Elect. Comm. in Probab. **14** (2009), 192–201

ELECTRONIC COMMUNICATIONS in PROBABILITY

A NOTE ON STOCHASTIC INTEGRATION WITH RESPECT TO OPTIONAL SEMIMARTINGALES

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Submitted August 11, 2008, *accepted in final form* March 3, 2009

AMS 2000 Subject classification: 60G48, 60H05, 91B28
Keywords: stochastic integration theory, optional semimartingales, dynamic portfolio choice

Abstract

In this note we discuss the extension of the elementary stochastic Itô-integral w.r.t. an optional semimartingale. The paths of an optional semimartingale possess limits from the left and from the right, but may have double jumps. This leads to quite interesting phenomena in integration theory.

We find a mathematically tractable domain of general integrands. The simple integrands are embedded into this domain. Then, we characterize the integral as the unique continuous and linear extension of the elementary integral and show completeness of the space of integrals. Thus our integral possesses desirable properties to model dynamic trading gains in mathematical finance when security price processes follow optional semimartingales.

1 Introduction

In this note we discuss the extension of the elementary stochastic Itô-integral in a general framework where the integrator is an optional semimartingale. The paths of an optional semimartingale possess limits from the left and from the right, but may have double jumps. Such processes have been studied extensively by Lenglart [9] and Galtchouk [3, 4, 5, 6].

It turns out that the extension of the elementary integral to all predictable integrands is too small. Namely, the space of integrals for (suitably integrable) predictable integrands is still not complete (even w.r.t. the uniform convergence). This is of course in contrast to the standard framework with a càdlàg integrator, cf. [2].

Galtchouk [4] has introduced a stochastic integral w.r.t. an optional martingale with a larger domain. But the integral of [4] is not the *unique* (continuous and linear) extension of the elementary integral. There are stochastic integrals that can in no way be approximated by elementary integrals. This is an undesirable feature in some applications, e.g. if one wants to model trading gains from dynamic strategies by the integral. As real-world investment strategies are of course piecewise constant, it would not make sense to optimize over a set of integrals including some elements that cannot be approximated by elementary integrals.

In this note we introduce a mathematically tractable domain of integrands which is somehow between the small set of predictable integrands and the large domain in [4]. The latter is a two-dimensional product space of predictable and optional processes.

The simple strategies are embedded into our domain. Then, in the usual manner, we characterize the integral defined on this domain as the unique continuous and linear extension of the elementary integral and show its completeness. In mathematical finance completeness of the space of achievable trading gains guarantees that the supremum in a portfolio optimization problem is attained and in “complete markets” derivatives can be replicated and not only be approximated by gains from dynamic trading in the underlying securities.

In addition, this note may also provide another abstract view to the extension of the elementary integral and the identification of $1_{]]\tau_1,\tau_2]]} \bullet X_t$ with $X_{t\wedge\tau_2} - X_{t\wedge\tau_1}$ in the usual situation of a càdlàg integrator X.

2 Notation

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, P)$ be a complete filtered probability space, where the family $(\mathscr{F}_t)_{t\in[0,T]}$ is not necessarily right-continuous. $\mathscr{P}$ and $\mathscr{O}$ denote the predictable resp. the optional σ-algebra on $\Omega \times [0, T]$, i.e. $\mathscr{P}$ is generated by all left-continuous adapted processes and $\mathscr{O}$ is generated by all càdlàg adapted processes (considered as mappings on $\Omega \times [0, T]$). If X and Y are two optional processes and we write $X = Y$, we mean equality up to indistinguishability.

The following definitions are from [6]. Adjusted to our finite time horizon setting, we repeat them here for convenience of the reader. We add a localization procedure based on stopping which preserves the martingale property of a process. The results of Galtchouk that we use still hold when localization is done in the way chosen here.

Definition 2.1. *A stochastic process $X = (X_t)_{t\in[0,T]}$ is called an* optional martingale *(resp.* square integrable optional martingale*), and we write $X \in \mathscr{M}$ (resp. $X \in \mathscr{M}^2$), if X is an optional process and there exists an $\mathscr{F}_T$-measurable random variable $\widetilde{X}$ with $E[|\widetilde{X}|] < \infty$ (resp. $E[\widetilde{X}^2] < \infty$) such that $X_\tau = E[\widetilde{X}|\mathscr{F}_\tau]$ a.s. for every $[0, T]$-valued stopping time τ.*

Galtchouk has shown in [3] that for any $\mathscr{F}_T$-measurable integrable random variable Z there exists an optional martingale $(X_t)_{t\in[0,T]}$ with terminal value $X_T = Z$. Almost all paths of X possess limits from the left and the right (see e.g. Theorem 4 in Appendix I of [2]). Thus if one considers general filtrations, optional martingales emerge quite naturally. For a làglàd process X we denote $\Delta^- X_t := X_t - X_{t-}$ and $\Delta^+ X_t := X_{t+} - X_t$.

Definition 2.2. *Denote by $\mathscr{T}$ (resp. $\mathscr{T}_+$) the set of all $[0, T] \cup \{+\infty\}$-valued $(\mathscr{F}_t)_{t\in[0,T]}$-stopping times (resp. $(\mathscr{F}_{t+})_{t\in[0,T]}$-stopping times). Let $\mathscr{C}$ be a class of stochastic processes. A stochastic process X with right-hand limits is in the localized class of $\mathscr{C}$, and we write $X \in \mathscr{C}_{loc}$ if there exists an increasing sequence $(\tau_n, \sigma_n)_{n\in\mathbb{N}} \subset \mathscr{T} \times \mathscr{T}_+$ such that $\lim_{n\to\infty} P(\tau_n \wedge \sigma_n = T) = 1$ and the stopped processes $X^{(\tau_n,\sigma_n)}$ defined by*

$$X_t^{(\tau_n,\sigma_n)} := X_t 1_{\{t\le\tau_n\wedge\sigma_n\}} + X_{\tau_n} 1_{\{t>\tau_n,\ \tau_n\le\sigma_n\}} + X_{\sigma_n+} 1_{\{t>\sigma_n,\ \tau_n>\sigma_n\}}$$

are in $\mathscr{C}$ for all n.

Definition 2.3. *Let $\mathscr{V}$ denote the set of adapted finite variation processes (that is P-a.a. paths are of finite variation) with $A_0 = 0$. We say that $A \in \mathscr{V}$ is in $\mathscr{A}$ if $E[\sum_{0\leq s<T} |\Delta^+ A_s| + \int_{[0,T]} |dA^r_s|] < \infty$.*

Galtchouk has shown that it is possible to uniquely decompose a local martingale M into a càdlàg part M^r and an orthogonal part M^g, i.e. $M^g \widetilde{M}$ is a local martingale for any càdlàg martingale $\widetilde{M}$. M^g possesses càglàd paths (see Theorem 4.10 in [4] for details). Furthermore, any $A \in \mathscr{V}$ can obviously be decomposed uniquely into a càglàd part $A^g := \sum_{0\leq s<t} \Delta^+ A_s$ and a càdlàg part $A^r := A - A^g$. Note however that for processes which are both local martingales and of finite variation the decompositions usually differ.

Definition 2.4. *A stochastic process X is called* strongly predictable *if its trajectories have right limits, $(X_t)_{t\in[0,T]}$ is $\mathscr{P}$-measurable, and $(X_{t+})_{t\in[0,T]}$ is $\mathscr{O}$-measurable.*

Definition 2.5. *A stochastic process X is called an* optional semimartingale *if it can be written as*

$$X = X_0 + M + A, \qquad M \in \mathscr{M}_{loc}, \qquad A \in \mathscr{V}, \qquad M_0 = 0. \tag{2.1}$$

A semimartingale X is called special if there exists a representation (2.1) with a strongly predictable process $A \in \mathscr{A}_{loc}$.

Note that any optional semimartingale has limits from the left and the right, i.e. almost all paths are làglàd (again by [2] this assertion holds for the local martingale component; for the finite variation component the assertion is trivial).

3 Results

Suppose X is the (for simplicity deterministic) evolution of a stock price given by $X_t := t - 1_{[t_0]}(t) + 1_{]t_0,T]}(t)$, where $t_0 \in (0,T)$ is the time of a double jump. $]t_0,T]$ denotes an interval on $\mathbb{R}$ whereas for τ_1,τ_2 stopping times $]]\tau_1,\tau_2]] := \{(\omega,t) \in \Omega\times[0,T] \mid \tau_1(\omega) < t \leq \tau_2(\omega)\}$ is a stochastic interval. Now consider the strategies A^n where we buy one unit of the stock at time $t_0 - 1/n$ and sell it at time t_0. The (negative) trading gain would be $1/n - 1$, and as $n \to \infty$ the trading loss would go to 1 and occur exactly at time t_0. Other possible strategies B^n would be to buy one unit of the stock at time t_0 and sell it at time $t_0 + 1/n$. The trading gain would be $2 + 1/n$, which would converge to a trading gain of 2 also occuring at time t_0. If we wanted the space of trading strategies to be complete, for the two sequences of trading strategies there should be limit trading strategies $\widetilde{A}$ and $\widetilde{B}$ reproducing the limit trading gain such that it occured exactly at time t_0. If we wanted to use one-dimensional processes to specify our trading strategy, we would run into a dilemma because something like $1_{[t_0]}$ would have to represent both $\widetilde{A}$ and $\widetilde{B}$, but this is clearly impossible since the trading gains from $\widetilde{A}$ and $\widetilde{B}$ are completely different.
Put differently, since the process has double jumps, there might be a left jump $\Delta^- X_t$ and a right jump $\Delta^+ X_t$ at the same time. Using a one-dimensional integrand, an investor cannot differentiate between what should be invested in the left jump and what should be invested in the right jump, because at each point in time he only has a single value of the integrand at his disposal. For example, in the considerations above, the limit strategy $\widetilde{A}$ would have to invest 1 in $\Delta^- X_{t_0}$ but 0 in $\Delta^+ X_{t_0}$.
This explains why Galtchouk [4] introduced two-dimensional integrands (H,G) where H is a $\mathscr{P}$-measurable process and G is an $\mathscr{O}$-measurable process. Unfortunately, *this* expansion of the space of integrands to two dimensions leads to a new problem. The integrals of *these* two-dimensional integrands can in general no longer be approximated by integrals of simple predictable integrands as the following example shows.

Example 3.1. *Consider the process $M = M^r + M^g$, where M^r is a compensated Poisson process with jump rate 1 and jump size 1 (so it is càdlàg), and M^g is the left-continuous modification of a compensated Poisson process with jump rate 1 and jump size -1, i.e. $M_t^r = N_t - t$ and $M_t^g = -\widetilde{N}_{t-} + t$ where N and $\widetilde{N}$ are Poisson processes. Assume that N and $\widetilde{N}$ are independent of each other and let $(\mathscr{F}_t)_{t\in[0,T]}$ be the (not right-continuous) natural filtration of (M^r, M^g). If we consider the integrand $(H,G)_t \equiv (2,1)$, the integral $Y := (H,G)\cdot M = H\cdot M^r + G\cdot M^g$ is an optional martingale linearly decreasing with rate -1 (if no jump occurs), $\Delta^- Y$ jumps of size 2 and $\Delta^+ Y$ jumps of size -1. Clearly Y cannot be approximated by any sequence $Z^n\cdot M$, where (Z^n) is a sequence of simple predictable integrands because $Z^n\cdot M_1 = 0$ if no jump occurs up to time 1. Furthermore, it is impossible to approximate the left jumps of Y (which are of size 2) and the right jumps of Y (with size -1) by the same process $Z^n\cdot M$. This is because the jumps of M cannot be anticipated.*

For two sets A, B we define $A\Delta B := (A\setminus B)\cup(B\setminus A)$. Let $\widetilde{\Omega} := \Omega\times[0,T]$. Define a collection $\mathscr{A}$ of subsets of $\{1,2\}\times\widetilde{\Omega}$ by

$$\begin{aligned}\mathscr{A} \quad := \quad \Big\{ (\{1\}\times A)\cup(\{2\}\times B) \quad | \quad (A,B)\in\mathscr{P}\times\mathscr{O} \text{ with} \\ A\Delta B = \bigcup_{n\in\mathbb{N}} [[\tau_n]] \text{ for some } (\tau_n)_{n\in\mathbb{N}}\subset\mathscr{T}\Big\},\end{aligned} \tag{3.1}$$

i.e. the symmetric difference $A\Delta B$ has to be a thin set. Note that τ is $[0,T]\cup\{+\infty\}$-valued, but $[[\tau]] = \{(\omega,t)\in\Omega\times[0,T] \mid \tau(\omega) = t\}$. Our general integrands will be $\mathscr{A}/\mathscr{B}(\mathbb{R})$-measurable functions.

Proposition 3.2. *$\mathscr{A}$ is a σ-field.*

Proof. Obvious as $\mathscr{P}$ and $\mathscr{O}$ are σ-fields and countable unions of thin sets are thin sets. ☐

An immediate observation is that if H is $\mathscr{A}/\mathscr{B}(\mathbb{R})$-measurable, then $H^1 := H(1,\cdot,\cdot)$ is a predictable process and $H^2 := H(2,\cdot,\cdot)$ is an optional process. Furthermore, H^1 and H^2 differ only at countably many $(\mathscr{F}_t)_{t\in[0,T]}$-stopping times (as H can be approximated pointwise by simple functions).

Proposition 3.3. *Define the set*

$$\mathscr{C} := \{\{1\}\times\widetilde{A}\times\{0\} : \widetilde{A}\in\mathscr{F}_0\}\cup\{\{1\}\times]]\tau_1,\tau_2]]\cup\{2\}\times[[\tau_1,\tau_2[[: \tau_1,\tau_2\in\mathscr{T}, \tau_1\le\tau_2\}.$$

Then $\sigma(\mathscr{C}) = \mathscr{A}$.

Proof. $\sigma(\mathscr{C})\subset\mathscr{A}$ holds by $\mathscr{C}\subset\mathscr{A}$. Since $\bigcap_{i=1}^{\infty}(\{1\}\times]]\tau,\tau+\frac{1}{n}]]\cup\{2\}\times[[\tau,\tau+\frac{1}{n}[[)\in\sigma(\mathscr{C})$, we have that $\{1\}\times\emptyset\cup\{2\}\times[[\tau]]\in\sigma(\mathscr{C})$ for any $\tau\in\mathscr{T}$. Therefore also $\{1\}\times]]\tau_1,\tau_2]]\cup\{2\}\times]]\tau_1,\tau_2]]\in\sigma(\mathscr{C})$ for all $\tau_1,\tau_2\in\mathscr{T}$. Because $\mathscr{P}$ is generated by the family of sets $\{\widetilde{A}\times\{0\} : \widetilde{A}\in\mathscr{F}_0\}\cup\{]]\tau_1,\tau_2]] : \tau_1,\tau_2\in\mathscr{T}\}$ and since $\widetilde{A}\times\{0\}$ is the graph of a stopping time, we have $\{1\}\times A\cup\{2\}\times A\in\sigma(\mathscr{C})$ for any $A\in\mathscr{P}$. Now let $F\in\mathscr{A}$, i.e. $F = \{1\}\times A\cup\{2\}\times B$, where $A\in\mathscr{P}$, $B\in\mathscr{O}$. $A\setminus B$ and $B\setminus A$ are both thin sets by Theorem 3.19 in [7], thus there exist two sequences of stopping times $(\tau_i)_{i\in\mathbb{N}}$ and $(\nu_j)_{j\in\mathbb{N}}$ such that $B = (A\setminus\bigcup[[\tau_i]])\cup(\bigcup[[\nu_j]])$. Therefore $F\in\sigma(\mathscr{C})$ as required. ☐

Consider simple integrands of the form

$$H = Z^0 1_{\{1\}\times\widetilde{A}\times\{0\}\cup\{2\}\times\widetilde{A}\times\{0\}} + \sum_{i=1}^{n} Z^i 1_{\{1\}\times]]\tau_i,\tau_{i+1}]]\cup\{2\}\times[[\tau_i,\tau_{i+1}[[}, \tag{3.2}$$

where $\tau_i \in \mathscr{T}$, $\tau_1 \leq \tau_2 \ldots \leq \tau_{n+1}$, Z^0 is $\mathscr{F}_0$-measurable, and each Z^i is a $\mathscr{F}_{\tau_i}$-measurable random variable. Let $\mathscr{E}$ denote the class of simple integrands. Note that the simple integrands are indeed $\mathscr{A}$-measurable, and that there is a one-to-one correspondence between the simple integrands defined in (3.2) and the usual one-dimensional simple predictable integrands. By Proposition 3.3 $\mathscr{E}$ generates the σ-field $\mathscr{A}$ on $\{1,2\} \times \widetilde{\Omega}$. We call simple integrands simple $\mathscr{A}$-measurable.
We now define for $H \in \mathscr{E}$ the elementary stochastic integral in the usual way by

$$(H \bullet X)_t := \sum_{i=1}^{n} Z^i (X_{\tau_{i+1} \wedge t} - X_{\tau_i \wedge t}), \qquad t \in [0, T].$$

Remark 3.4. *The second summand in (3.2) can be motivated as follows: To obtain* $Z^i (X_{\tau_{i+1}} - X_{\tau_i})$ *one weights the right jump of* X *at* τ_i *already with* Z^i *whereas the left jumps are weighted with* Z^i *only immediately after* τ_i.

The next theorem shows that the elementary integral possesses a unique continuous and linear extension to general integrands defined as $\mathscr{A}/\mathscr{B}(\mathbb{R})$-measurable functions.

Theorem 3.5. *Suppose* X *is an optional semimartingale. The mapping* $H \mapsto H \bullet X$ *on* $\mathscr{E}$ *has a unique extension (also denoted* $H \mapsto H \bullet X$*) to all locally bounded* $\mathscr{A}$*-measurable processes* $H : \{1,2\} \times \widetilde{\Omega} \to \mathbb{R}$ *such that*

(i) $H \mapsto H \bullet X$ *is linear;*

(ii) if an $\mathscr{A}$*-measurable sequence* $(H^n)_{n \in \mathbb{N}}$ *converges pointwise to* H *and* $|H^n| \leq K$*, where* K *is a locally bounded* $\mathscr{A}$*-measurable process, then* $\sup_{s \in [0,T]} |(H^n \bullet X)_s - (H \bullet X)_s|$ *converges in probability to* 0.

Proof. Step 1 (uniqueness). Let $H \bullet X$ and $H \circ X$ be two extensions satisfying (i) and (ii). Then (i) and (ii) imply that $\mathscr{G} := \{F \in \mathscr{A} : 1_F \bullet X = 1_F \circ X\}$ is a Dynkin system. Since $\mathscr{C} \subset \mathscr{G}$ and $\mathscr{C}$ is a $\cap$-stable generator of $\mathscr{A}$, by a Dynkin argument we have $\mathscr{A} = \mathscr{G}$. A locally bounded $\mathscr{A}$-measurable process H can be approximated pointwise by the sequence $(H^n)_{n \in \mathbb{N}}$, where

$$H^n := \sum_{k=-n^2}^{n^2} \frac{k}{n} 1_{\{\frac{k-1}{n} < H \leq \frac{k}{n}\}}.$$

Because of the linearity requirement (i) we know that $H^n \bullet X = H^n \circ X$ for all n. In addition it is true that $|H^n| \leq |H| + 1$. Thus from (ii) follows $H \bullet X = H \circ X$ and the uniqueness of the extension is established.
Step 2 (existence). Let $X = X_0 + M + A$ with $M \in \mathscr{M}_{loc}$ and $A \in \mathscr{V}$ be any decomposition of X. Consider the integral (once again denoted by $H \mapsto H \bullet X$)

$$H \bullet X := H^1 \bullet M^r + H^1 \bullet A^r + H^2 \bullet M^g + H^2 \bullet A^g, \tag{3.3}$$

which is by Galtchouk defined for any locally bounded $H^1 \in \mathscr{P}$ and $H^2 \in \mathscr{O}$, thus in particular when H is locally bounded and $\mathscr{A}$-measurable. Note that (3.3) generally depends on the decomposition of the optional semimartingale into a local martingale and a process of finite variation (Thus in Galtchouk $H^1 \bullet M^r + H^2 \bullet M^g$ and $H^1 \bullet A^r + H^2 \bullet A^g$ are seen as separate integrals. But, later on by the uniqueness of the extension it will turn out that for $\mathscr{A}/\mathscr{B}(\mathbb{R})$-measurable integrands the choice of the decomposition is not relevant).

If H is a simple integrand this integral is equal to our definition of the simple integral, i.e. it is an extension. From the standard theory (see e.g. [2], chapter VIII) we know that the first half of the right-hand side of (3.3) fulfils properties (i) and (ii). For the left-continuous parts $H^2 \cdot M^g$ and $H^2 \cdot A^g$ the same line of argument holds true: M^g can be decomposed into a locally square integrable martingale and a local martingale of finite variation (by considering the process $\sum_{0\le s\le \cdot} \Delta^+ M_s 1_{\{|\Delta^+ M_s|>1\}} \in \mathscr{A}_{loc}$ and using the existence of strongly predictable càglàd compensators, see Lemma 1.10 in [6]). Because a version of Doob's inequality still holds for optional square-integrable martingales (see Appendix I in [2] on how to prove such inequalities using the optional section-theorem, which still holds under non-usual conditions), the usual arguments for the càdlàg case can be reproduced for the locally square integrable part. The martingale part of finite variation is treated like $(H^2 \cdot A^g)_t = \int_{[0,t[} H_s^2 dA_{s+}^g$ which is a Lebesgue-Stieltjes integral. Thus it is known that it is linear and has the continuity property. □

Remark 3.6. *We have shown that it is possible to extend the integral in a unique way from all simple $\mathscr{A}$-measurable integrands (which are in a one-to-one correspondence with the (one-dimensional) simple predictable integrands) to all locally bounded $\mathscr{A}$-measurable integrands. Note that the elementary integral does not depend on the decomposition in (3.3). In Galtchouk's framework [6] the integral is extended uniquely from all two-dimensional simple $\mathscr{P} \otimes \mathscr{O}$-measurable integrands to all locally bounded $\mathscr{P} \otimes \mathscr{O}$-measurable integrands. What cannot be done is to extend the integral* uniquely *from one-dimensional simple predictable integrands to all locally bounded $\mathscr{P} \otimes \mathscr{O}$-measurable integrands. To see this note that besides $H \cdot X := H^1 \cdot M^r + H^1 \cdot A^r + H^2 \cdot M^g + H^2 \cdot A^g$ the mapping $H \circ X := H \cdot X + H^1 \cdot I - H^2 \cdot I$, where $I_t(\omega) := t$, is also a continuous and linear extension of the elementary integral. But generally for $\mathscr{P} \otimes \mathscr{O}$-measurable integrands $H \cdot X$ and $H \circ X$ are different. Confer this with Example 3.1.*

Any special semimartingale Y for which the canonical decomposition $Y_0 + N + B$ satisfies $N \in \mathscr{M}^2$ and $B \in \mathscr{A}$, can be considered an element of the Banach space $\mathscr{M}^2 \oplus \mathscr{A}$, where the norm is given by $E[N_T^2]^{1/2} + E[\mathrm{Var}(B)_T]$. Now we show a completeness property for the space of integrands for which the integrals are in $\mathscr{M}^2 \oplus \mathscr{A}$. At first we define analogously to the standard theory the space of general integrands (cf. Definition III.6.17 in [8]).

Definition 3.7. *We say that a $\mathscr{A}$-measurable process $H = (H^1, H^2)$ is integrable w.r.t. an optional semimartingale X if there exists a decomposition $X = X_0 + M + A$ with $M \in \mathscr{M}^2_{loc}$ and $A \in \mathscr{V}$ such that*

$$(H^1)^2 \cdot [M^r, M^r] \in \mathscr{A}_{loc}, \quad (H^2)^2 \cdot [M^g, M^g] \in \mathscr{A}_{loc}$$

and the Lebesgue-Stieltjes integrals $|H^1| \cdot \mathrm{Var}(A^r)$, $|H^2| \cdot \mathrm{Var}(A^g)$ are finite-valued. We denote by $L(X)$ the set of these processes.

Let $H \in L(X)$. By Theorem 3.5 the integral $\left(H 1_{\{|H|\le n\}}\right) \cdot X = \left(H^1 1_{\{|H|\le n\}}\right) \cdot M^r + \left(H^1 1_{\{|H|\le n\}}\right) \cdot A^r + \left(H^2 1_{\{|H|\le n\}}\right) \cdot M^g + \left(H^2 1_{\{|H|\le n\}}\right) \cdot A^g$ is well-defined (i.e. it does not depend on the decomposition $X = X_0 + M + A$). By Theorem I.4.40 and Lemma III.6.15 in [8] and Theorem 7.3 in [4] all four integrals converge uniformly in probability against the corresponding integrals without truncation. Thus $H \cdot X$ is also well-defined.

Theorem 3.8. *Let X be a special semimartingale. If $(H^n)_{n\in\mathbb{N}} \subset L(X)$ such that $(H^n \cdot X)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{M}^2 \oplus \mathscr{A}$, then there exists a $H \in L(X)$ such that $H^n \cdot X \to H \cdot X$ in $\mathscr{M}^2 \oplus \mathscr{A}$.*

Proof. Step 1. We start by showing that for all n the canonical decomposition of $H^n \cdot X$ can be written as $H^n \cdot M + H^n \cdot A$, where $X = X_0 + M + A$ is the canonical decomposition of X. The reasoning is similar to the proof of Lemma III.3 in [10], but we present it here for the convenience of the reader. Some facts about (strongly predictable) compensators are used; they can be found in the appendix. Let n be fixed. There exists a decomposition $X = N + B$ such that $(H^n \cdot N) \in \mathscr{M}^2_{loc}$ and $(H^n \cdot B) \in \mathscr{V}$. Since $H^n \cdot X$ is in $\mathscr{M}^2 \oplus \mathscr{A}$, we have by Lemma 4.2 in [6] that $H^n \cdot B \in \mathscr{A}_{loc}$. As X is special, we have with the same argument that $B \in \mathscr{A}_{loc}$. Again by Lemma 4.2 in [6], $H^n \cdot X$ is special and hence it possesses a canonical decomposition $L + D$. By Proposition 4.3 the unique compensators of B and $H^n \cdot B$ are given by A and D. But since B and $H^n \cdot B$ are both in $\mathscr{A}_{loc}$, by Proposition 4.5 $(H^n \cdot B)^p = H^n \cdot B^p = H^n \cdot A$, i.e. the compensator of $H^n \cdot B$ is $H^n \cdot A$. Thus $D = H^n \cdot A$, which in turn implies $L = H^n \cdot M$.

Step 2. For any local martingale M, we define a non-negative measure m on $(\{1,2\} \times \widetilde{\Omega}, \mathscr{A})$ by

$$m(F) := E[1_B \cdot [M^r, M^r]_T + 1_C \cdot [M^g, M^g]_T], \quad \forall F = \{1\} \times B \cup \{2\} \times C \in \mathscr{A}.$$

Similarly, for $A \in \mathscr{A}_{loc}$ let

$$n(F) := E[1_B \cdot \mathrm{Var}(A^r)_T + 1_C \cdot \mathrm{Var}(A^g)_T].$$

By the decomposition of M (resp. A) into a right- and a left-continuous part we ensure that m (resp. n) is a measure. Note that m and n are in general not σ-finite. Let $H \cdot M \in \mathscr{M}^2$; then we have that

$$\begin{aligned} E[(H \cdot M)^2_T] &= E[(H^1 \cdot M^r + H^2 \cdot M^g)^2_T] \\ &= E[(H^1 \cdot M^r)^2_T + (H^2 \cdot M^g)^2_T + 2(H^1 \cdot M^r)_T (H^2 \cdot M^g)_T] \\ &= E[(H^1 \cdot M^r)^2_T + (H^2 \cdot M^g)^2_T] \\ &= E[(H^1)^2 \cdot [M^r, M^r]_T + (H^2)^2 \cdot [M^g, M^g]_T] \\ &= \int_0^T (H)^2 dm. \end{aligned} \tag{3.4}$$

The crucial third equality follows because $H^1 \cdot M^r$ and $H^2 \cdot M^g$ are orthogonal optional martingales, which is due to fact that

$$[H^1 \cdot M^r, H^2 \cdot M^g] = H^2 \cdot [(H^1 \cdot M^r)^g, M^g] = [0, M^g] = 0$$

(see [4], Theorem 7.11). The fourth equality is valid since there are Itô isometries for both the standard stochastic integral and the optional stochastic integral w.r.t. to a càglàd optional martingale (see [4], Section 7).

Let us verify an isometry property for the integrable variation part. Note that for the finite variation part A, the process A^g is just the sum of the jumps $\Delta^+ A$. The total variation can thus be split into two parts by

$$\mathrm{Var}(A)_t = \int_0^t |dA^r_s| + \sum_{s<t} |\Delta^+ A_s| = \mathrm{Var}(A^r) + \mathrm{Var}(A^g),$$

and the following isometry holds for any $A \in \mathscr{A}$

$$\begin{aligned} E[\mathrm{Var}(H \cdot A)_T] &= E[\mathrm{Var}(H^1 \cdot A^r + H^2 \cdot A^g)_T] \\ &= E[\mathrm{Var}(H^1 \cdot A^r)_T + \mathrm{Var}(H^2 \cdot A^g)_T] \\ &= E[|H^1| \cdot \mathrm{Var}(A^r)_T + |H^2| \cdot \mathrm{Var}(A^g)_T] \\ &= \int_0^T |H| dn. \end{aligned} \tag{3.5}$$

By (3.4) and (3.5), $\left(L^2(\{1,2\}\times\widetilde{\Omega},\mathscr{A},m)\cap L^1(\{1,2\}\times\widetilde{\Omega},\mathscr{A},n)\right)\subset L(X)$ and $H\mapsto H\cdot X$ is an isometry mapping from $L^2(\{1,2\}\times\widetilde{\Omega},\mathscr{A},m)\cap L^1(\{1,2\}\times\widetilde{\Omega},\mathscr{A},n)$ to $\mathscr{M}^2\oplus\mathscr{A}$ (surjective onto the subspace of $\mathscr{M}^2\oplus\mathscr{A}$ whose elements can be represented by stochastic integrals). As $L^2(\{1,2\}\times\widetilde{\Omega},\mathscr{A},m)\cap L^1(\{1,2\}\times\widetilde{\Omega},\mathscr{A},n)$ is a complete vector space this implies the assertion. □

Remark 3.9. *Suppose for any $(\mathscr{F}_t)_{t\in[0,T]}$-stopping time τ we have $P(\Delta^+X_\tau\neq 0,\tau<T)=0$ (we call such a process* quasi-right-continuous*). Then for any locally bounded $\mathscr{A}$-measurable process H the stochastic integrals $H\cdot X=H^1\cdot X^r+H^2\cdot X^g$ and $H^1\cdot X^r+H^1\cdot X^g$ are indistinguishable. To see this, note that we only have to check that $(H^1-H^2)\cdot X^g=0$. Now H^1-H^2 is equal to 0 on the complement of a thin set and according to the condition above there are a.s. no jumps of X^g on this thin set. Thus if X is quasi-right-continuous, the set of locally bounded predictable integrands is adequate, as in the usual right-continuous setting.*

Remark 3.10. *In mathematical finance a similar problem arises in the standard model with càdlàg-price processes when portfolio adjustments cause transaction costs. At time t the value of a portfolio may change due to a jump of the asset prices between $t-$ and t. In addition, any portfolio adjustments (which may be seen as taking place at time $t-$) reduce the wealth of the investor (in contrast to the model without transaction costs). Thus, the wealth process may have double jumps. However, the portfolio holdings in each asset can still be represented by a one-dimensional process, cf. [1].*

4 Appendix

Lemma 4.1. *Suppose $A\in\mathscr{V}$. Then A is strongly predictable if and only if $(A^r_t)_{t\in[0,T]}$ is predictable and $(A^g_{t+})_{t\in[0,T]}$ is optional.*

Proof. Obvious, as $A_t=A^r_t+A^g_t=A^r_t+A^g_{t-}$ and $A_{t+}=A^r_{t+}+A^g_{t+}=A^r_t+A^g_{t+}$. □

Lemma 4.2. *Let $A\in\mathscr{V}$ be strongly predictable and $H=(H^1,H^2)$ be an $\mathscr{A}$-measurable function s.t. $H^1\cdot A^r$ and $H^2\cdot A^g$ exist. Then $H\cdot A$ is strongly predictable.*

Proof. By Lemma 4.1 and $H\cdot A=H^1\cdot A^r+H^2\cdot A^g$ we only have to check that $(H^1\cdot A^r_t)$ is predictable and $(H^2\cdot A^g_{t+})$ is optional. Since H^1 is predictable and again by Lemma 4.1 (A^r_t) is also predictable, Proposition I.3.5 in [8] ensures that $H^1\cdot A^r$ is predictable, too. Once more by Lemma 4.1 $(A^g_{t+})_{t\in[0,T]}$ is optional, thus Δ^+A_s is $\mathscr{F}_t$ measurable for all $s\le t$. As $H^2 1_{\Omega\times[0,t]}$ is $\mathscr{F}_t\otimes\mathscr{B}([0,t])$-measurable, by Fubini's theorem for transition measures this implies that $(H^2\cdot A^g)_{t+}=\sum_{0\le s\le t}H^2_s\Delta^+A_s$ is $\mathscr{F}_t$-measurable and therefore optional. □

Proposition 4.3. *Let $A\in\mathscr{A}_{loc}$. There exists a process, called the compensator of A and denoted by A^p, which is unique up to indistinguishability, and which is characterized by being a strongly predictable process of $\mathscr{A}_{loc}$ such that $A-A^p$ is a local martingale.*

Proof. $A\in\mathscr{A}_{loc}$ implies $A^r,A^g\in\mathscr{A}_{loc}$. By Theorem I.3.18 in [8], there exists a unique predictable càdlàg process $(A^r)^p$ such that $A^r-(A^r)^p\in\mathscr{M}_{loc}$ (formally we apply the theorem to A^r under the right-continuous filtration $(\mathscr{F}_{t+})_{t\in[0,T]}$ and use that the $(\mathscr{F}_{t+})_{t\in[0,T]}$-predictable processes coincide with the $(\mathscr{F}_t)_{t\in[0,T]}$-predictable processes). By Lemma 1.10 in [6], there exists a unique strongly predictable càglàd process $(A^g)^p$ such that $A^g-(A^g)^p\in\mathscr{M}_{loc}$. The process $A^p:=(A^r)^p+(A^g)^p$ is strongly predictable and $A-A^p\in\mathscr{M}_{loc}$. If two strongly predictable processes B and C are compensators of A, $B-C$ is in $\mathscr{M}_{loc}\cap\mathscr{A}_{loc}$, i.e. $B-C=0$ (since as in the standard model, using Theorem 3.5 in [5] it can be shown that if $X\in\mathscr{M}_{loc}\cap\mathscr{A}_{loc}$, then $X=0$.) □

Proposition 4.4. *Let $A \in \mathscr{A}^+_{loc}$. The compensator A^p can then be characterized as being a strongly predictable process in $\mathscr{A}^+_{loc}$ meeting any of the two following equivalent statements*

(i) $E[A^p_\tau] = E[A_\tau]$ for all $\tau \in \mathscr{T}$;

(ii) $E[(H \bullet A^p)_T] = E[(H \bullet A)_T]$ for all non-negative $\mathscr{A}$-measurable processes H.

Proof. The proof is similar to the proof of Theorem I.3.17 in [8]. Just note that (ii) implies (i) because $H := 1_{\{1,2\}\times[[0,\tau]]}$ is $\mathscr{A}$-measurable. (i) implies for all $\tau \in \mathscr{T}$ that

$$\begin{aligned} E[(1_{\{1\}\times[[0,\tau]]\cup\{2\}\times[[0,\tau[[} \bullet A^p)_T] &= E[(A^p)^r_\tau + (A^p)^g_\tau] \\ &= E[A^r_\tau + A^g_\tau] \\ &= E[(1_{\{1\}\times[[0,\tau]]\cup\{2\}\times[[0,\tau[[} \bullet A)_T]. \end{aligned}$$

Since $\mathscr{A}$ is also generated by $\{\{1\}\times\widetilde{A}\times\{0\} : \widetilde{A}\in\mathscr{F}_0\}\cup\{\{1\}\times[[0,\tau]]\cup\{2\}\times[[0,\tau[[: \tau\in\mathscr{T}\}$ and because $A_0 = A^p_0 = 0$, we have (ii) by monotone convergence and a monotone class argument. ☐

Proposition 4.5. *Let $A \in \mathscr{A}_{loc}$. For each $\mathscr{A}$-measurable process H such that $H \bullet A \in \mathscr{A}_{loc}$, we have that $H \bullet A^p \in \mathscr{A}_{loc}$ and $H \bullet A^p = (H \bullet A)^p$, and in particular $H \bullet A - H \bullet A^p$ is a local martingale.*

Proof. The proof of the second half of Theorem I.3.18 in [8] can be reproduced without any major changes (using Proposition 4.4 and Lemma 4.2). Note that the associativity of the integral used in the proof holds because

$$\begin{aligned} H \bullet (G \bullet A) &= H^1 \bullet (G \bullet A)^r + H^2 \bullet (G \bullet A)^g \\ &= H^1 \bullet (G^1 \bullet A^r + G^2 \bullet A^g)^r + H^2 \bullet (G^1 \bullet A^r + G^2 \bullet A^g)^g \\ &= H^1 \bullet (G^1 \bullet A^r) + H^2 \bullet (G^2 \bullet A^g) \\ &= (H^1 G^1) \bullet A^r + (H^2 G^2) \bullet A^g \\ &= (HG)^1 \bullet A^r + (HG)^2 \bullet A^g = (HG) \bullet A, \end{aligned}$$

where the crucial third equality is true because for any $A \in \mathscr{V}$ we obviously have $(A^r)^g = (A^g)^r = 0$. The fourth equality follows from the associativity of the one-dimensional Lebesgue-Stieltjes integral. ☐

Acknowledgements

We would like to thank an anonymous referee for valuable comments.

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