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GEOMETRY OF STOCHASTIC DELAY DIFFERENTIAL EQUATIONS

PEDRO JOSÉ CATUOGNO[1]
Departamento de Matemática, Universidade Estadual de Campinas, 13.081-970-Campinas - SP, Brazil.
email: `pedrojc@ime.unicamp.br`

PAULO R. C. RUFFINO[2]
Departamento de Matemática, Universidade Estadual de Campinas, 13.081-970-Campinas - SP, Brazil.
email: `ruffino@ime.unicamp.br`

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Abstract

Stochastic delay differential equations (SDDE) on a manifold M depend intrinsically on a connection ∇ in this space. The main geometric result in this notes concerns the horizontal lift of solutions of SDDE on a manifold M to an SDDE in the frame bundle BM, hence the lifted equation should come together with the prolonged horizontal connection ∇^H on BM. We show that every horizontal semimartingale can be represented as a solution of an SDDE.

1 Delay differential equations on Manifolds

Delay equations in differentiable manifolds involve a parallel transport in order to map vectors from a tangent spaces to another. Hence, in the geometrical context, delay equations depend strongly on a chosen connection. Let M be a differentiable manifold, X a vector field, ∇ a connection on M and $\alpha : [-1, 0] \to M$ an initial continuous trajectory. The solution of a delay equation on M (with retard $r = 1$, say), when it exists, is a curve $\gamma(t)$ such that the derivative $\dot{\gamma}(t)$ equals the parallel transport of $X(\gamma(t-1))$ along γ from $T_{\gamma(t-1)}M$ to $T_{\gamma(t)}M$, for $t \geq 0$. In, symbols:

$$\begin{cases} \frac{d\gamma}{dt}(t) &= P^{\nabla}_{t,t-1}(\gamma)(X(\gamma(t-1))) \\ \gamma(t) &= \alpha(t) \quad \text{for all } t \in [-1, 0] \end{cases}$$

where $P^{\nabla}_{t,s}(\gamma) : T_{\gamma(s)}M \to T_{\gamma(t)}M$ is the parallel transport along γ induced by ∇.

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We extend the previous definition to a stochastic delay differential equation (SDDE) on a differentiable manifold M endowed with a connection ∇ in the following way: let $A_1, \ldots, A_m$ be vector fields in M and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ be a complete filtered probability space satisfying the usual conditions. Let $(M_t)_{t\geq 0}$ be an $\mathbb{R}^m$-semimartingale adapted to $(\mathcal{F}_t)_{t\geq 0}$, we suppose that $M_0 = 0$. Finally, let (α_t) be a deterministic (initial) trajectory in M. The corresponding SDDE on the manifold M writes:

$$\left\{ \begin{array}{lcl} dx_t & = & \sum_{k=1}^m P^{\nabla}_{t,t-1}(x)(A_k(x_{t-1})) \circ dM^k_t, \\ \\ x_t & = & \alpha_t \ \text{ for } \ t \in [-1, 0]. \end{array} \right. \tag{1}$$

A stochastic process ϕ_t on M adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ is called a local solution of the (1) if for all $t \in [-1, 0]$, $\phi_t = \alpha_t$, there exists a stopping time $T > 0$ such that for all $t \leq T$ and for any $F \in \mathcal{C}^\infty(M)$:

$$F(\phi_t) = F(\phi_0) + \sum_{k=1}^m \int_0^t P^{\nabla}_{r,r-1}(\phi)(A_k(\phi_{r-1}))F(\phi_r) \circ dM^k_r.$$

We remark that existence and uniqueness of solutions of SDDE, as presented here, are a particular case of the theory of stochastic functional differential equations (see [6], [7]).
Horizontal processes in the frame bundle BM of a manifold M is a structural concept in stochastic geometry and stochastic dynamical systems, e.g. parallel transport, development, anti-development and others, they are all constructed based on horizontal processes. The main question we address in this article is the following: is the horizontal lift of an SDDE again an SDDE in BM? As we said before, once an SDDE depends on a connection in the manifold, this question carries intrinsically another one: the lifted SDDE in BM (if exists at all!) is taken with respect to which (prolonged) connection in BM?

2 Horizontal lift of SDDE on manifolds

In this section we provide answers to the questions proposed above: with the so called horizontal connection ∇^H (to be defined later), the horizontal lift of a solution of equation (1) is a solution of the following SDDE on BM:

$$\left\{ \begin{array}{lcl} dx_t & = & \sum_{k=1}^m P^{\nabla^H}_{t,t-1}(x)(A^H_k(x_{t-1})) \circ dM^k_t, \\ x_t & = & \alpha^H_{p\ t} \ \text{ for } t \in [-1, 0], \end{array} \right. \tag{2}$$

where $p \in \pi^{-1}(\alpha(0))$, α^H_p is the horizontal lift of α such that $\alpha^H_p(0) = p$, A^H_k is the horizontal lift of A_k for $k = 1, 2, \ldots, m$. Still, by the end of this section we prove that every horizontal semimartingale is solution of an SDDE.
We begin by recalling some fundamental facts on differential geometry, we indicate e.g. Bishop and Crittenden [1], Cordero et al. [2] or Kobayashi and Nomizu [5]. Let M be a differentiable manifold. The frame bundle BM of M consists of all linear isomorphism $p : \mathbb{R}^n \to T_xM$ for some $x \in M$, with projection $\pi(p) = x$. The fibre bundle BM is a principal bundle over M with structure group $GL(n, \mathbb{R})$ and Lie algebra denoted by $\mathcal{G}l(n, \mathbb{R})$.
Let $\alpha : I \to M$ be a curve in M. The horizontal lift of α to BM, can be written as the composition

$$\alpha^H_p(t) := P^{\nabla}_{t,0}(\alpha) \circ p \tag{3}$$

where $P^{\nabla}_{t,s}(\alpha) : T_{\alpha(s)}M \to T_{\alpha(t)}M$ is the parallel transport along the curve α. A connection ∇ on M determines a decomposition of each tangent space T_pBM into the direct sum of the vertical subspace $V_pBM = Ker(\pi_*(p))$ and the horizontal subspace H_pBM of the tangent at p of horizontal lifts of curves in M. This decomposition naturally defines the horizontal lift of $v \in T_xM$ at $p \in BM$ ($\pi(p) = x$) as the unique tangent vector $v^H \in H_pBM$ such that $\pi_*(p)v^H = v$. Let $\{e_1, \ldots, e_d\}$ be the standard basis of $\mathbb{R}^n$, the standard vector fields $\{E[e_1], \ldots, E[e_n]\}$ in BM are the unique horizontal fields such that $\pi_*(p)E[e_i](p) = p(e_i)$ for every $p \in BM$. The distribution $\{H_p : p \in BM\}$ is the span of the standard vector fields $E[e_i]$. Let $A \in \mathcal{G}l(n,\mathbb{R})$, $A^*(p) = p_*(Id)A$ where p is considered as the mapping $p : GL(n,\mathbb{R}) \to BM$, $p(g) = p \circ g$. Obviously, $A^*(p)$ is a vertical vector. Let $\{E_{i,j} : 1 \le i, j \le n\}$ be the standard basis of $\mathcal{G}l(n,\mathbb{R})$, the distribution $\{V_pBM : p \in BM\}$ is the span of the vertical vector fields $E^*_{i,j}$. Note that $\{E[e_i], E^*_{i,j} : 1 \le i, j \le n\}$ parallelizes BM.
There are many ways of extending a connection ∇ of M to BM. We are particularly interested in the horizontal lift ∇^H (see e.g. Cordero et al. [2, Chap. 6]). Assuming that the connections are torsion free, the horizontal lift ∇^H is defined as the unique connection on BM which satisfies:

$$\left\{\begin{array}{lcl} \nabla^H_{A^*}B^* & = & (AB)^* \\ \nabla^H_{A^*}X^H & = & 0 \\ \nabla^H_{X^H}A^* & = & 0 \\ \nabla^H_{X^H}Y^H & = & (\nabla_X Y)^H \end{array}\right. \tag{4}$$

With connection ∇^H we have the following commutative property of the parallel transport and horizontal lift:

Lemma 2.1 *Let ∇ be a connection on M, ∇^H its horizontal lift to BM and α a curve in BM. Then, for any $v \in T_{\pi\circ\alpha(0)}M$ we have that*

$$P^{\nabla^H}_{0,t}(\alpha)(v^H) = (P^{\nabla}_{0,t}(\pi \circ \alpha)(v))^H$$

Proof: Initially, note that the projection π is an affine transformation because, by definition of the horizontal connection, for any pair of vector fields Z, W in BM we have:

$$\pi_*(\nabla^H_Z W) = \nabla_{\pi_* Z}\pi_* W,$$

which implies also that π preserves the parallel transport, i.e.

$$\pi_*(P^{\nabla^H}_{0,t}(\alpha)\ w) = P^{\nabla}_{0,t}(\pi\alpha)\ \pi_* w.$$

The covariant derivative of a horizontal vector field is horizontal (formulae (4)), hence the horizontal distribution is preserved by parallel transport. The result follows immediately by the uniqueness of the horizontal lift of $P^{\nabla}_{0,t}(\pi \circ \alpha)(v)$.

□

Now, we present a fundamental lemma for the main results of this section.

Proposition 2.1 *Let γ be a solution of the deterministic delay differential equation*

$$\left\{\begin{array}{lcl} \frac{dx}{dt}(t) & = & P^{\nabla}_{t,t-1}(x)(X(x(t-1))) \\ x(t) & = & \alpha(t) \quad for\ t \in [-1,0]. \end{array}\right.$$

where ∇ is a connection on M, X a vector field in M and $\alpha : [-1,0] \to M$ a differentiable curve. Then the horizontal lift γ^H_p is a solution of

$$\left\{\begin{array}{lcl} \frac{dx}{dt}(t) & = & P^{\nabla^H}_{t,t-1}(x)(X^H(x(t-1))) \\ x(t) & = & \alpha^H_p(t) \quad for\ \ t \in [-1,0] \end{array}\right.$$

Proof:
We apply Lemma 2.1 to γ_p^H and $X(\gamma(t-1))$, thus:

$$\begin{aligned}
\frac{d\gamma_p^H}{dt}(t) &= (\frac{d\gamma}{dt}(t))^H \\
&= (P^{\nabla}_{t,t-1}(\gamma)(X(\gamma(t-1)))^H \\
&= P^{\nabla^H}_{t,t-1}(\gamma_p^H)(X^H(\gamma_p^H(t-1)))
\end{aligned}$$

and obviously $\gamma_p^H(t) = \alpha_p^H(t)$ for $t \in [-1, 0]$.

□

Extending this result to stochastic case we have:

Theorem 2.2 *Let γ be a solution of the following SDDE on M (with connection ∇):*

$$\begin{cases} dx_t &= \sum_{k=1}^m P^{\nabla}_{t,t-1}(x)(A_k(x_{t-1})) \circ dM_t^k \\ x_t &= \alpha_t \;\; for \;\; t \in [-1, 0]. \end{cases}$$

Then γ_p^H is solution of the SDDE on BM (with connection ∇^H):

$$\begin{cases} dx_t &= \sum_{k=1}^m P^{\nabla^H}_{t,t-1}(x)(A_k^H(x_{t-1})) \circ dM_t^k, \\ x_t &= \alpha_{p\;t}^H \;\; for \; t \in [-1, 0], \end{cases}$$

Proof:
Apply the above proposition and the transfer principle (see e.g. Emery [3]).

□

We say that a process γ in BM is a horizontal semimartingale if there is no vertical component in the sense that $\int \omega \;\circ d\gamma = 0$ where ω is the connection form associated to the connection ∇ on M, i.e. $\omega(A^*) = A$ and $\omega(E[e_i]) = 0$. The second main result in this section is a representation theorem of horizontal semimartingales by SDDE on the frame bundle BM, that is, every horizontal semimartingale is a solution of an SDDE.

Theorem 2.3 (SDDE representation of horizontal semimartingales) *Let γ be a horizontal semimartingale in BM. Then γ is a solution of an SDDE.*

Proof:
By Shigekawa representation of horizontal semimartingales [8], we know that γ is solution of the SDE:

$$dx_t = \sum_{i=1}^d E[e_i](x_t) \circ dN_t^i, \tag{5}$$

where N_t is the $\mathbb{R}^d$-semimartingale $\int_0^t \theta \circ d\gamma$ and θ is the canonical 1-form on BM defined by $\theta(p) = p^{-1}\pi_*(p)$. We shall prove that equation (5) is also an SDDE with respect to the horizontal connection ∇^H.
It is enough to prove that the global fields of frames $E[v]$ (where $v \in \mathbb{R}^n$ and $E[v] = v_1 E[e_1] + \ldots + v_n E[e_n]$), are parallel for ∇^H along horizontal curves: let α be a curve in M, and $v \in \mathbb{R}^n$.

Then by Lemma 2.1, formula (3) and the definitions:

$$\begin{aligned}
P_{t,0}^{\nabla^H}(\alpha_p^H)(E[v](\alpha_p^H(0))) &= (P_{t,0}^{\nabla}(\alpha)(\pi_*(p)(E[v](\alpha(0)))))^H \\
&= (P_{t,0}^{\nabla}(\alpha)(p(v)))^H \\
&= (\alpha_p^H(t)(v))^H \\
&= E[v](\alpha_p^H(t)).
\end{aligned}$$

Hence, we have the invariance of $E[v]$ with respect to parallel transport:

$$P_{t,0}^{\nabla^H}(\alpha_p^H)(E[v](\alpha_p^H(0))) = E[v](\alpha_p^H(t)).$$

which implies that γ satisfies the SDDE:

$$dx_t = \sum_{i=1}^{d} P_{t,t-1}^{\nabla^H}\ E[e_i](x_{t-1})\ \circ dN_t^i.$$

□

Note that, in particular, the Shigekawa representation of horizontal semimartingales implies that horizontal lift of solutions of SDDE on M can be written as a solution of a non-delay equation on BM.

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