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SHARP BOUNDS FOR GREEN FUNCTIONS AND JUMPING FUNCTIONS OF SUBORDINATE KILLED BROWNIAN MOTIONS IN BOUNDED $C^{1,1}$ DOMAINS

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Abstract
In this paper we obtain sharp bounds for the Green function and jumping function of a subordinate killed Brownian motion in a bounded $C^{1,1}$ domain, where the subordinating process is a subordinator whose Laplace exponent has certain asymptotic behavior at infinity.

1 Introduction

Let $X = (X_t, \mathbb{P}_x)$ be a Brownian motion in $\mathbb{R}^d$ with generator Δ. Let $D \subset \mathbb{R}^d$ be a bounded domain, and we use $X^D = (X^D_t, \mathbb{P}_x)$ to denote the killed Brownian motion in D. Let $T = (T_t : t \geq 0)$ be a subordinator independent of X, and define $Y^D_t := X^D(T_t)$, $t \geq 0$. The process $Y^D = (Y^D_t : t \geq 0)$ is called a subordinate killed Brownian motion.
The study of the process Y^D was initiated in [7], where T is assumed to be an $\alpha/2$-stable subordinator, $\alpha \in (0, 2)$. Recently a lot of progress has been made in the study of the potential theory of Y^D. In [12], under the assumption that T is an $\alpha/2$-stable subordinator, upper and lower bounds on the Green function and jumping function of Y^D were established when D is a bounded $C^{1,1}$ domain. However, the upper and lower bounds provided in [12] were different near the boundary. In [11], new lower bounds for the Green function and jumping function of Y^D, that agree with the upper bounds of [12] up to multiplicative constants, were established. In this sense, sharp bounds of the Green function and jumping function of Y^D, in the case when T is an $\alpha/2$-stable subordinator, were obtained.

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The purpose of this paper is to obtain sharp bounds for the Green function and jumping function of Y^D for a much larger class of subordinating processes T. The sharp bounds obtained in this paper are very useful. For instance, they can be used to give concrete criteria for functions or measures to belong to the classes $\mathbf{S}_1(X)$ and $\mathbf{A}_\infty(X)$, defined in [2], [3] and [4], when the process X is the subordinate killed Brownian motion dealt with in this paper.
The content of this paper is organized as follows. In Section 2 we recall some basic facts about special subordinators and subordinate killed Brownian motion and in Section 3 we establish our main results.
In this paper we write $f \sim g$ as $x \to \infty$ (respectively, $x \to 0$), if $\lim_{x\to\infty} f(x)/g(x) = 1$ (respectively, $\lim_{x\to 0} f(x)/g(x) = 1$).

2 Preliminaries

Let $T = (T_t : t \geq 0)$ be a subordinator, that is, an increasing Lévy process taking values in $[0,\infty]$ with $T_0 = 0$. The Laplace exponent of the subordinator T is a function $\phi : (0,\infty) \to [0,\infty)$ such that

$$\mathbb{E}[\exp(-\lambda T_t)] = \exp(-t\phi(\lambda)) , \quad \lambda > 0. \tag{2.1}$$

The function ϕ has a representation

$$\phi(\lambda) = a + b\lambda + \int_0^\infty (1 - e^{-\lambda t})\, \mu(dt) . \tag{2.2}$$

Here $a, b \geq 0$, and μ is a σ-finite measure on $(0,\infty)$ satisfying

$$\int_0^\infty (t \wedge 1)\, \mu(dt) < \infty .$$

The constant a is called the killing rate, b the drift, and μ the Lévy measure of the subordinator T.
Recall that a C^∞ function $\phi : (0,\infty) \to [0,\infty)$ is called a Bernstein function if $(-1)^n D^n \phi \leq 0$ for every $n \in \mathbb{N}$. It is well known that a function $\phi : (0,\infty) \to \mathbb{R}$ is a Bernstein function if and only if it has the representation given by (2.2).
The potential measure of the subordinator T is defined by

$$U(A) = \mathbb{E} \int_0^\infty 1_{(T_t \in A)}\, dt , \tag{2.3}$$

and its Laplace transform is given by

$$\mathcal{L}U(\lambda) = \int_0^\infty e^{-\lambda t}\, dU(t) = \mathbb{E} \int_0^\infty \exp(-\lambda T_t)\, dt = \frac{1}{\phi(\lambda)} . \tag{2.4}$$

It is well known that if the drift b is strictly positive, then the potential measure U is absolutely continuous with a density $u : (0,\infty) \to \mathbb{R}$ that is continuous and positive, and $u(0+) = 1/b$ (e.g., [1], p.79). Moreover, for every $t > 0$, $u(t) \leq u(0+)$ (e.g., [10]). In order to establish the main results of this paper we will need the existence of a decreasing potential density for subordinators not necessarily having a strictly positive drift. A large class of such subordinators, called special subordinators, was introduced and studied in [13]. We recall the definition below.

Definition 2.1 *A Bernstein function ϕ is called a special Bernstein function if the function $\lambda/\phi(\lambda)$ is also a Bernstein function. A subordinator T is called a special subordinator if its Laplace exponent ϕ is a special Bernstein function.*

The family of special Bernstein functions is very large, and it contains in particular the family of complete Bernstein functions. Recall that a function $\phi : (0,\infty) \to \mathbb{R}$ is called a complete Bernstein function if there exists a Bernstein function η such that

$$\phi(\lambda) = \lambda^2 \mathcal{L}\eta(\lambda), \quad \lambda > 0.$$

The family of complete Bernstein functions is also very large and it contains the following well known Bernstein functions: (i) λ^α, $\alpha \in (0,1]$; (ii) $(\lambda+1)^\alpha - 1$, $\lambda \in (0,1)$, and (iii) $\ln(1+\lambda)$. It is known (see [8], for instance) that every complete Bernstein function is a special Bernstein function and that a Bernstein function ϕ of the form (2.2) is a complete Bernstein function if and only if its integral part

$$\phi_1(\lambda) = \int_0^\infty (1 - e^{-\lambda t})\mu(dt), \quad \lambda > 0 \tag{2.5}$$

is a complete Bernstein function.

One of the main results about special subordinators proved in [13] is the following: If T is a special subordinator such that $b > 0$ or $\mu((0,\infty)) = \infty$, then the potential measure U of T has a decreasing density u. For other results about special subordinators, as well as for many examples, we refer the reader to [13].

Suppose that $X = (X_t : t \geq 0)$ is a Brownian motion in $\mathbb{R}^d$ with generator Δ. Suppose that D is a bounded domain in $\mathbb{R}^d$ and that X^D is the killed Brownian motion in D. Suppose that $T = (T_t : t \geq 0)$ is a subordinator independent of X. The process $Y^D = (Y_t^D : t \geq 0)$ defined by $Y_t^D = X^D(T_t)$ is called a subordinate killed Brownian motion.

Let $p^D(t,x,y)$, $t \geq 0$, $x, y \in D$, denote the transition density of X^D and $(P_t^D : t \geq 0)$ the transition semigroup of X^D. It is well known that the potential operator of the subordinate process Y^D has a density U^D given by the formula

$$U^D(x,y) = \int_0^\infty p^D(t,x,y)\, U(dt)\,. \tag{2.6}$$

We call $U^D(x,y)$ the Green function of Y^D. In case the potential measure U of the subordinator T has a density u, we may write

$$U^D(x,y) = \int_0^\infty p^D(t,x,y) u(t)\, dt\,. \tag{2.7}$$

Let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form corresponding to Y^D, then $H^1(D) \subset \mathcal{F}$. When the drift b of T is positive, we have $\mathcal{F} = H^1(D)$ and for $u \in \mathcal{F}$,

$$\mathcal{E}(u,u) = b\int_D |\nabla u|^2(x)dx + \int_{D\times D} (u(x) - u(y))^2 J^D(x,y)dxdy + \int_D u^2(x)\kappa^D(x)dx,$$

where

$$J^D(x,y) = \frac{1}{2}\int_0^\infty p^D(s,x,y)\mu(ds),$$

and

$$\kappa^D(x) = a + \int_0^\infty (1 - P_s^D 1_D(x))\mu(ds).$$

When T has no drift,

$$\mathcal{F} = \{u \in L^2(D) : \int_0^\infty (u - P_s^D u, u)\mu(ds) < \infty\} \tag{2.8}$$

and for any $u \in \mathcal{F}$,

$$\mathcal{E}(u,u) = \int_{D\times D} (u(x) - u(y))^2 J^D(x,y)dxdy + \int_D u^2(x)\kappa^D(x)dx,$$

with J^D and κ^D given above. The functions J^D and κ^D are called the jumping function and killing functions of Y^D respectively. For the above facts on the Dirichlet form of Y^D, please see [9].

We recall now the definition of a $C^{1,1}$ domain. A bounded domain $D \subset \mathbb{R}^d$, $d \geq 2$, is called a bounded $C^{1,1}$ domain if there exist positive constants r_0 and M with the following property: for every $z \in \partial D$ and every $r \in (0, r_0]$, there exist a function $\Gamma_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying the condition $|\nabla\Gamma_z(\xi) - \nabla\Gamma_z(\eta)| \leq M|\xi - \eta|$ for all $\xi, \eta \in \mathbb{R}^{d-1}$, and an orthonormal coordinate system CS_z such that if $y = (y_1, \ldots, y_d)$ in CS_z coordinates, then

$$B(z,r) \cap D = B(z,r) \cap \{y : y_d > \Gamma_z(y_1, \ldots, y_{d-1}\}.$$

When we speak of a bounded $C^{1,1}$ domain in $\mathbb{R}$ we mean a finite open interval.

3 Main results

In this section we will always assume that D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$. For any $x \in D$, we use $\rho(x)$ to denote the distance between x and ∂D.

In this section we will establish sharp estimates on the Green function and jumping function of Y^D following the method of [11]. Our basic information is the Laplace exponent ϕ of the subordinator T and our basic assumption will be the asymptotic behavior of the Laplace exponent $\phi(\lambda)$ at infinity. We assume that ϕ has the representation (2.2).

For the Green function estimates, we will consider two cases: In the first case we will consider special subordinators T with Laplace exponent ϕ satisfying $\phi(\lambda) \sim \gamma^{-1}\lambda^{\alpha/2}$ as $\lambda \to \infty$ for $\alpha \in (0,2)$ and a positive constant γ. Note that in this case the drift of the subordinator must be zero. In the second case, the drift b is strictly positive, but we do not assume that the subordinator T is special. Note that the drift b is strictly positive if and only if $\phi(\lambda) \sim b\lambda$ as $\lambda \to \infty$. We put these two cases into the following assumption:

Assumption A: *The Laplace exponent ϕ of T satisfies*

$$\phi(\lambda) \sim \gamma^{-1}\lambda^{\alpha/2}, \quad \lambda \to \infty, \tag{3.1}$$

for some $\alpha \in (0,2]$, and in the case $\alpha \in (0,2)$ we always assume that ϕ is a special Bernstein function.

Note that in the case $\alpha = 2$, we have that $\gamma^{-1} = b$, the drift of the subordinator.

Assumption A implies that, in the case $\alpha \in (0,2)$, the Lévy measure μ of T must satisfy $\mu((0,\infty)) = \infty$. So under the Assumption A, the subordinator T has a potential density $u(t)$.

In the case $\alpha \in (0,2)$, it follows from (2.4) and Assumption A by use of the Tauberian theorem and the monotone density theorem, that

$$u(t) \sim \frac{\gamma}{\Gamma(\alpha/2)} t^{\alpha/2-1}, \quad t \to 0+ \, .$$

Therefore, for each $A > 0$, there exists a positive constant $c_1 = c_1(A)$ such that

$$\frac{u(t)}{t^{\alpha/2-1}} \geq c_1 \, , \quad 0 < t \leq A \, . \tag{3.2}$$

Moreover, there exists a positive constant c_2 such that

$$\frac{u(t)}{t^{\alpha/2-1}} \leq \begin{cases} c_2, & t < 1 \\ c_2 t^{-\alpha/2+1}, & t \geq 1 \end{cases} \tag{3.3}$$

Note also that both (3.2) and (3.3) are valid for the case of a strictly positive drift (i.e., $\alpha = 2$). For the jumping kernel estimates we will need the following assumption.
Assumption B: *The Laplace exponent ϕ of T is a complete Bernstein function and satisfies one of the following two conditions: (i) The drift b is positive and the integral part ϕ_1 of ϕ has the following asymptotic behavior*

$$\phi_1(\lambda) \sim \gamma^{-1} \lambda^{\beta/2}, \quad \lambda \to \infty, \tag{3.4}$$

for some $\beta \in (0,2)$. (ii) $b = 0$ and ϕ has the following asymptotic behavior

$$\phi(\lambda) \sim \gamma^{-1} \lambda^{\beta/2}, \quad \lambda \to \infty, \tag{3.5}$$

for some $\beta \in (0,2)$.
It is known (see [8] for instance) that the Lévy measure of any complete Bernstein function has a completely monotone density. The asymptotic relations (3.4) and (3.5) imply that the Lévy measure μ of T must satisfy $\mu((0,\infty)) = \infty$. Corollary 2.4 of [13] implies that the Lévy measure ν of the complete Bernstein function $\lambda/\phi_1(\lambda)$ in case (i), and of the complete Bernstein function $\lambda/\phi(\lambda)$ in case (ii), satisfies $\nu((0,\infty)) = \infty$. Hence the potential measure of the subordinator with Laplace exponent $\lambda/\phi_1(\lambda)$ in case (i), and with Laplace exponent $\lambda/\phi(\lambda)$ in case (ii), has a decreasing density v. Again from Corollary 2.4 of [13] we know that the Lévy measure μ of ϕ and the potential density v are related as follows

$$v(t) = \tilde{a} + \mu((t,\infty)) \, , \quad t > 0,$$

with $\tilde{a} = 0$ in case (i) and $\tilde{a} = a$ in case (ii). It follows from (3.4), (3.5), the Tauberian theorem and the monotone density theorem that

$$v(t) \sim \frac{1}{\gamma \Gamma(1-\beta/2)} t^{-\beta/2}, \quad t \to 0+ \, ,$$

hence we have

$$\mu((t,\infty)) \sim \frac{1}{\gamma \Gamma(1-\beta/2)} t^{-\beta/2}, \quad t \to 0+ \, .$$

Using the monotone density theorem again we get that

$$\mu(t) \sim \frac{\beta}{2\gamma \Gamma(1-\beta/2)} t^{-\beta/2-1}, \quad t \to 0+ \, , \tag{3.6}$$

where $\mu(t)$ denotes the density of the measure μ. Therefore, for each $A > 0$, there exists a positive constant $c_3 = c_3(A)$ such that

$$\frac{\mu(t)}{t^{-\beta/2-1}} \geq c_3\,, \quad 0 < t \leq A\,. \tag{3.7}$$

Moreover, there exists a positive constant c_4 such that

$$\frac{\mu(t)}{t^{-\beta/2-1}} \leq \begin{cases} c_4, & t < 1 \\ c_4 t^{\beta/2+1}, & t \geq 1 \end{cases} \tag{3.8}$$

The sharp estimates on the Green function of Y^D are included in the following result.

Theorem 3.1 *Suppose that D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ and that the subordinator $T = (T_t : t \geq 0)$ satisfies Assumption A. If $d > \alpha$, then there exist positive constants $C_1 \leq C_2$ such that for all $x, y \in D$*

$$C_1\left(\frac{\rho(x)\rho(y)}{|x-y|^2} \wedge 1\right)\frac{1}{|x-y|^{d-\alpha}} \leq U^D(x,y) \leq C_2\left(\frac{\rho(x)\rho(y)}{|x-y|^2} \wedge 1\right)\frac{1}{|x-y|^{d-\alpha}}\,.$$

Proof. Upper bound. It is known (see [5] and [12]) that there exists a positive constant c_5 such that for all $t > 0$ and any $x, y \in D$,

$$p^D(t,x,y) \leq c_5 t^{-d/2-1}\rho(x)\rho(y)\exp\left(-\frac{|x-y|^2}{6t}\right)\,. \tag{3.9}$$

Hence by the formula for U^D,

$$\begin{aligned} U^D(x,y) &\leq c_5\rho(x)\rho(y)\int_0^\infty t^{-d/2-1}e^{-|x-y|^2/6t}u(t)\,dt \\ &= c_6\rho(x)\rho(y)|x-y|^{-d}\int_0^\infty s^{d/2-1}e^{-s}u\left(\frac{|x-y|^2}{6s}\right)ds \\ &= c_7\rho(x)\rho(y)|x-y|^{-d-2+\alpha}\int_0^\infty s^{d/2-\alpha/2}e^{-s}\frac{u(|x-y|^2/6s)}{(|x-y|^2/6s)^{\alpha/2-1}}\,ds\,. \end{aligned}$$

For $\alpha = 2$, the last integral is clearly bounded by a positive constant. For $0 < \alpha < 2$, we estimate the integral by use of (3.3):

$$\begin{aligned} &\int_0^\infty s^{d/2-\alpha/2}e^{-s}\frac{u(|x-y|^2/6s)}{(|x-y|^2/6s)^{\alpha/2-1}}\,ds = \int_0^{|x-y|^2/6} + \int_{|x-y|^2/6}^\infty \\ \leq\;& c_2\int_0^{|x-y|^2/6} s^{d/2-\alpha/2}e^{-s}(|x-y|^2/6s)^{-\alpha/2+1}\,ds + c_2\int_{|x-y|^2/6}^\infty s^{d/2-\alpha/2}e^{-s}\,ds \\ \leq\;& c_8\left(|x-y|^{-\alpha+2}\int_0^{|x-y|^2/6} s^{d/2-1}e^{-s}\,ds + \int_{|x-y|^2/6}^\infty s^{d/2-\alpha/2}e^{-s}\,ds\right) \\ \leq\;& c_9\left(\int_0^\infty s^{d/2-1}e^{-s}\,ds + \int_0^\infty s^{d/2-\alpha/2}e^{-s}\,ds\right) = c_{10}\,. \end{aligned}$$

Here we used that $|x-y| \leq \operatorname{diam}(D)$. Hence we have shown that there exists a positive constant c_{11} such that for all $x, y \in D$,

$$U^D(x,y) \leq c_{11}\frac{\rho(x)\rho(y)}{|x-y|^{d+2-\alpha}}\,. \tag{3.10}$$

Let $p(t,x,y)$, $t \ge 0$, $x,y \in \mathbb{R}^d$, be the transition density of the Brownian motion X in $\mathbb{R}^d$. Then $p^D(t,x,y) \le p(t,x,y)$, implying

$$U^D(x,y) \le \int_0^\infty p(t,x,y)u(t)\,dt\,.$$

A similar (but easier) argument to the one in the paragraph above shows that there is a constant $c_{12} > 0$ such that

$$\int_0^\infty p(t,x,y)u(t)\,dt \le c_{12}|x-y|^{\alpha-d}, \quad x,y \in D.$$

Therefore,

$$U^D(x,y) \le c_{12}|x-y|^{\alpha-d}, \quad x,y \in D. \tag{3.11}$$

Combining (3.10) and (3.11) we get the upper bound of the theorem.
Lower bound. It was proved in [14] and [11] that for any $A > 0$, there exist positive constants c_{13} and c_{14} such that for any $t \in (0,A]$ and any $x,y \in D$,

$$p^D(t,x,y) \ge c_{13}\left(\frac{\rho(x)\rho(y)}{t} \wedge 1\right) t^{-d/2} \exp\left(-\frac{c_{14}|x-y|^2}{t}\right). \tag{3.12}$$

Hence by the formula for U^D,

$$U^D(x,y) \ge c_{13}\int_0^A \left(\frac{\rho(x)\rho(y)}{t} \wedge 1\right) t^{-d/2} \exp\{-c_{14}|x-y|^2/t\} u(t)\,dt\,.$$

Assume $x \ne y$. Let R be the diameter of D and assume that A has been chosen so that $A = R^2$. Then for any $x,y \in D$, $\rho(x)\rho(y) < R^2 = A$. The lower bound is proved by considering two separate cases:
(i) $\dfrac{|x-y|^2}{\rho(x)\rho(y)} < \dfrac{2R^2}{A}$. In this case we have:

$$\begin{aligned}
U^D(x,y) &\ge c_{13}\int_0^{\rho(x)\rho(y)} \left(\frac{\rho(x)\rho(y)}{t} \wedge 1\right) t^{-d/2} \exp\{-c_{14}|x-y|^2/t\} u(t)\,dt \\
&\ge c_{15}|x-y|^{-d+2}\int_{\frac{c_{14}|x-y|^2}{\rho(x)\rho(y)}}^\infty s^{d/2-2}e^{-s}u(c_{14}|x-y|^2/s)\,ds \\
&\ge c_{15}|x-y|^{-d+2}\int_{\frac{2c_{14}R^2}{A}}^\infty s^{d/2-2}e^{-s}u(c_{14}|x-y|^2/s)\,ds \\
&= c_{16}|x-y|^{-d+\alpha}\int_{\frac{2c_{14}R^2}{A}}^\infty s^{d/2-\alpha/2-1}e^{-s}\frac{u(c_{14}|x-y|^2/s)}{(c_{14}|x-y|^2/s)^{\alpha/2-1}}\,ds\,.
\end{aligned}$$

For $s > 2c_{14}R^2/A$, we have that $c_{14}|x-y|^2/s \le A/2$, hence we can estimate the fraction in the above integral by c_1, see (3.2). Hence,

$$U^D(x,y) \ge c_{17}|x-y|^{-d+\alpha}\,.$$

(ii) $\dfrac{|x-y|^2}{\rho(x)\rho(y)} \geq \dfrac{2R^2}{A}$. In this case we have:

$$\begin{aligned}
U^D(x,y) &\geq c_{13}\rho(x)\rho(y)\int_{\rho(x)\rho(y)}^{A} t^{-d/2-1}\exp\{-c_{14}|x-y|^2/t\}u(t)\,dt \\
&= c_{18}\rho(x)\rho(y)|x-y|^{-d}\int_{\frac{c_{14}|x-y|^2}{A}}^{\frac{c_{14}|x-y|^2}{\rho(x)\rho(y)}} s^{d/2-1}e^{-s}u(c_{14}|x-y|^2/s)\,ds \\
&= c_{19}\rho(x)\rho(y)|x-y|^{-d+\alpha-2}\int_{\frac{c_{14}|x-y|^2}{A}}^{\frac{c_{14}|x-y|^2}{\rho(x)\rho(y)}} s^{-d/2-\alpha/2}e^{-s}\frac{u(c_{14}|x-y|^2/s)}{(c_{14}|x-y|^2/s)^{\alpha/2-1}}\,ds \\
&\geq c_{19}\rho(x)\rho(y)|x-y|^{-d+\alpha-2}\int_{\frac{c_{14}R^2}{A}}^{\frac{2c_{14}R^2}{A}} s^{-d/2-\alpha/2}e^{-s}\frac{u(c_{14}|x-y|^2/s)}{(c_{14}|x-y|^2/s)^{\alpha/2-1}}\,ds\,.
\end{aligned}$$

Again, if $s > c_{14}R^2/A \geq c_{14}|x-y|^2/A$, then $c_{14}|x-y|^2/A < s$, and we use (3.2) to estimate the fraction above by c_1. Hence,

$$U^D(x,y) \geq c_{20}\frac{\rho(x)\rho(y)}{|x-y|^{d-\alpha+2}}\,.$$

Combining the two cases above we arrive at the lower bound of the theorem. □

Remark 3.2 *Note that the theorem above implies that when b is positive, the Green function U^D of Y^D is comparable to the Green function of X^D.*

The sharp estimates on the jumping function of Y^D are included in the following result.

Theorem 3.3 *Suppose that D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ and that the subordinator $T = (T_t : t \geq 0)$ satisfies Assumption B. Then there exist positive constants $C_3 \leq C_4$ such that for all $x, y \in D$*

$$C_3\left(\frac{\rho(x)\rho(y)}{|x-y|^2}\wedge 1\right)\frac{1}{|x-y|^{d+\beta}} \leq J^D(x,y) \leq C_4\left(\frac{\rho(x)\rho(y)}{|x-y|^2}\wedge 1\right)\frac{1}{|x-y|^{d+\beta}}\,.$$

Proof. The proof of this theorem is the same as that of Theorem 3.1, the only differences are that we use Assumption B, the formula for J, (3.7) and (3.8) instead of Assumption A, the formula for U^D, (3.2) and (3.3). We omit the details. □

Using arguments similar to that of Proposition 3.2 of [12], we can get the following estimates on the killing function of Y^D.

Theorem 3.4 *Suppose that D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ and that the subordinator $T = (T_t : t \geq 0)$ satisfies Assumption B. Then there exist positive constants $C_5 \leq C_6$ such that*

$$C_5(\rho(x))^{-\beta} \leq \kappa^D(x) \leq C_6(\rho(x))^{-\beta}, \quad x \in D.$$

Proof. Let $Z = (Z_t : t \geq 0)$ be the subordinate Brownian motion defined by $Z_t = X(T_t)$. Clearly, the killing function $\kappa(x)$ of Z is given by $\kappa(x) = a$. It follows from [9] that the jumping function $J(x, y)$ of this process is given by

$$J(x,y) = \frac{1}{2}\int_0^\infty p(t,x,y)\mu(t)dt, \tag{3.13}$$

where $p(t, x, y)$ is the transition density of X. Let $Z^D = (Z_t^D, \mathbb{P}_x)$ be the process Z killed upon exiting D. Then it is known (see [6], for instance) that the killing function $\tilde{\kappa}^D(x)$ of Z^D is given by

$$\tilde{\kappa}^D(x) = a + 2\int_{D^c} J(x,y)dy, \quad x \in D.$$

Now using (3.6), (3.13) and an argument similar to that of Theorem 3.1 of [10] we get that

$$J(x,y) \sim c_{21}|x-y|^{-d-\beta}, \quad |x-y| \to 0$$

for some constant $c_{21} > 0$. From this we immediately get that there exist constants $0 < c_{22} < c_{23}$ such that

$$c_{22}(\rho(x))^{-\beta} \leq \tilde{\kappa}^D(x) \leq c_{23}(\rho(x))^{-\beta}, \quad x \in D.$$

Repeating the argument of Proposition 3.2 of [12] we get that there exist constants $0 < c_{24} < c_{25}$ such that

$$c_{24}\tilde{\kappa}^D(x) \leq \kappa^D(x) \leq c_{25}\tilde{\kappa}^D(x), \quad x \in D.$$

The proof is now finished. □

References

[1] J. Bertoin. *Lévy Processes*. Cambridge University Press, Cambridge, 1996.

[2] Z.-Q. Chen. Gaugeability and conditional gaugeability. *Trans. Amer. Math. Soc.* **354** (2002), 4639–4679

[3] Z.-Q. Chen and R. Song. General gauge and conditional gauge theorems. *Ann. Probab.* **30** (2002), 1313–1339.

[4] Z.-Q. Chen and R. Song. Conditional gauge theorem for non-local Feynman-Kac transforms. *Probab. Theory Rel. Fields* **125** (2003), 45–72.

[5] E. B. Davies. *Heat kernels and spectral theory*. Cambridge University Press, Cambridge, 1989.

[6] M. Fukushima, Y. Oshima and M. Takeda. *Dirichlet forms and symmetric Markov processes*. Walter De Gruyter, Berlin, 1994.

[7] J. Glover, M. Rao, H. Šikić and R. Song. Γ-potentials. in *Classical and modern potential theory and applications* (Chateau de Bonas, 1993). 217–232, Kluwer Acad. Publ., Dordrecht, 1994.

[8] N. Jacob. *Pseudo Differential Operators and Markov Processes*. Vol. 1, Imperial College Press, London, 2001.

[9] K. Okura. Recurrence and transience criteria for subordinated symmetric Markov processes. *Forum Math.* **14** (2002), 121–146.

[10] M. Rao, R. Song and Z. Vondraček. Green function estimates and Harnack inequality for subordinate Brownian motions. Preprint, 2004.

[11] R. Song. Sharp bounds on the density, Green function and jumping function of subordinate killed BM. *Probab. Theory Rel. Fields* **128** (2004), 606-628.

[12] R. Song and Z. Vondraček. Potential theory of subordinate killed Brownian motion in a domain. *Probab. Theory Rel. Fields* **125** (2003), 578–592.

[13] R. Song and Z. Vondraček. Potential theory of special subordinators and subordinate killed stable processes. Preprint, 2004

[14] Q. S. Zhang. The boundary behavior of heat kernels of Dirichlet Laplacians. *J. Differential Equations* **182** (2002), 416–430.