3.2}$$

and this proves the theorem.

4 The second proof

The second argument I give looks somewhat more complicated, but it is interesting that it avoids the need to apply the Ray-Knight theorem to the pieces of the excursion. It is a variant of the argument used in [5].

Suppose ν is a positive finite measure supported on $(0,\infty)$. Define the excursion measure n_ν via the absolute continuity relation

$$n_\nu(de) = \mathcal{E}^\nu(l(e)) \cdot n(de),$$

where

$$\mathcal{E}^\nu(l) = \exp\left\{-\tfrac{1}{2}\int_0^\infty l(z)\nu(dz)\right\}.$$

Now for a given ν there exists a unique continuous increasing function $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ with $g(0) = 0$ such that if we take e according to Itô measure then e_g defined by

$$g\big(e_g(u)\big) = e\left(\int_0^u g'(e_g(v))^2 dv\right)$$

is distributed according to n_ν. In fact (although we do not use this) g is obtained via $g(x) = \int_0^x dy/\Phi(y)^2$, where Φ is the unique, positive, decreasing solution of the Sturm-Liouville equation

$$\phi'' = \nu\phi,$$

with $\Phi(0) = 1$, see Revuz and Yor ([4], appendix).

Using the space-time transform we can compute the distribution of $T_p(e, t_1, \dots t_p)$ under the measure

$$1_{[0,\sigma(e)]}(t_1)\dots 1_{[0,\sigma(e)]}(t_p)\; n_\nu(de)\; dt_1 \dots dt_p$$

and find it to be given by

$$2^{(p-1)}\frac{g'(y_1)\dots g'(y_{p-1})}{g'(x_1)\dots g'(x_p)} \cdot \Lambda_p(dT). \tag{4.1}$$

Elementary manipulations using $g'(y)l(e_g)(y) = l(e)(g(y))$ show that the density D transforms in the same way. More exactly

$$D(T, l(e_g)) = \frac{g'(y_1)\dots g'(y_{p-1})}{g'(x_1)\dots g'(x_p)} D(T_g, l(e)), \tag{4.2}$$

were T_g is the tree with the same shape as T, but the kth leaf moves from height x_k to $g(x_k)$ and similarly the heights of the branchpoints change. We have already observed in the previous section that

$$\int q(dl)D(T,l) \equiv 1, \tag{4.3}$$

and combining this with (4.2) we obtain that for any tree T with leaves at x_i and branchpoints at y_i,

$$\int q(dl)D(T,l)\mathcal{E}^\nu(l) = \int n(de)D(T,l(e_g)) = \int n(de)\frac{g'(y_1)\dots g'(y_{p-1})}{g'(x_1)\dots g'(x_p)}D(T_g,l(e)) = \frac{g'(y_1)\dots g'(y_{p-1})}{g'(x_1)\dots g'(x_p)}. \quad (4.4)$$

From this, and (4.1), we obtain that, for any positive test function F defined on the space of trees, and for arbitrary ν,

$$\int 2^{p-1}D(T,l)\,\Lambda(dT)\,q(dl)\,F(T)\mathcal{E}^\nu(l) = \int 1_{[0,\sigma(e)]}(t_1)\dots 1_{[0,\sigma(e)]}(t_p)\,n(de)\,dt_1\dots dt_p\,F(T)\mathcal{E}^\nu(l), \quad (4.5)$$

from which the theorem follows.

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