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CONTINUOUS-TIME TRADING AND THE EMERGENCE OF VOLATILITY

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Abstract

This note continues investigation of randomness-type properties emerging in idealized financial markets with continuous price processes. It is shown, without making any probabilistic assumptions, that the strong variation exponent of non-constant price processes has to be 2, as in the case of continuous martingales.

1 Introduction

This note is part of the recent revival of interest in game-theoretic probability (see, e.g., [7, 8, 4, 2, 3]). It concentrates on the study of the "$\sqrt{\mathrm{d}t}$ effect", the fact that a typical change in the value of a non-degenerate diffusion process over short time period $\mathrm{d}t$ has order of magnitude $\sqrt{\mathrm{d}t}$. Within the "standard" (not using non-standard analysis) framework of game-theoretic probability, this study was initiated in [9]. In our definitions, however, we will be following [11], which also establishes some other randomness-type properties of continuous price processes. The words such as "positive", "negative", "before", and "after" will be understood in the wide sense of $\geq$ or $\leq$, respectively; when necessary, we will add the qualifier "strictly".

2 Null and almost sure events

We consider a perfect-information game between two players, Reality (a financial market) and Sceptic (a speculator), acting over the time interval $[0, T]$, where T is a positive constant fixed throughout. First Sceptic chooses his trading strategy and then Reality chooses a continuous function $\omega : [0, T] \to \mathbb{R}$ (the price process of a security).

Let Ω be the set of all continuous functions $\omega : [0, T] \to \mathbb{R}$. For each $t \in [0, T]$, $\mathcal{F}_t$ is defined to be the smallest σ-algebra that makes all functions $\omega \mapsto \omega(s)$, $s \in [0, t]$, measurable. A

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process S is a family of functions $S_t : \Omega \to [-\infty, \infty]$, $t \in [0,T]$, each S_t being $\mathcal{F}_t$-measurable (we drop the adjective "adapted"). An *event* is an element of the σ-algebra $\mathcal{F}_T$. Stopping times $\tau : \Omega \to [0,T] \cup \{\infty\}$ w.r. to the filtration $(\mathcal{F}_t)$ and the corresponding σ-algebras $\mathcal{F}_\tau$ are defined as usual; $\omega(\tau(\omega))$ and $S_{\tau(\omega)}(\omega)$ will be simplified to $\omega(\tau)$ and $S_\tau(\omega)$, respectively (occasionally, the argument ω will be omitted in other cases as well).

The class of allowed strategies for Sceptic is defined in two steps. An *elementary trading strategy* G consists of an increasing sequence of stopping times $\tau_1 \le \tau_2 \le \cdots$ and, for each $n = 1, 2, \ldots$, a bounded $\mathcal{F}_{\tau_n}$-measurable function h_n. It is required that, for any $\omega \in \Omega$, only finitely many of $\tau_n(\omega)$ should be finite. To such G and an *initial capital* $c \in \mathbb{R}$ corresponds the *elementary capital process*

$$\mathcal{K}_t^{G,c}(\omega) := c + \sum_{n=1}^{\infty} h_n(\omega)\bigl(\omega(\tau_{n+1} \wedge t) - \omega(\tau_n \wedge t)\bigr), \quad t \in [0,T]$$

(with the zero terms in the sum ignored); the value $h_n(\omega)$ will be called the *portfolio* chosen at time τ_n, and $\mathcal{K}_t^{G,c}(\omega)$ will sometimes be referred to as Sceptic's capital at time t.

A *positive capital process* is any process S that can be represented in the form

$$S_t(\omega) := \sum_{n=1}^{\infty} \mathcal{K}_t^{G_n,c_n}(\omega), \tag{1}$$

where the elementary capital processes $\mathcal{K}_t^{G_n,c_n}(\omega)$ are required to be positive, for all t and ω, and the positive series $\sum_{n=1}^{\infty} c_n$ is required to converge. The sum (1) is always positive but allowed to take value ∞. Since $\mathcal{K}_0^{G_n,c_n}(\omega) = c_n$ does not depend on ω, $S_0(\omega)$ also does not depend on ω and will sometimes be abbreviated to S_0.

The *upper probability* of a set $E \subseteq \Omega$ is defined as

$$\overline{\mathbb{P}}(E) := \inf\bigl\{S_0 \bigm| \forall \omega \in \Omega : S_T(\omega) \ge \mathbb{I}_E(\omega)\bigr\},$$

where S ranges over the positive capital processes and $\mathbb{I}_E$ stands for the indicator of E. Notice that $\overline{\mathbb{P}}$ is not a probability measure, even if restricted to $\mathcal{F}_T$: for example, the events $E^{(c)} := \{\omega \in \Omega \mid \omega(t) = c, \forall t \in [0,T]\}$ are disjoint for different c, but $\overline{\mathbb{P}}(E^{(c)}) = 1$ for all c; see [11] for less trivial examples.

We say that $E \subseteq \Omega$ is *null* if $\overline{\mathbb{P}}(E) = 0$. A property of $\omega \in \Omega$ will be said to hold *almost surely* (a.s.), or for *almost all* ω, if the set of ω where it fails is null.

Upper probability is countably (and finitely) subadditive:

Lemma 1. *For any sequence of subsets $E_1, E_2, \ldots$ of Ω,*

$$\overline{\mathbb{P}}\left(\bigcup_{n=1}^{\infty} E_n\right) \le \sum_{n=1}^{\infty} \overline{\mathbb{P}}(E_n).$$

In particular, a countable union of null sets is null.

3 Main result

For each $p \in (0, \infty)$, the *strong p-variation* of $\omega \in \Omega$ is

$$\mathrm{var}_p(\omega) := \sup_{\kappa} \sum_{i=1}^{n} \left|\omega(t_i) - \omega(t_{i-1})\right|^p,$$

where n ranges over all positive integers and κ over all subdivisions $0 = t_0 < t_1 < \cdots < t_n = T$ of the interval $[0, T]$. It is obvious that there exists a unique number $\mathrm{vex}(\omega) \in [0, \infty]$, called the *strong variation exponent* of ω, such that $\mathrm{var}_p(\omega)$ is finite when $p > \mathrm{vex}(\omega)$ and infinite when $p < \mathrm{vex}(\omega)$; notice that $\mathrm{vex}(\omega) \notin (0, 1)$.
The following is a game-theoretic counterpart of the well-known property of continuous semimartingales (Lepingle [5], Theorem 1 and Proposition 3; Lévy [6] in the case of Brownian motion).

Theorem 1. *For almost all $\omega \in \Omega$,*

$$\mathrm{vex}(\omega) = 2 \ \textit{or } \omega \textit{ is constant.} \tag{2}$$

(Alternatively, (2) can be expressed as $\mathrm{vex}(\omega) \in \{0, 2\}$.)

4 Proof

The more difficult part of this proof ($\mathrm{vex}(\omega) \le 2$ a.s.) will be modelled on the proof in [1], which is surprisingly game-theoretic in character. The proof of the easier part is modelled on [10]. (Notice, however, that our framework is very different from those of [1] and [10], which creates additional difficulties.) Without loss of generality we impose the restriction $\omega(0) = 0$.

Proof that $\mathrm{vex}(\omega) \ge 2$ for non-constant ω a.s.

We need to show that the event $\mathrm{vex}(\omega) < 2$ & $\mathrm{nc}(\omega)$ is null, where $\mathrm{nc}(\omega)$ stands for "ω is not constant". By Lemma 1 it suffices to show that $\mathrm{vex}(\omega) < p$ & $\mathrm{nc}(\omega)$ is null for each $p \in (0, 2)$. Fix such a p. It suffices to show that $\mathrm{var}_p(\omega) < \infty$ & $\mathrm{nc}(\omega)$ is null and, therefore, it suffices to show that the event $\mathrm{var}_p(\omega) < C$ & $\mathrm{nc}(\omega)$ is null for each $C \in (0, \infty)$. Fix such a C. Finally, it suffices to show that the event

$$E_{p,C,A} := \left\{ \omega \in \Omega \;\middle|\; \mathrm{var}_p(\omega) < C \;\&\; \sup_{t \in [0,T]} |\omega(t)| > A \right\}$$

is null for each $A > 0$. Fix such an A.
Choose a small number $\delta > 0$ such that $A/\delta \in \mathbb{N}$ (in this note, $\mathbb{N} := \{1, 2, \ldots\}$), and let $\Gamma := \{k\delta \mid k \in \mathbb{Z}\}$ be the corresponding grid. Define a sequence of stopping times τ_n inductively by

$$\tau_{n+1} := \inf\bigl\{t > \tau_n \bigm| \omega(t) \in \Gamma \setminus \{\omega(\tau_n)\}\bigr\}, \quad n = 0, 1, \ldots,$$

with $\tau_0 := 0$ and $\inf \emptyset$ understood to be ∞. Set $T_A := \inf\{t \mid |\omega(t)| = A\}$, again with $\inf \emptyset := \infty$, and

$$h_n(\omega) := \begin{cases} 2\omega(\tau_n) & \text{if } \tau_n(\omega) < T \wedge T_A(\omega) \text{ and } n + 1 < C/\delta^p \\ 0 & \text{otherwise.} \end{cases}$$

The elementary capital process corresponding to the elementary gambling strategy $G := (\tau_n, h_n)_{n=1}^{\infty}$ and initial capital $c := \delta^{2-p} C$ will satisfy

$$\begin{aligned} \omega^2(\tau_{n+1}) - \omega^2(\tau_n) &= 2\omega(\tau_n)\left(\omega(\tau_{n+1}) - \omega(\tau_n)\right) + \left(\omega(\tau_{n+1}) - \omega(\tau_n)\right)^2 \\ &= \mathcal{K}^{G,c}_{\tau_{n+1}}(\omega) - \mathcal{K}^{G,c}_{\tau_n}(\omega) + \delta^2 \end{aligned}$$

provided $\tau_{n+1}(\omega) \le T \wedge T_A(\omega)$ and $n+1 < C/\delta^p$, and so satisfy

$$\omega^2(\tau_N) = \mathcal{K}^{G,c}_{\tau_N}(\omega) - \mathcal{K}^{G,c}_0 + N\delta^2 = \mathcal{K}^{G,c}_{\tau_N}(\omega) - \delta^{2-p}C + \delta^{2-p}N\delta^p \le \mathcal{K}^{G,c}_{\tau_N}(\omega) \tag{3}$$

provided $\tau_N(\omega) \le T \wedge T_A(\omega)$ and $N < C/\delta^p$. On the event $E_{p,C,A}$ we have $T_A(\omega) < T$ and $N < C/\delta^p$ for the N defined by $\tau_N = T_A$. Therefore, on this event

$$A^2 = \omega^2(T_A) \le \mathcal{K}^{G,c}_{T_A}(\omega) = \mathcal{K}^{G,c}_T(\omega).$$

We can see that $\mathcal{K}^{G,c}_t(\omega)$ increases from $\delta^{2-p}C$, which can be made arbitrarily small by making δ small, to A^2 over $[0,T]$; this shows that the event $E_{p,C,A}$ is null.
The only remaining gap in our argument is that $\mathcal{K}^{G,c}_t$ may become strictly negative strictly between some $\tau_n < T \wedge T_A$ and τ_{n+1} with $n+1 < C/\delta^p$ (it will be positive at all $\tau_N \in [0, T\wedge T_A]$ with $N < C/\delta^p$, as can be seen from (3)). We can, however, bound $\mathcal{K}^{G,c}_t$ for $\tau_n < t < \tau_{n+1}$ as follows:

$$\mathcal{K}^{G,c}_t(\omega) = \mathcal{K}^{G,c}_{\tau_n}(\omega) + 2\omega(\tau_n)\left(\omega(t) - \omega(\tau_n)\right) \ge 2|\omega(\tau_n)|\,(-\delta) \ge -2A\delta,$$

and so we can make the elementary capital process positive by adding the negligible amount $2A\delta$ to Sceptic's initial capital.

Proof that $\mathrm{vex}(\omega) \le 2$ a.s.

We need to show that the event $\mathrm{vex}(\omega) > 2$ is null, i.e., that $\mathrm{vex}(\omega) > p$ is null for each $p > 2$. Fix such a p. It suffices to show that $\mathrm{var}_p(\omega) = \infty$ is null, and therefore, it suffices to show that the event

$$E_{p,A} := \left\{ \omega \in \Omega \;\middle|\; \mathrm{var}_p(\omega) = \infty \;\&\; \sup_{t\in[0,T]} |\omega(t)| < A \right\}$$

is null for each $A > 0$. Fix such an A.
The rest of the proof follows [1] closely. Let $M_t(f,(a,b))$ be the number of upcrossings of the open interval (a,b) by a continuous function $f \in \Omega$ during the time interval $[0,t]$, $t \in [0,T]$. For each $\delta > 0$ we also set

$$M_t(f,\delta) := \sum_{k\in\mathbb{Z}} M_t\left(f,(k\delta,(k+1)\delta)\right).$$

The strong p-variation $\mathrm{var}_p(f,[0,t])$ of $f \in \Omega$ over an interval $[0,t]$, $t \le T$, is defined as

$$\mathrm{var}_p(f,[0,t]) := \sup_\kappa \sum_{i=1}^n |f(t_i) - f(t_{i-1})|^p\,,$$

where n ranges over all positive integers and κ over all subdivisions $0 = t_0 < t_1 < \cdots < t_n = t$ of the interval $[0,t]$ (so that $\mathrm{var}_p(f) = \mathrm{var}_p(f,[0,T])$). The following key lemma is proved in [1] (Lemma 1; in fact, this lemma only requires $p > 1$).

Lemma 2. *For all $f \in \Omega$, $t > 0$, and $q \in [1,p)$,*

$$\mathrm{var}_p(f,[0,t]) \le \frac{2^{p+q+1}}{1-2^{q-p}}\left(2c_{q,\lambda,t}(f)+1\right)\lambda^p,$$

where

$$\lambda \ge \sup_{s\in[0,t]} |f(s) - f(0)| \quad \text{and} \quad c_{q,\lambda,t}(f) := \sup_{k\in\mathbb{N}} 2^{-kq} M_t(f,\lambda 2^{-k}).$$

Another key ingredient of the proof is the following game-theoretic version of Doob's upcrossings inequality:

Lemma 3. *Let $c < a < b$ be real numbers. For each elementary capital process $S \geq c$ there exists a positive elementary capital process S^* that starts from $S_0^* = a - c$ and satisfies, for all $t \in [0, T]$ and $\omega \in \Omega$,*

$$S_t^*(\omega) \geq (b - a) M_t(S(\omega), (a, b)),$$

where $S(\omega)$ stands for the sample path $t \mapsto S_t(\omega)$.

Proof. The following standard argument is easy to formalize. Let G be an elementary gambling strategy leading to S (when started with initial capital S_0). An elementary gambling strategy G^* leading to S^* (with initial capital $a - c$) can be defined as follows. When S first hits a, G^* starts mimicking G until S hits b, at which point G^* chooses portfolio 0; after S hits a, G^* mimics G until S hits b, at which point G^* chooses portfolio 0; etc. Since $S \geq c$, S^* will be positive. $\square$

Now we are ready to finish the proof of the theorem. Let $T_A := \inf\{t \mid |\omega(t)| = A\}$ be the hitting time for $\{-A, A\}$ (with $T_A := T$ if $\{-A, A\}$ is not hit). By Lemma 3, for each $k \in \mathbb{N}$ and each $i \in \{-2^k + 1, \ldots, 2^k\}$ there exists a positive elementary capital process $S^{k,i}$ that starts from $A + (i-1)A2^{-k}$ and satisfies

$$S_{T_A}^{k,i} \geq A2^{-k} M_{T_A}\left(\omega, \left((i-1)A2^{-k}, iA2^{-k}\right)\right).$$

Summing $2^{-kq} S^{k,i}/A2^{-k}$ over $i \in \{-2^k + 1, \ldots, 2^k\}$, we obtain a positive elementary capital process S^k such that

$$S_0^k = 2^{-kq} \sum_{i=-2^k+1}^{2^k} \frac{A + (i-1)A2^{-k}}{A2^{-k}} \leq 2^{-kq} 2^{2k+1} \quad \text{and} \quad S_{T_A}^k \geq 2^{-kq} M_{T_A}(\omega, A2^{-k}).$$

Next, assuming $q \in (2, p)$ and summing over $k \in \mathbb{N}$, we obtain a positive capital process S such that

$$S_0 = \sum_{k=1}^{\infty} 2^{-kq} 2^{2k+1} = \frac{2^{3-q}}{1 - 2^{2-q}} \quad \text{and} \quad S_{T_A} \geq c_{q,A,T_A}(\omega).$$

On the event $E_{p,A}$ we have $T_A = T$ and so, by Lemma 2, $c_{q,A,T_A}(\omega) = \infty$. This shows that $S_T = \infty$ on $E_{p,A}$ and completes the proof.

5 Conclusion

Theorem 1 says that, almost surely,

$$\operatorname{var}_p(\omega) \begin{cases} < \infty & \text{if } p > 2 \\ = \infty & \text{if } p < 2 \text{ and } \omega \text{ is not constant.} \end{cases}$$

The situation for $p = 2$ remains unclear. It would be very interesting to find the upper probability of the event $\{\operatorname{var}_2(\omega) < \infty \text{ and } \omega \text{ is not constant}\}$. (Lévy's [6] result shows that this event is null when ω is the sample path of Brownian motion, while Lepingle [5] shows this for continuous, and some other, semimartingales.)

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