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A NON-COMMUTATIVE SEWING LEMMA

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Abstract

A non-commutative version of the sewing lemma [1] is proved, with some applications

Introduction

In a preceding paper [1] we proved a sewing lemma which was a key result for the study of Hölder continuous functions. In this paper we give a non-commutative version of this lemma.

In the first section we recall the commutative version, and give some applications (Young integral and stochastic integral).

In the second section we prove the non-commutative version. This last result has interesting applications: an extension of the so-called integral product, an application to a Lie-Trotter type formula, and a sharpening of the Lyons theorem about multiplicative functionals [3,4,5].

Note that we replaced the Hölder modulus of continuity t^α by a more general modulus $V(t)$.

This paper was elaborated with the regretted G. Mokobodzki. The writing has only been done after his death.

1 The additive sewing lemma

0 Definition: *We say that a function $V(t)$ defined on $[0,T[$ is a control function if it is non decreasing, $V(0)=0$ and $\sum_{n\geq 1} V(1/n) < \infty$.*

As easily seen, this is equivalent to the property

$$\overline{V}(t) = \sum_{n\geq 0} 2^n V(t.2^{-n}) < \infty$$

for every $t \geq 0$. For example, t^α and $t/(\log t^{-1})^\alpha$ with $\alpha > 1$ are control functions.

Observe that we have

$$\overline{V}(t) = V(t) + \cdots + 2^n V(t.2^{-n}) + 2^{n+1}\overline{V}(t.2^{-n-1})$$

from which follows that $\operatorname*{Lim}_{t\to 0} \overline{V}(t)/t = 0$.

1 Theorem: *Consider a continuous function $\mu(a,b)$ defined for $0 \leq a \leq b < T$ satisfying the relation*

$$|\mu(a,b) - \mu(a,c) - \mu(c,b)| \leq V(b-a)$$

for every $c \in [a,b]$, where V is a control function. Then there exists a unique continuous function $\varphi(t)$ on $[0,T[$, up to an additive constant, such that

$$|\varphi(b) - \varphi(a) - \mu(a,b)| \leq \overline{V}(b-a)$$

Proof : Put $\mu'(a,b) = \mu(a,c) + \mu(c,b)$ for $c = (a+b)/2$, $\mu^{(0)} = \mu$ and $\mu^{(n+1)} = \mu^{(n)\prime}$. We easily get for $n \geq 0$

$$|\mu^{(n)}(a,b) - \mu^{(n+1)}(a,b)| \leq 2^n V(2^{-n}|b-a|)$$

so that the series $\sum_{n\geq 0} |\mu^{(n)}(a,b) - \mu^{(n+1)}(a,b)| \leq \overline{V}(b-a)$ converges, and the sequence $\mu^{(n)}(a,b)$ converges to a limit $u(a,b)$. For $c = (a+b)/2$ we have $\mu^{(n+1)}(a,b) = \mu^{(n)}(a,c) + \mu^{(n)}(c,b)$ which implies

$$u(a,b) = u(a,c) + u(c,b)$$

We say that u is midpoint-additive.

Now, we prove that u is the unique midpoint-additive function with the inequality $|u(a,b) - \mu(a,b)| \leq \mathrm{Cst}\,\overline{V}(b-a)$. Indeed if we have another one v, we get

$$|v(a,b) - u(a,b)| \leq K.\overline{V}(b-a)$$

and by induction $|v(a,b) - u(a,b)| \leq 2^n K.\overline{V}[2^{-n}(b-a)]$ which vanishes as $n \to \infty$ as mentioned above. Let k be an integer $k \geq 3$, and take the function

$$w(a,b) = \sum_{i=0}^{k-1} u(t_i, t_{i+1})$$

with $t_i = a + i.(b-a)/k$. It follows that w also is midpoint-additive, and satisfies

$$|w(a,b) - \mu(a,b)| \leq \mathrm{Cst}_k\, \overline{V}(b-a)$$

hence we have $w = u$, that is u is in fact rationnally-additive. As μ is continuous, then so also is u, as the defining series converges uniformly for $0 \leq a \leq b < T$. Then u is additive, and it suffices to put $\varphi(t) = u(0,t)$. $\square$

2 Proposition: *(Riemann sums) Let* $\sigma = \{t_i\}$ *some finite subdivision of* $[a,b]$. *Put* $\delta = \operatorname{Sup}_i |t_{i+1} - t_i|$. *Then*

$$\operatorname*{Lim}_{\delta \to 0} \sum_i \mu(t_i, t_{i+1}) = \varphi(b) - \varphi(a)$$

Proof : We have

$$\varphi(b) - \varphi(a) - \sum_i \mu(t_i, t_{i+1}) = \sum_i [\varphi(t_{i+1}) - \varphi(t_i) - \mu(t_i, t_{i+1})]$$

$$\left| \varphi(b) - \varphi(a) - \sum_i \mu(t_i, t_{i+1}) \right| \leq \sum_i \overline{V}(t_{i+1} - t_i)) \leq \varepsilon \sum_i (t_{i+1} - t_i) = (b-a)\varepsilon$$

since $\overline{V}(\delta)/\delta \leq \varepsilon$ as $\delta \to 0$.

3 Remarks: a) In fact the result holds even in the case of discontinuous μ, as proved in the appendix.

b) The result obviously extends to Banach spaces valued functions μ.

In the case $V(t) = t^\alpha$ with $\alpha > 1$, we get $\overline{V}(t) = \dfrac{2^\alpha\, t^\alpha}{2^\alpha - 2}$.

Example 1: The Young integral

Take $V(t) = t^{2\alpha}$ with $\alpha > 1/2$. If x and y are two α-Hölder continuous functions on $[0,1]$, put

$$\mu(a,b) = x_a(y_b - y_a)$$

We get

$$\mu(a,b) - \mu(a,c) - \mu(c,b) = -(x_c - x_a)(y_b - y_c)$$

so that

$$|\mu(a,b) - \mu(a,c) - \mu(c,b)| \leq \|x\|_\alpha \|y\|_\alpha |b-a|^{2\alpha}$$

where $\|x\|_\alpha$ is the norm in the space $\mathcal{C}^\alpha$. Let φ be the function of theorem 2, put

$$\int_a^b x_t\, dy_t = \varphi(b) - \varphi(a)$$

This is a Young integral (*cf.* also [7]).

Remark: We could take $x \in \mathcal{C}^\alpha$, $y \in \mathcal{C}^\beta$ with $\alpha + \beta > 1$.

Example 2: The stochastic integral

Let X_t be the standard $\mathbb{R}^m$-valued Brownian motion. As is well known, $t \to X_t$ is $\mathcal{C}^{1/2}$ with values in L^2. Let f be a tensor-valued $\mathcal{C}^2$-function with bounded derivatives of order 2 on $\mathbb{R}^m$. Put

$$\mu(a,b) = f(X_a) \otimes (X_b - X_a) + \nabla f(X_a) \otimes \int_a^b (X_t - X_a) \otimes dX_t$$

where the last integral is taken in the Ito or in the Stratonovitch sense. Let N_2 be the natural norm on L^2. By straightforward computations, we get

$$N_2[\mu(a,b) - \mu(a,c) - \mu(c,b)] \leq K.\|\nabla^2 f\|_\infty |b-a|^{3/2}$$

By the additive sewing lemma, there exists a unique L^2-valued function $\varphi(t)$ such that $N_2[\varphi(b) - \varphi(a) - \mu(a,b)] \leq \mathrm{Cst}\,.|b-a|^{3/2}$ (the control function is $V(t) = t^{3/2}$). It is easily seen that

$$\varphi(b) - \varphi(a) = \int_a^b f(X_t) \otimes dX_t$$

in the Ito or in the Stratonovitch sense.

Observe that as the stochastic integral $\int_a^b X_t \otimes dX_t$ has $\mathcal{C}^\alpha$-trajectories almost surely for $1/3 < \alpha < 1/2$, analoguous computations as above yield $\mathcal{C}^\alpha$-trajectories for $\int_a^b f(X_t) \otimes dX_t$ on the same set of paths as $\int_a^b X_t \otimes dX_t$.

4 Remark: For the FBM with $\alpha > 1/4$, the reader is referred to our previous paper [1].

2 The multiplicative sewing lemma

Here we need a strong notion of control function

5 Definition: *We say that a function $V(t)$ defined on $[0,T[$ is a strong control function if it is a control function and there exists a $\theta > 2$ such that for every t*

$$\overline{V}(t) = \sum_{n\geq 0} \theta^n V(t.2^{-n}) < \infty$$

We consider an associative monoide $\mathcal{M}$ with a unit element I, and we assume that $\mathcal{M}$ is complete under a distance d satisfying

$$d(xz,yz) \leq |z|\, d(x,y), \qquad d(zx,zy) \leq |z|\, d(x,y)$$

for every $x,y,z \in \mathcal{M}$, where $z \to |z|$ is a Lipschitz function on $\mathcal{M}$ with $|I| = 1$.

Let $\mu(a,b)$ be an $\mathcal{M}$-valued function defined for $0 \leq a \leq b < T$. We assume that μ is continuous, that $\mu(a,a) = I$ for every a, and that for every $a \leq c \leq b$ we have

$$d(\mu(a,b), \mu(a,c)\mu(c,b)) \leq V(b-a) \tag{1}$$

We say that an $\mathcal{M}$-valued $u(a,b)$ is multiplicative if $u(a,b) = u(a,c)u(c,b)$ for every $a \leq c \leq b$.

6 Theorem: *There exists a unique continuous multiplicative function u such that $d(\mu(a,b), u(a,b)) \leq \mathrm{Cst}\,\overline{V}(b-a)$ for every $a \leq b$.*

Proof : Put $\mu_0 = \mu$ and by induction

$$\mu_{n+1}(a,b) = \mu_n(a,c)\mu_n(c,b) \quad \text{where} \quad c = (a+b)/2$$

$$h_n(t) = \operatorname*{Sup}_{b-a\le t} |\mu_n(a,b)|, \qquad U_n(t) = \operatorname*{Sup}_{b-a\le t} d(\mu_{n+1}(a,b), \mu_n(a,b))$$

The functions h_n and U_n are continuous non decreasing with $h_n(0) = 1$ and $U_n(0) = 0$. Let κ be the Lipschitz constant of $z \to |z|$. One has

$$h_{n+1}(t) \le h_n(t) + \kappa U_n(t) \le h_0(t) + \kappa U_0(t) + \cdots + \kappa U_n(t)$$

$$U_{n+1}(t) \le [h_n(t/2) + h_{n+1}(t/2)]\, U_n(t/2) \tag{2}$$

Let $\tau > 0$ be such that $h_0(\tau) + \kappa \overline{V}(\tau) \le \theta/2$. Assume that $U_i(t) \le \theta^i V(t/2^i)$ for $t \le \tau$ and $i \le n$. One has $h_{n+1}(t) \le \theta/2$, then

$$U_{n+1}(t) \le \theta U_n(t/2) \le \theta^{n+1} V(t/2^{n+1})$$

for $t \le \tau$ and every n by induction.

Hence for $t \le \tau$ the series $U_n(t)$ converges, so that the sequence $h_n(\tau)$ is bounded. By inequality (2) the series $U_n(2\tau)$ converges, and the sequence $h_n(2\tau)$ is bounded. From one step to the other we see that the sequence h_n is locally bounded, and that the series U_n converges locally uniformly on $[0, T[$. It follows that the sequence $\mu_n(a,b)$ converges locally uniformly to a continuous function $u(a,b)$ which is midpoint-multiplicative, that is $u(a,b) = u(a,c)u(c,b)$ for $c = (a+b)/2$. One has $d(u,\mu) \le \mathrm{Cst}\, \overline{V}$.

Next we prove the uniqueness of u. Let v be a continuous function with the same properties as u. Put $K(t) = \operatorname*{Sup}_{b-a\le t} \operatorname{Sup}[|u(a,b)|, |v(a,b)|]$. Let $\tau_1 > 0$ be such that $K(\tau_1) \le \theta/2$. One has $d(u(a,b), v(a,b)) \le k\, \overline{V}(b-a)$ with some constant k, then $d(u(a,b), v(a,b)) \le 2K(t/2)k\overline{V}(t/2) \le k\,\theta \overline{V}(t/2)$ for $b - a \le t \le \tau_1$, and by induction $d(u(a,b), v(a,b)) \le k\, \theta^n \overline{V}(t/2^n)$. It follows that $u(a,b) = v(a,b)$ for $b - a \le \tau_1$. This equality extends to every $b - a$ by midpoint-multiplicativity.

Finally we prove that u is multiplicative. We argue as in the additive case, and we put for an integer m

$$w(a,b) = \prod_{i=0}^{m-1} u(t_i, t_{i+1})$$

where $t_i = a + i.(b-a)/m$. For simplicity we limit ourselves to the case $m = 3$, that is

$$w(a,b) = u(a,c')u(c',c'')u(c'',b)$$

with $c' = a+(b-a)/3$, $c'' = a+2(b-a)/3$. Observe that w is obviously midpoint-multiplicative. Take $a \le b \le T_0 < T$, we get with successive constants k_i

$$\begin{aligned} d(w(a,b), \mu(a,b)) \le& k_1 \overline{V}(b-a) + d(w(a,b), \mu(a,c')\mu(c',b)) \\ &\le k_2 \overline{V}(b-a) + d(u(a,c')u(c',c'')u(c'',b), u(a,c')\mu(c',b)) \\ &\qquad + d(u(a,c')\mu(c',b), \mu(a,c')\mu(c',b)) \\ &\le k_3 \overline{V}(b-a) + kd(u(c',b), \mu(c',b)) + kd(u(a,c'), \mu(a,c')) \\ &\le k_4 \overline{V}(b-a) \end{aligned}$$

By the second step of the proof, we get $w = u$. The same proof extends to every m, so that u is in fact rationally multiplicative. As u is continuous, it is multiplicative. ∎

7 Remark: Let ν be a function with the same properties as μ. Moreover suppose that $d(\nu(a,b),\mu(a,b)) \leq \mathrm{Cst}\,\overline{V}(b-a)$ for every $a \leq b$. Then ν defines the same multiplicative function u as μ.

Example 3: The integral product

Let $t \to A_t$ a $\mathcal{C}^\alpha$ function with values in a Banach algebra $\mathcal{A}$ with a unit I. Put $A_{ab} = A_b - A_a$ and

$$\mu(a,b) = I + A_{ab}$$

We get

$$\mu(a,b) - \mu(a,c)\mu(c,b) = -A_{ac}A_{cb}$$

Suppose that $\alpha > 1/2$, then the multiplicative sewing lemma applies with the obvious distance, and there exists a unique multiplicative function $u(a,b)$ with values in $\mathcal{A}$ such that

$$|u(a,b) - \mu(a,b)| \leq \mathrm{Cst}\,|b-a|^{2\alpha}$$

In view of the remark 8, we get the same $u(a,b)$ by taking $\nu(a,b) = \mathrm{e}^{A_{ab}}$. A good notation for $u(a,b)$ is

$$u(a,b) = \prod_a^b (I + dA_t) = \prod_a^b \mathrm{e}^{dA_t}$$

8 Theorem: *Put $H_t = u(0,t)$. Then this is the solution of the EDO*

$$H_t = I + \int_0^t H_s\,dA_s$$

where the integral is taken in the Young sense.

Proof : We have only to verify that $|H_b - H_a - H_aA_{ab}| \leq \mathrm{Cst}\,|b-a|^{2\alpha}$. The first member is worth

$$u(0,a)[u(a,b) - I - A_{ab}] = u(0,a)[u(a,b) - \mu(a,b)]$$

so that we are done.

Example 4: A Trotter type formula

Let $t \to A_t$ and $t \to B_t$ as in the previous paragraph, and put

$$\mu(a,b) = [I + A_{ab}][I + B_{ab}]$$

It is straightforward to verify the good inequality

$$|\mu(a,b) - \mu(a,c)\mu(c,b)| \leq \mathrm{Cst}\,|b-a|^{2\alpha}$$

so that we get a multiplicative $u(a,b)$ such that $|u(a,b) - \mu(a,b)| \leq \mathrm{Cst}\,|b-a|^{2\alpha}$ or equivalently

$$|u(a,b) - I - A_{ab} - B_{ab}| \leq \mathrm{Cst}\,|b-a|^{2\alpha}$$

$$u(a,b) = \prod_a^b (I + dA_t + dB_t) = \prod_a^b \mathrm{e}^{dA_t}\,\mathrm{e}^{dB_t}$$

that is

$$u(a,b) = \operatorname*{Lim}_{n\to\infty} \prod_{i=1}^{2^n} \mathrm{e}^{A_{t_i t_{i+1}}} \, \mathrm{e}^{B_{t_i t_{i+1}}}$$

for $t_{i+1} - t_i = (b-a)/2^n$.

Particularly we can take $A_t = tA$ and $B_t = tB$ with $\alpha = 1$, this yields

$$\mathrm{e}^{A+B} = u(0,1) = \operatorname*{Lim}_{n\to\infty} \prod_{i=1}^{2^n} \mathrm{e}^{A/2^n} \, \mathrm{e}^{B/2^n} = \operatorname*{Lim}_{n\to\infty} \left[\mathrm{e}^{A/2^n} \, \mathrm{e}^{B/2^n}\right]^{2^n}$$

Indeed, in view of the remark 8 as above, we have $u(0,1) = \mathrm{e}^{A+B}$, which is the classical Lie-Trotter formula.

Example 5: Extending the Lyons theorem

Let $\mathcal{A}$ be a Banach algebra with a unit I. Take $\mu(a,b)$ of the form

$$\mu(a,b) = \sum_{k=0}^{N} \lambda^k A^{(k)}_{ab}$$

where $A^{(0)}_{ab} = I$, $(a,b) \to A^{(k)}_{ab} \in \mathcal{A}$ is continuous, λ is a real parameter and N a given integer. We have

$$\mu(a,c)\mu(c,b) = \sum_{k=0}^{N} \lambda^k B^{(k)}_{acb} + \sum_{k=N+1}^{2N} \lambda^k C^{(k)}_{acb}$$

Following [5], we suppose the algebraic hypothesis for $k \leq N$

$$A^{(k)}_{ab} = \sum_{i=0}^{k} A^{(i)}_{ac} A^{(k-i)}_{cb} \tag{3}$$

that is

$$\mu(a,c)\mu(c,b) = \mu(a,b) + \sum_{k=N+1}^{2N} \lambda^k C^{(k)}_{acb}$$

9 Theorem: *Under the condition (3) and the inequality*

$$|A^{(k)}_{ab}| \leq M|b-a|^{k\alpha}$$

for every $k \leq N$, where $\alpha > 1/(N+1)$, there exists a unique multiplicative function $u(a,b)$ such that

$$|u(a,b) - \mu(a,b)| \leq \mathrm{Cst}\,|b-a|^{(N+1)\alpha}$$

Moreover we have

$$u(a,b) = \sum_{k=0}^{N} \lambda^k A^{(k)}_{ab} + \sum_{k>N} \lambda^k B^{(k)}_{ab} \tag{4}$$

where the series is normally convergent for every λ.

Proof : The only problem is to prove formula (4), that is to prove that u is the sum of its Taylor expansion with respect to λ. In the case where $\mathcal{A}$ is a complex Banach algebra, the proof of the multiplicative sewing lemma yields a sequence of holomorphic functions which converges uniformly with respect to λ in every compact set of $\mathbb{C}$. Hence $u(a,b)$ in holomorphic in $\lambda \in \mathbb{C}$. If $\mathcal{A}$ is only a real Banach algebra, we get a sequence of holomorphic functions with values in the complexified Banach space of $\mathcal{A}$, and the result follows. It remains to observe that the $N+1$ first terms of the Taylor expansion are the same for every function of the sequence μ_n converging to u.

Application to the Lyons theorem: Let E be a Banach space. Denote $E^n = E^{\otimes n}$. Suppose that every E^n has a cross-norm such that

$$\|u \otimes v\|_{n+m} \leq \|u\|_n \|v\|_m$$

for every $u \in E^n$, $v \in E^m$. Let $\mathcal{A}$ be the completed tensor algebra under the norm

$$\|t\| = \sum_{n \geq 0} \|t_n\|_n$$

This is a Banach algebra.

Let N be given, and for every $k \leq N$, let $(a,b) \to X_{ab}^{(k)}$ be an E^k-valued continuous function such that

$$X_{ab}^{(k)} = \sum_{i=0}^{k} X_{ac}^{(i)} \otimes X_{cb}^{(k-i)}$$

for $a \leq c \leq b$. Suppose that $\alpha > 1/(N+1)$, and that we have for $k \leq N$

$$\|X_{ab}^{(k)}\|_k \leq M.|b-a|^{k\alpha}$$

The previous theorem applies to the function

$$\mu(a,b) = \sum_{k=1}^{N} \lambda^k X_{ab}^{(k)}$$

so that there exists a unique continuous function $(a,b) \to Y_{ab}^{(k)}$ for every k such that $Y^{(k)} = X^{(k)}$ for $k \leq N$,

$$Y_{ab}^{(k)} = \sum_{i=0}^{k} Y_{ac}^{(i)} \otimes Y_{cb}^{(k-i)}$$

for every integer k and every $a \leq c \leq b$, and

$$\sum_{k>N} \|Y_{ab}^{(k)}\|_k \leq \mathrm{Cst}\, |b-a|^{(N+1)\alpha}$$

Remark: This theorem sharpens the theorem 3.2.1 of [5].

Some estimations

We return to formula (4) of theorem 10

$$u(a,b) = \sum_{k=0}^{N} \lambda^k A_{ab}^{(k)} + \sum_{k=N+1}^{\infty} \lambda^k B_{ab}^{(k)}$$

for $N = \mathrm{Ent}(1/\alpha)$, and we put $B_{ab}^{(k)} = A_{ab}^{(k)}$ for simplification, so that we have

$$u(a,b) = \sum_{k=0}^{\infty} \lambda^k A_{ab}^{(k)}$$

There exist best constants K_n for every n such that $|A_{ab}^{(n)}| \leq K_n |b-a|^{n\alpha}$. We have

$$A_{ab}^{(n+1)} = A_{ac}^{(n+1)} + A_{cb}^{(n+1)} + \sum_{k=1}^{n} A_{ac}^{(k)} A_{cb}^{(n-k+1)}$$

By taking $c = (a+b)/2$ we get

$$|A_{ab}^{(n+1)}| \leq 2^{-(n+1)\alpha} \left[2K_{n+1} + \sum_{k=1}^{n} K_k K_{n-k+1} \right] |b-a|^{(n+1)\alpha}$$

and then

$$(2^{(n+1)\alpha} - 2) K_{n+1} \leq \sum_{k=1}^{n} K_k K_{n-k+1}$$

Let $0 < \beta < \alpha$, and introduce the entire function

$$\mathrm{e}(x) = \mathrm{e}_\beta(x) = \sum_{n \geq 0} \frac{x^n}{n!^\beta} \qquad \Rightarrow \qquad \mathrm{e}(x)^2 = \sum_{n \geq 0} E_{n,\beta} \frac{x^n}{n!^\beta}$$

where

$$E_{n,\beta} = \sum_{k=0}^{n} \left[C_n^k \right]^\beta \leq 2^{n\beta}(n+1)$$

There exist $c \geq 0$ and $x > 0$ such that for $1 \leq m \leq N$

$$K_m \leq c.x^m / m!^\beta \tag{5}$$

Hence we have for $n \geq N$

$$(2^{(n+1)\alpha} - 2) K_{n+1} \leq c^2 x^{n+1} \sum_{k=1}^{n} (k!)^{-\beta} (n-k+1)^{-\beta} \leq c^2 x^{n+1} [(n+1)!]^{-\beta} E_{n+1,\beta}$$

In order that (5) holds for every n, it suffices that

$$\frac{1}{c} \geq \operatorname*{Sup}_{n > N} \frac{E_{n+1,\beta}}{2^{(n+1)\alpha} - 2}$$

which is realizable since the fraction in the second hand member shrinks to 0 as $n \to \infty$ (first choose c and afterward choose x).

10 Corollary: *Put* $c' = \mathrm{Max}(c,1)$, *we have*

$$|u(a,b)| \leq c'\,\mathrm{e}_\beta(|\lambda|x|b-a|^\alpha)$$

11 Remarks: a) Note that for $\alpha = 1$ one can take $\beta = \alpha = 1$ so that we recover the classical inequality.

b) For $\beta < 1$, the function $\mathrm{e}_\beta(x)$ increases faster than the exponential function (*cf.* Schwartz [6] for a comparison when $\beta = 1/2$).

c) there are some analoguous estimates in Gubinelli [2].

Appendix: the discontinuous case

As announced in Remark 4a), we extend the additive sewing lemma in the case where μ is discontinuous. We go back to the proof of the lemma: we get a unique function $u(a,b)$ which is rationally additive and such that $|u(a,b) - \mu(a,b)| \leq \mathrm{Cst}\,\overline{V}(b-a)$. Put

$$v_n(a,b) = u(a_n,b_n) - u(a_n,a) + u(b_n,b)$$

where $a_n \leq a$ and $b_n \leq b$ are the classical dyadic approximations of a and b. As easily verified, v_n is additive for every $a \leq c \leq b$. Besides, we have

$$|v_n(a,b) - u(a,b)| \leq 2\overline{V}(b-a) + 2\overline{V}(b_n-a_n) + \overline{V}(a-a_n) + \overline{V}(b-b_n)$$

so that the sequence $v_n(a,b) - u(a,b)$ is bounded. Let $v(a,b)$ be the limit of $v_n(a,b)$ according to an ultrafilter $\mathcal{U} \to \infty$. We first have $v(a,b) = v(a,c) + v(c,b)$ for every $a \leq c \leq b$. Then we get

$$|v(a,b) - \mu(a,b)| \leq 3\overline{V}(b-a) + 2\,\operatorname*{Lim}_{\mathcal{U}}\overline{V}(b_n-a_n) \leq 5\overline{V}(2(b-a))$$

As $V(2t)$ is also a control function for μ, v is the unique additive function such that $|v(a,b) - \mu(a,b)| \leq 5\overline{V}(2(b-a))$, which implies that $v = u$. Hence u is completely additive. $\square$

Here we point out the important fact that the result also holds if μ takes values in a Banach space B. Indeed, the proof is exactly the same, the last limit according to $\mathcal{U}$ must be taken in the bidual B'' with the topology $\sigma(B'',B')$, but the result u belongs to B.

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