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INTERSECTION PROBABILITIES FOR A CHORDAL SLE PATH AND A SEMICIRCLE

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Abstract

We derive a number of estimates for the probability that a chordal SLE_κ path in the upper half plane $\mathbb{H}$ intersects a semicircle centred on the real line. We prove that if $0 < \kappa < 8$ and $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is a chordal SLE_κ in $\mathbb{H}$ from 0 to ∞, then $P\{\gamma[0, \infty) \cap \mathcal{C}(x; rx) \neq \emptyset\} \asymp r^{4a-1}$ where $a = 2/\kappa$ and $\mathcal{C}(x; rx)$ denotes the semicircle centred at $x > 0$ of radius rx, $0 < r \leq 1/3$, in the upper half plane. As an application of our results, for $0 < \kappa < 8$, we derive an estimate for the diameter of a chordal SLE_κ path in $\mathbb{H}$ between two real boundary points 0 and $x > 0$. For $4 < \kappa < 8$, we also estimate the probability that an entire semicircle on the real line is swallowed at once by a chordal SLE_κ path in $\mathbb{H}$ from 0 to ∞.

1 Introduction

The Schramm-Loewner evolution (SLE) is a one-parameter family of random growth processes introduced by O. Schramm [9] which has been successfully used to establish a number of rigorous mathematical results about various two-dimensional models from statistical mechanics including percolation, loop-erased random walk, and the Ising model. There are actually two variants of SLE that one can consider. Radial SLE describes the growth of a curve connecting a given boundary point to a given interior point whereas chordal SLE describes the growth of a curve connecting two distinct boundary points. It is assumed that the reader is familiar with the basic properties of SLE as found in [7].

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The primary purpose of this paper is to derive estimates for the probability that a chordal SLE path in $\mathbb{H}$ from 0 to ∞ intersects a semicircle in the upper half plane centred at a fixed point $x > 0$ on the real line. Specifically, suppose that $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is (the trace of) a chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞ where $\kappa \in (0, 8)$. For $\epsilon > 0$ and $x \in \mathbb{R}$, denote the semicircle of radius ϵ centred at x in the upper half plane by $\mathcal{C}(x; \epsilon)$. In the $0 < \kappa < 8$ regime, we will derive estimates for the intersection probability

$$P\{\gamma[0, \infty) \cap \mathcal{C}(x; rx) \neq \emptyset\}$$

where $0 < r \leq 1/3$ and $x > 0$. An example is shown in Figure 1.

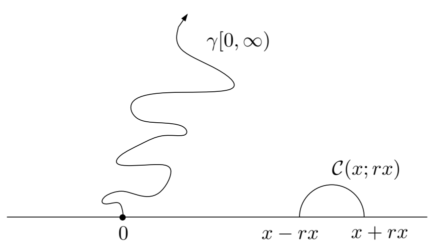

Figure 1: The event $\{\gamma[0, \infty) \cap \mathcal{C}(x; rx) = \emptyset\}$ in the $0 < \kappa \leq 4$ case.

We conclude the introduction with the statement of our primary theorems. Most of this paper is devoted to their proof. In the final section we give some applications of our results. Recall that $g(r) \asymp h(r)$ if there exist non-zero, finite constants c_1 and c_2 such that $c_1 h(r) \leq g(r) \leq c_2 h(r)$. Furthermore, $g(r) \sim h(r)$ if $g(r)/h(r) \to 1$ as $r \downarrow 0$.

Theorem 1.1. *Suppose $x > 0$ is a real number, $0 < r \leq 1/3$, and $\mathcal{C}(x; rx) = \{x + rxe^{i\theta} : 0 < \theta < \pi\}$ denotes the semicircle of radius rx centred at x in the upper half plane. If $0 < \kappa < 8$ and $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is a chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞, then*

$$P\{\gamma[0, \infty) \cap \mathcal{C}(x; rx) \neq \emptyset\} \asymp r^{\frac{8-\kappa}{\kappa}}.$$

By an appropriate conformal transformation, an equivalent formulation of Theorem 1.1 gives an estimate for the diameter of a chordal SLE$_\kappa$ path in $\mathbb{H}$ from 0 to $x > 0$. This is illustrated in Figure 2.

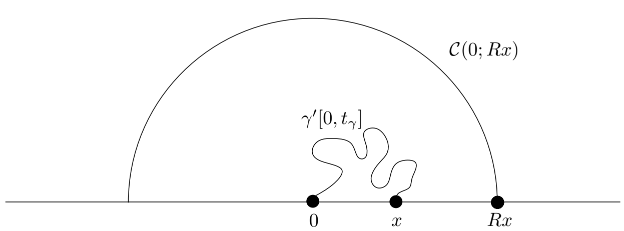

Figure 2: The event $\{\gamma'[0, t_{\gamma'}] \cap \mathcal{C}(0; Rx) = \emptyset\}$ in the $0 < \kappa \leq 4$ case.

Corollary 1.2. *Suppose $x > 0$ is a real number, $R \geq 3$, and $\mathcal{C}(0; Rx) = \{Rxe^{i\theta} : 0 < \theta < \pi\}$ denotes the circle of radius Rx centred at 0 in the upper half plane. If $0 < \kappa < 8$ and $\gamma' : [0, t_{\gamma'}] \to \overline{\mathbb{H}}$ is a chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to x, then*

$$P\{\gamma'[0, t_{\gamma'}] \cap \mathcal{C}(0; Rx) \neq \emptyset\} \asymp R^{\frac{\kappa-8}{\kappa}}.$$

Remark. In the particular case when $\kappa = 8/3$ it is possible to compute the intersection probabilities in Theorem 1.1 and Corollary 1.2 exactly. We show in Section 5 that if $0 < r < 1$, then

$$P\{\gamma[0, \infty) \cap \mathcal{C}(x; rx) \neq \emptyset\} = 1 - (1 - r^2)^{5/8}, \tag{1}$$

and if $R > 1$, then

$$P\{\gamma'[0, t_{\gamma'}] \cap \mathcal{C}(0; Rx) \neq \emptyset\} = 1 - (1 - R^{-2})^{5/8}.$$

As well, (1) implies

$$P\{\gamma[0, \infty) \cap \mathcal{C}(x; rx) \neq \emptyset\} \sim \frac{5}{8} r^2$$

as $r \downarrow 0$ which is consistent with Theorem 1.1.

The outline of the remainder of the paper is as follows. In Section 2, we introduce some notation. The proof of Theorem 1.1 is then given in Section 3. In Section 4 we derive Corollary 1.2 and then conclude in Section 6 by using Theorem 1.1 to derive two other intersection probabilities for a chordal SLE path and a semicircle centred on the real line. (These are given by Theorem 6.1 and Corollary 6.2.) In particular, for $4 < \kappa < 8$, we estimate the probability that an entire semicircle on the real line is swallowed at once by a chordal SLE$_\kappa$ path in $\mathbb{H}$ from 0 to ∞.

2 Notation

We now introduce the notation that will be used throughout the remainder of the paper. Let $\mathbb{C}$ denote the set of complex numbers and write $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ to denote the upper half plane. If $\epsilon > 0$ and $z \in \mathbb{C}$, we write $\mathcal{B}(z; \epsilon) = \{w \in \mathbb{C} : |z - w| < \epsilon\}$ for the ball of radius ϵ centred at z. If $x \in \mathbb{R}$, then the half disk and semicircle of radius ϵ centred at x in the upper half plane are given by

$$\mathcal{D}(x; \epsilon) = \mathcal{B}(x; \epsilon) \cap \mathbb{H} = \{x + \rho e^{i\theta} : 0 < \theta < \pi, 0 < \rho < \epsilon\} \tag{2}$$

and

$$\mathcal{C}(x; \epsilon) = \partial\mathcal{B}(x; \epsilon) \cap \mathbb{H} = \{x + \epsilon e^{i\theta} : 0 < \theta < \pi\}, \tag{3}$$

respectively.
The chordal Schramm-Loewner evolution in $\mathbb{H}$ from 0 to ∞ with parameter $\kappa = 2/a$ is the solution of the differential equation

$$\partial_t g_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z, \tag{4}$$

where $z \in \mathbb{H}$ and $U_t = -B_t$ is a standard one-dimensional Brownian motion with $B_0 = 0$. It is a hard theorem to prove that there exists a curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ with $\gamma(0) = 0$ which generates the maps $\{g_t,\ t \geq 0\}$. More precisely, for $z \in \mathbb{H}$, let T_z denote the first time of explosion of the chordal Loewner equation (4), and define the hull K_t by $K_t = \overline{\{z \in \mathbb{H} : T_z < t\}}$. The hulls $\{K_t,\ t \geq 0\}$ are an increasing family of compact sets in $\overline{\mathbb{H}}$ and g_t is a conformal transformation of $\mathbb{H} \setminus K_t$ onto $\mathbb{H}$. For all $\kappa > 0$, there is a continuous curve $\{\gamma(t),\ t \geq 0\}$ with $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ and $\gamma(0) = 0$ such that $\mathbb{H} \setminus K_t$ is the unbounded connected component of $\mathbb{H} \setminus \gamma(0, t]$ a.s. The behaviour of the curve γ depends on the parameter κ (or, equivalently, the value of a). If $a \geq 1/2$ (i.e., $0 < \kappa \leq 4$), then γ is a simple curve with $\gamma(0, \infty) \subset \mathbb{H}$ and $K_t = \gamma(0, t]$. If $1/4 < a < 1/2$ (i.e., $4 < \kappa < 8$), then γ is a non-self-crossing curve with self-intersections and $\gamma(0, \infty) \cap \mathbb{R} \neq \emptyset$. Although the present work will not be concerned with the case $a \leq 1/4$ (i.e., $\kappa \geq 8$), it is worth recalling that for this regime γ is a space-filling, non-self-crossing curve. Let $\mu^{\#}_{\mathbb{H}}(0, \infty)$ denote the chordal SLE$_\kappa$ probability measure on paths in $\mathbb{H}$ from 0 to ∞. If $D \subset \mathbb{C}$ is a simply connected domain and z, w are distinct points in ∂D, then $\mu^{\#}_D(z, w)$, the chordal SLE$_\kappa$ probability measure on paths in D from z to w, is defined to the image of $\mu^{\#}_{\mathbb{H}}(0, \infty)$ under a conformal transformation $f : \mathbb{H} \to D$ with $f(0) = z$ and $f(\infty) = w$. In other words, SLE$_\kappa$ in D from z to w is simply the conformal image of SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞. For further details about SLE, consult [7].

3 Proof of Theorem 1.1

In this section we prove Theorem 1.1. The proof is divided into three subsections. For the lower bound in both the $0 < \kappa \leq 4$ and $4 < \kappa < 8$ cases, we are able to give an explicit value for the constant. For the upper bound, however, all that can be determined is the existence of a constant.

3.1 The upper bound

Throughout this section, suppose that $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is a chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞ with $0 < \kappa < 8$ and $a = 2/\kappa$. The primary tool we need to establish the upper bound in Theorem 1.1 is originally due to Beffara [3, Proposition 2]. We briefly recall the statement here and refer the reader to [3] for further details.

Proposition 3.1. *If $z \in \mathbb{H}$, $0 < \epsilon \leq \Im\{z\}/2$, and $\mathcal{B}(z; \epsilon) = \{w \in \mathbb{C} : |z - w| < \epsilon\}$ denotes the ball of radius ϵ centred at z, then*

$$P\{\gamma[0, \infty) \cap \mathcal{B}(z; \epsilon) \neq \emptyset\} \asymp \left(\frac{\epsilon}{\Im\{z\}}\right)^{1 - \frac{1}{4a}} \left(\frac{\Im\{z\}}{|z|}\right)^{4a-1}$$

where the constants implied by $\asymp$ may depend on a.

We would like to stress that this proposition holds for all $a > 1/4$ (equivalently, all $0 < \kappa < 8$). The following theorem gives a careful statement of the upper bound that we will establish.

Theorem 3.2. *Let $0 < r \leq 1/3$ and $x > 0$. If $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is a chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞ with $0 < \kappa < 8$ and $a = 2/\kappa$, then there exists a constant c_a such that*

$$P\{\gamma[0, \infty) \cap \mathcal{C}(x; rx) \neq \emptyset\} \leq c_a r^{4a-1}.$$

Our general strategy for Theorem 3.2 will be to cover the semicircle $\mathcal{C}(x; rx)$ with a sequence of balls and then apply Proposition 3.1 to each ball. This is illustrated in Figure 3.

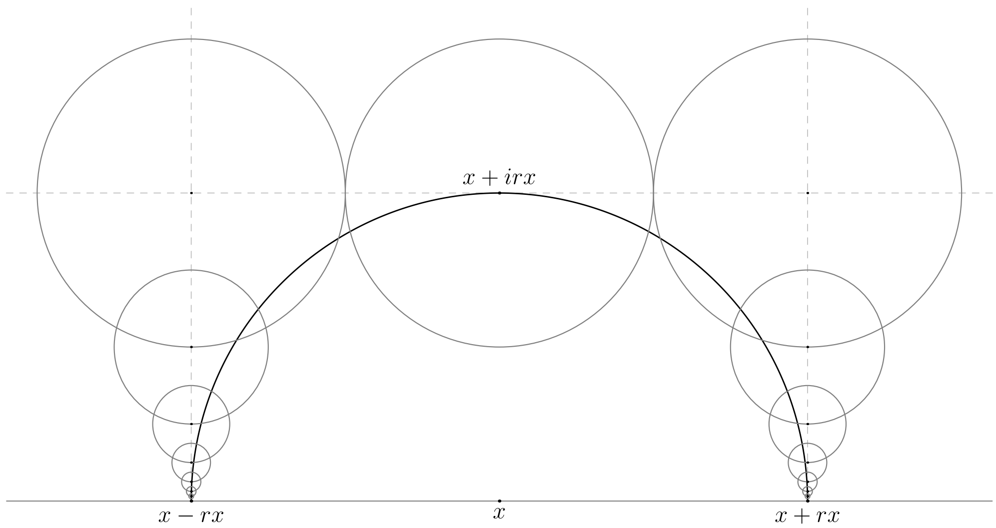

Figure 3: The semicircle $\mathcal{C}(x; rx)$ covered by a sequence of balls centred at $\{z_{\pm n}, n = 0, 1, \ldots\}$.

Proof of Theorem 3.2. Set $z_0 = x + irx$ and for $n = 1, 2, \ldots$, let $z_{\pm n} = x \pm rx + irx2^{-|n|+1}$. Using Proposition 3.1, it follows that

$$P\left\{\gamma[0,\infty) \cap \mathcal{B}\left(z_{\pm n}; \frac{\Im\{z_{\pm n}\}}{2}\right) \neq \emptyset\right\} \asymp 2^{\frac{1}{4a}-1}\left(\frac{\Im\{z_{\pm n}\}}{|z_{\pm n}|}\right)^{4a-1} \asymp \frac{r^{4a-1}}{2^{(4a-1)|n|}}$$

since $|z_{\pm n}| \asymp x$ for $0 < r \le 1/3$. Hence,

$$\sum_{n=-\infty}^{\infty} P\left\{\gamma[0,\infty) \cap \mathcal{B}\left(z_n; \frac{\Im\{z_n\}}{2}\right) \neq \emptyset\right\} \asymp r^{4a-1}. \tag{5}$$

But if $\gamma[0,\infty)$ intersects $\mathcal{C}(x; rx)$, then it also must intersect at least one of $\mathcal{B}\left(z_{\pm n}; \frac{\Im\{z_{\pm n}\}}{2}\right)$, as is clear from Figure 3. Hence, (5) implies that there exists a constant c_a such that

$$P\left\{\gamma[0,\infty) \cap \mathcal{C}(x; rx) \neq \emptyset\right\} \le \sum_{n=-\infty}^{\infty} P\left\{\gamma[0,\infty) \cap \mathcal{B}\left(z_n; \frac{\Im\{z_n\}}{2}\right) \neq \emptyset\right\} \le c_a r^{4a-1}$$

and the proof is complete. $\square$

3.2 The lower bound for $4 < \kappa < 8$

In this section we establish the lower bound in Theorem 1.1 for $\kappa \in (4, 8)$.

Theorem 3.3. *Let $0 < r \le 1/3$ and $x > 0$. If $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ is a chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞ with $4 < \kappa < 8$ and $a = 2/\kappa$, then there exists a constant c_a such that*

$$P\{\gamma[0,\infty) \cap \mathcal{C}(x; rx) \neq \emptyset\} \ge c_a r^{4a-1}.$$

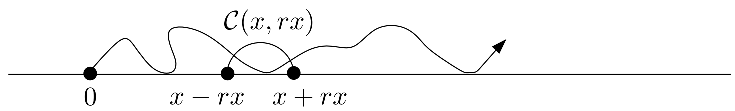

Figure 4: The event that $\gamma[0,\infty)$ intersects the interval $[x - rx, x + rx]$.

Proof. It is clear that if $\gamma[0,\infty)$ intersects the interval $[x - rx, x + rx]$ then it also intersects the semicircle $\mathcal{C}(x; rx)$, as Figure 4 shows. By Proposition 6.34 of [7] and the scale invariance of SLE,

$$\begin{aligned} P\{\gamma[0,\infty) \cap [x - rx, x + rx] \neq \emptyset\} &= \frac{\Gamma(2a)}{\Gamma(1-2a)\Gamma(4a-1)} \int_0^{\frac{2r}{1+r}} \frac{dt}{t^{2-4a}(1-t)^{2a}} \\ &\geq \frac{\Gamma(2a)}{\Gamma(1-2a)\Gamma(4a-1)} \int_0^{\frac{2r}{1+r}} \frac{dt}{t^{2-4a}(1/2)^{2a}} \\ &\geq \frac{\Gamma(2a)2^{2a}}{\Gamma(1-4a)\Gamma(4a)} (2r)^{4a-1}. \end{aligned}$$

The first and second inequalities use $0 < r \leq 1/3$. (Actually, all that is required here is that $r > 0$ be bounded away from 1. We took $0 < r \leq 1/3$ simply for convenience.) $\square$

3.3 The lower bound for $0 < \kappa \leq 4$

In this section we establish the lower bound in Theorem 1.1 for $\kappa \in (0, 4)$.

Theorem 3.4. *Let $0 < r < 1$ and $x > 0$. If $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ is a chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞ with $0 < \kappa \leq 4$ and $a = 2/\kappa$, then there exists a constant c_a such that*

$$P\{\gamma[0,\infty) \cap \mathcal{C}(x; rx) \neq \emptyset\} \geq c_a r^{4a-1}.$$

To prove the theorem we recall the probability that a fixed point $z \in \mathbb{H}$ lies to the left of $\gamma[0,\infty)$. This result is originally due to Schramm [10], although the form that we include is from Garban and Trujillo Ferreras [5].

Proposition 3.5. *Let $z = \rho e^{i\theta} \in \mathbb{H}$, and set $f(z) = P\{z$ is to the left of $\gamma[0,\infty)\}$. By scaling, the function f only depends on θ and is given by*

$$f(\theta) = \frac{\int_0^\theta (\sin\alpha)^{4a-2}\, d\alpha}{\int_0^\pi (\sin\alpha)^{4a-2}\, d\alpha}.$$

Proof of Theorem 3.4. Figure 5 clearly shows that

$$P\{\gamma[0,\infty) \cap \mathcal{C}(x; rx) \neq \emptyset\} \geq P\{x + irx \text{ is to the left of } \gamma[0,\infty)\}.$$

Since $\arg(x + irx) = \arctan(r)$ and since $2\sin t \geq t$ for $0 \leq t \leq \pi/4$, we conclude from

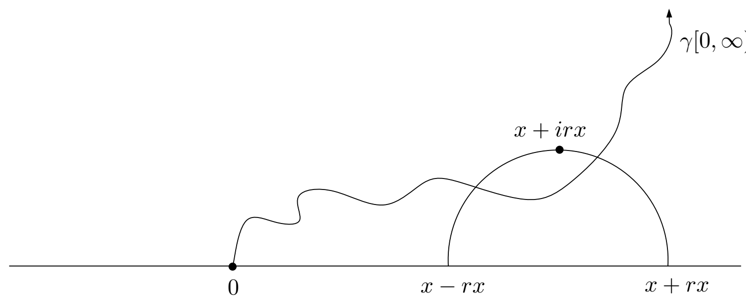

Figure 5: The point $z = x + irx$ is to the left of $\gamma[0, \infty)$.

Proposition 3.5 that

$$\begin{aligned} P\{x + irx \text{ is to the left of } \gamma[0,\infty)\} \cdot \int_0^{\pi} (\sin\alpha)^{4a-2}\, d\alpha &= \int_0^{\arctan(r)} (\sin\alpha)^{4a-2}\, d\alpha \\ &\geq \frac{1}{2}\int_0^{\arctan(r)} \alpha^{4a-2}\, d\alpha \\ &= \frac{\arctan^{4a-1}(r)}{8a-2}. \end{aligned} \tag{6}$$

Since $8 \arctan t \geq \pi t$ for $0 \leq t \leq 1$, we see that (6) implies that there exists a constant c_a, namely

$$c_a = \frac{\pi^{4a-1}}{4^{6a-1}(4a-1)\int_0^{\pi} (\sin\alpha)^{4a-2}\, d\alpha},$$

such that $P\{x + irx \text{ is to the left of } \gamma[0,\infty)\} \geq c_a r^{4a-1}$. □

4 Estimating the diameter of a chordal SLE path

In this section, we derive Corollary 1.2 from Theorem 1.1. The proof is not difficult; the basic idea is to determine the appropriate conformal transformation and use the conformal invariance of chordal SLE. Recall that if $D \subset \mathbb{C}$ is a simply connected domain and z, w are two distinct points in ∂D, then chordal SLE$_\kappa$ in D from z to w is defined to be the conformal image of chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞ as discussed in Section 2. Let $x > 0$ be real, and suppose that $\gamma' : [0, t_{\gamma'}] \to \overline{\mathbb{H}}$ is an SLE$_\kappa$ in $\mathbb{H}$ from 0 to x. We also note that we are not interested in the parametrization of the SLE path, but only in the set of points visited by its trace. Suppose that $R \geq 3$, and consider $\mathcal{C}(0; Rx) = \{Rxe^{i\theta} : 0 < \theta < \pi\}$. For $z \in \mathbb{H}$, let

$$h(z) = \frac{R^2}{R^2-1}\frac{z}{x-z}$$

so that $h : \mathbb{H} \to \mathbb{H}$ is a conformal (Möbius) transformation with $h(0) = 0$ and $h(x) = \infty$. It is straightforward (although a bit tedious) to verify that

$$h\left(\mathcal{C}(0; Rx)\right) = \mathcal{C}\left(-1; \frac{1}{R}\right).$$

If $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ is a chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞, then the conformal invariance of SLE implies that

$$P\{\gamma'[0,t_{\gamma'}]\cap\mathcal{C}(0;Rx) \neq \emptyset\} = P\{h(\gamma'[0,t_{\gamma'}])\cap h(\mathcal{C}(0;Rx)) \neq \emptyset\} = P\left\{\gamma[0,\infty)\cap\mathcal{C}\left(-1,\frac{1}{R}\right)\neq\emptyset\right\}.$$

By the symmetry of SLE about the imaginary axis,

$$P\left\{\gamma[0,\infty)\cap\mathcal{C}\left(-1,\frac{1}{R}\right)\neq\emptyset\right\} = P\left\{\gamma[0,\infty)\cap\mathcal{C}\left(1,\frac{1}{R}\right)\neq\emptyset\right\} \asymp R^{1-4a}$$

where the last bound follows from Theorem 1.1 with $r = 1/R$.

5 The $\kappa = 8/3$ case

In this section we derive the facts given in the remark following Corollary 1.2. The key result that is needed is the restriction property of chordal SLE$_{8/3}$. Indeed, the following remarkable formula due to Lawler, Schramm, and Werner solves the $\kappa = 8/3$ case immediately. See Theorem 6.17 of [7] for a proof; compare this with Proposition 9.4 and Example 9.7 of [7] as well.

Proposition 5.1. *If $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ is a chordal SLE$_{8/3}$ in $\mathbb{H}$ from 0 to ∞, and A is a bounded subset of $\mathbb{H}$ such that $\mathbb{H}\setminus A$ is simply connected, $A = \mathbb{H}\cap\overline{A}$, and $0 \notin \overline{A}$, then*

$$P\{\gamma[0,\infty)\cap A = \emptyset\} = [\Phi_A'(0)]^{5/8}$$

where $\Phi_A : \mathbb{H}\setminus A \to \mathbb{H}$ is the unique conformal transformation of $\mathbb{H}\setminus A$ to $\mathbb{H}$ with $\Phi_A(0) = 0$ and $\Phi_A(z) \sim z$ as $z \to \infty$.

Applying Proposition 5.1 to our situation implies that if $0 < r < 1$, then

$$P\{\gamma[0,\infty)\cap\mathcal{C}(x;rx) = \emptyset\} = P\{\gamma[0,\infty)\cap\mathcal{D}(x;rx) = \emptyset\} = \left[\Phi'_{\mathcal{D}(x;rx)}(0)\right]^{5/8}$$

where $\mathcal{D}(x;rx)$ is the half disk of radius rx centred at x in the upper half plane as given by (2) and $\Phi_{\mathcal{D}(x;rx)}(z)$ is the conformal transformation from $\mathbb{H}\setminus\mathcal{D}(x;rx)$ onto $\mathbb{H}$ with $\Phi_{\mathcal{D}(x;rx)}(0) = 0$ and $\Phi_{\mathcal{D}(x;rx)}(z) \sim z$ as $z \to \infty$. In fact, the exact form of $\Phi_{\mathcal{D}(x;rx)}(z)$ is given by

$$\Phi_{\mathcal{D}(x;rx)}(z) = z + \frac{r^2x^2}{z-x} + r^2x.$$

Note that $\Phi_{\mathcal{D}(x;rx)}(0) = 0$, $\Phi_{\mathcal{D}(x;rx)}(\infty) = \infty$, and $\Phi'_{\mathcal{D}(x;rx)}(\infty) = 1$. We easily calculate $\Phi'_{\mathcal{D}(x;rx)}(0) = 1 - r^2$ and therefore conclude that

$$P\{\gamma[0,\infty)\cap\mathcal{C}(x;rx) = \emptyset\} = (1-r^2)^{5/8}$$

as required.

Remark. It is worth noting that Proposition 5.1 with the exact form of the conformal transformation $\Phi_{\mathcal{D}(x;rx)} : \mathbb{H}\setminus\mathcal{D}(x;rx) \to \mathbb{H}$ was used by Kennedy [6] to produce strong numerical evidence that the scaling limit of planar self-avoiding walk is chordal SLE$_{8/3}$.

6 An application of Theorem 1.1

In this section, we derive estimates for two more intersection probabilities for a chordal SLE path and a semicircle centred on the real line. In particular, Corollary 6.2 gives an estimate in the $4 < \kappa < 8$ regime for the probability that an entire semicircle is swallowed at once by a chordal SLE_κ path in $\mathbb{H}$ from 0 to ∞. By the scaling properties of SLE, we may rewrite Theorem 1.1 in terms of a semicircle centred at $x > 0$ of radius ϵ, $0 < \epsilon \le x/3$. For the convenience of the reader, we repeat the statement of Theorem 1.1 in this slightly different form, and note that it is seen to generalize a result due to Rohde and Schramm [8, Lemma 6.6].

Theorem 1.1. *Let $x > 0$ be a fixed real number, and suppose $0 < \epsilon \le x/3$. If $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is a chordal SLE_κ in $\mathbb{H}$ from 0 to ∞ with $0 < \kappa < 8$ and $a = 2/\kappa$, then*

$$P\{\gamma[0,\infty) \cap \mathcal{C}(x;\epsilon) \neq \emptyset\} \asymp \left(\frac{\epsilon}{x}\right)^{4a-1}$$

where $\mathcal{C}(x;\epsilon)$ is the semicircle of radius ϵ centred at x in the upper half plane as given by (3).

We conclude with an application of Theorem 1.1 by combining it with a method due to Dubédat [4]. For the remainder of the paper suppose that $4 < \kappa < 8$; as before, let $a = 2/\kappa$. Suppose that $0 < r \le 1/3$ and consider the two semicircles

$$\mathcal{C}_r = \mathcal{C}\left(1-r;\frac{r}{2}\right) = \left\{z \in \mathbb{H} : |z-1+r| = \frac{r}{2}\right\} \tag{7}$$

and

$$\mathcal{C}'_r = \mathcal{C}\left(1-\frac{3r}{4};\frac{3r}{4}\right) = \left\{z \in \mathbb{H} : \left|z-1+\frac{3r}{4}\right| = \frac{3r}{4}\right\} \tag{8}$$

as illustrated in Figure 6.

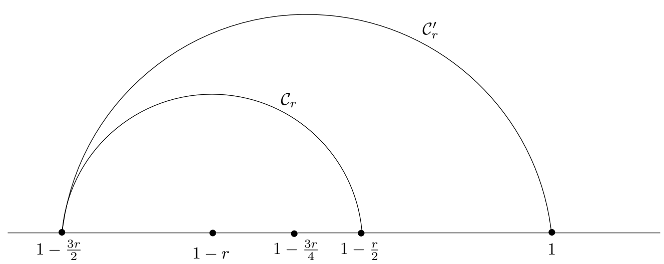

Figure 6: The semicircles $\mathcal{C}'_r$ and $\mathcal{C}_r$.

It follows from Theorem 1.1 that $P\{\gamma[0,\infty) \cap \mathcal{C}'_r \neq \emptyset\} \asymp r^{4a-1}$ and so there exists a constant c'_a such that $1 - c'_a r^{4a-1} \le P\{\gamma[0,\infty) \cap \mathcal{C}'_r = \emptyset\}$. Besides,

$$P\{\gamma[0,\infty) \cap \mathcal{C}'_r = \emptyset\} \le \inf_{z \in \mathcal{C}_r} P\{T_z = T_1\}$$

where T_z is the swallowing time of the point $z \in \overline{\mathbb{H}}$ (and the infimum is over all $z \in \mathcal{C}_r$ *not* $z \in \mathcal{C}'_r$). From this we conclude that there exists a constant c'_a such that

$$1 - c'_a r^{4a-1} \le \inf_{z \in \mathcal{C}_r} P\{T_z = T_1\}. \tag{9}$$

In order to derive an upper bound for the expression in (9), we use a method from Dubédat [4]. We now outline this method referring the reader to that paper for further details. Let g_t denote the solution to the chordal Loewner equation (4) with driving function $U_t = -B_t$ where B_t is a standard one-dimensional Brownian motion with $B_0 = 0$. For $t < T_1$, the swallowing time of the point 1, consider the conformal transformation $\tilde{g}_t : \mathbb{H} \setminus K_t \to \mathbb{H}$ given by

$$\tilde{g}_t(z) = \frac{g_t(z) + B_t}{g_t(1) + B_t}, \quad \tilde{g}_0(z) = z.$$

Note that $\tilde{g}_t(\gamma(t)) = 0$, $\tilde{g}_t(1) = 1$, $\tilde{g}_t(\infty) = \infty$, and that $\tilde{g}_t(z)$ satisfies the stochastic differential equation

$$d\tilde{g}_t(z) = \left[\frac{a}{\tilde{g}_t(z)} + (1-a)\tilde{g}_t(z) - 1\right] \frac{dt}{(g_t(1) + B_t)^2} + [1 - \tilde{g}_t(z)] \frac{dB_t}{g_t(1) + B_t}.$$

If we now perform the time-change

$$\sigma(t) = \int_0^t \frac{ds}{(g_s(1) + B_s)^2},$$

then $\tilde{g}_{\sigma(t)}(z)$ satisfies the stochastic differential equation

$$d\tilde{g}_t(z) = \left[\frac{a}{\tilde{g}_t(z)} + (1-a)\tilde{g}_t(z) - 1\right] dt + [1 - \tilde{g}_t(z)]\, dB_t \tag{10}$$

For ease of notation, and because it does not concern us at present, we have also denoted the time-changed flow by $\{\tilde{g}_t(z),\ t \geq 0\}$. Furthermore, it is shown in detail in [4] that for all $\kappa > 0$, the time-changed stochastic flow $\{\tilde{g}_t(z),\ t \geq 0\}$ given by (10) does not explode in finite time a.s. Therefore, if F is an analytic function on $\mathbb{H}$ such that $\{F(\tilde{g}_t(z)),\ t \geq 0\}$ is a local martingale, then Itô's formula (at $t = 0$) implies that F must be a solution to the differential equation

$$w(1-w)F''(w) + [2a - (2-2a)w]F'(w) = 0. \tag{11}$$

An explicit solution to (11) is given by

$$F(w) = \frac{\Gamma(2a)}{\Gamma(1-2a)\Gamma(4a-1)} \int_0^w \zeta^{-2a}(1-\zeta)^{4a-2} d\zeta \tag{12}$$

which is normalized so that $F(0) = 0$ and $F(1) = 1$. Note that (12) is a Schwarz-Christoffel transformation of the upper half plane onto the isosceles triangle whose interior angles are $(1-2a)\pi$, $(1-2a)\pi$, and $(4a-1)\pi$. The boundary values $F(0) = 0$ and $F(1) = 1$ imply that two of the vertices of the triangle are at 0 and 1, and from (12) we conclude that the third vertex of the triangle is at

$$F(\infty) = \frac{\Gamma(2a)\Gamma(1-2a)}{\Gamma(2-4a)\Gamma(4a-1)} e^{(1-2a)\pi i}$$

which follows from (6.2.1) and (6.2.2) of [1]. Using (6.1.17) of [1], namely $\Gamma(z)\Gamma(1-z) = \pi \csc(\pi z)$, one can deduce that

$$2\cos((1-2a)\pi) = \frac{\Gamma(2a)\Gamma(1-2a)}{\Gamma(2-4a)\Gamma(4a-1)}$$

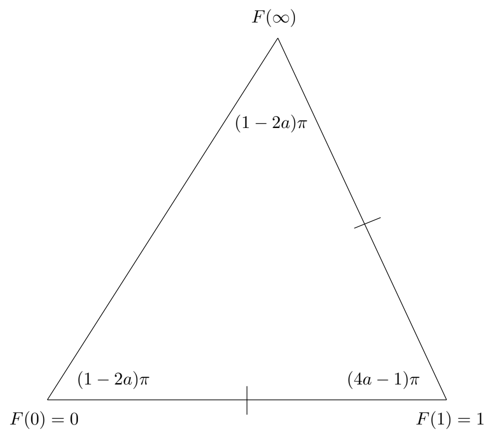

Figure 7: The isosceles triangle with vertices at 0, 1, and $F(\infty)$.

from which it follows that $\Re(F(\infty)) \geq 0$ and that $|F(\infty)-1| = 1$ as is to be expected for this isosceles triangle. The image of $\mathbb{H}$ under F is illustrated in Figure 7. We now apply the optional sampling theorem to the martingale $F(\tilde{g}_{t\wedge T_z \wedge T_1}(z))$ to find (see the discussion surrounding Proposition 1 of [4]) that for $z \in \mathbb{H}$,

$$\begin{aligned} F(\tilde{g}_0(z)) = F(z) &= F(0)P\{T_z < T_1\} + F(1)P\{T_z = T_1\} + F(\infty)P\{T_z > T_1\} \\ &= P\{T_z = T_1\} + F(\infty)P\{T_z > T_1\}. \end{aligned} \tag{13}$$

Consequently, identifying the imaginary and real parts of the previous equation (13) implies that

$$\Re\{F(z)\} = P\{T_z = T_1\} + \Re\{F(\infty)\}P\{T_z > T_1\}.$$

Since $\Re\{F(\infty)\} \geq 0$, we conclude $P\{T_z = T_1\} \leq \Re\{F(z)\} \leq |F(z)|$. But now integrating along the straight line from 0 to z (i.e., letting $\theta = \arg(z)$, $\zeta = \rho e^{i\theta}$, $0 \leq \rho \leq |z|$) gives

$$\begin{aligned} |F(z)| &= \frac{\Gamma(2a)}{\Gamma(1-2a)\Gamma(4a-1)} \left| \int_0^{|z|} (\rho e^{i\theta})^{-2a}(1-\rho e^{i\theta})^{4a-2} e^{i\theta} d\rho \right| \\ &\leq \frac{\Gamma(2a)}{\Gamma(1-2a)\Gamma(4a-1)} \int_0^{|z|} \rho^{-2a}|1-\rho|^{4a-2} d\rho \\ &= 1 - \frac{\Gamma(2a)}{\Gamma(1-2a)\Gamma(4a-1)} \int_{|z|}^1 \rho^{-2a}(1-\rho)^{4a-2} d\rho \end{aligned}$$

which relied on the fact that $4a-2<0$. If $z \in \mathcal{C}_r$ so that $0 < 1 - \frac{3r}{2} \leq |z| \leq 1 - \frac{r}{2} < 1$ by definition, then

$$\int_{|z|}^1 \rho^{-2a}(1-\rho)^{4a-2} d\rho \geq \int_{|z|}^1 (1-\rho)^{4a-2} d\rho = \frac{(1-|z|)^{4a-1}}{4a-1} \geq \frac{2^{1-4a}}{4a-1} r^{4a-1}.$$

Hence,

$$P\{T_z = T_1\} \leq |F(z)| \leq 1 - c_a'' r^{4a-1}$$

where

$$c_a'' = \frac{2^{1-4a}\tilde{c}_a}{4a-1} \quad \text{and} \quad \tilde{c}_a = \frac{\Gamma(2a)}{\Gamma(1-2a)\Gamma(4a-1)}.$$

Taking the supremum of the previous expression over all $z \in \mathcal{C}_r$ gives us the required upper bound to (9). Hence, we have proved the following theorem.

Theorem 6.1. *Let $0 < r \le 1/3$. If $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ is a chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞ with $4 < \kappa < 8$ and $a = 2/\kappa$, then there exist constants c_a' and c_a'' such that*

$$1 - c_a' r^{4a-1} \le \inf_{z \in \mathcal{C}_r} P\{T_z = T_1\} \le \sup_{z \in \mathcal{C}_r} P\{T_z = T_1\} \le 1 - c_a'' r^{4a-1}$$

where

$$\mathcal{C}_r = \mathcal{C}\left(1-r; \frac{r}{2}\right) = \left\{z \in \mathbb{H} : |z - 1 + r| = \frac{r}{2}\right\}$$

denotes the circle of radius $r/2$ centred at $1-r$ in the upper half plane as in (7).

This theorem now yields the following corollary.

Corollary 6.2. *Let $0 < r \le 1/3$. If $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ is a chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞ with $4 < \kappa < 8$ and $a = 2/\kappa$, then there exist constants c_a' and c_a'' such that*

$$1 - c_a' r^{4a-1} \le P\{T_z = T_1 \text{ for all } z \in \mathcal{C}_r\} \le 1 - c_a'' r^{4a-1}$$

where $\mathcal{C}_r$ is given by (7) as above.

Proof. Let $z_0 = 1 - r + \frac{ir}{2}$ so that $z_0 \in \mathcal{C}_r$. Theorem 6.1 implies that there exists a constant c_a'' such that

$$P\{T_z = T_1 \text{ for all } z \in \mathcal{C}_r\} \le P\{T_{z_0} = T_1\} \le \sup_{z \in \mathcal{C}_r} P\{T_z = T_1\} \le 1 - c_a'' r^{4a-1}. \tag{14}$$

As noted earlier, it follows from Theorem 1.1 that $P\{\gamma[0,\infty) \cap \mathcal{C}_r' \ne \emptyset\} \asymp r^{4a-1}$ where $\mathcal{C}_r'$ is given by (8), and so there exists a constant c_a' such that

$$P\{T_z = T_1 \text{ for all } z \in \mathcal{C}_r\} \ge P\{\gamma[0,\infty) \cap \mathcal{C}_r' = \emptyset\} \ge 1 - c_a' r^{4a-1}. \tag{15}$$

Taking (14) and (15) together completes the proof. □

Addendum

After this paper was completed, two preprints relevant to the subject at hand were released. Both Alberts and Sheffield [2] and Schramm and Zhou [11] prove, independently and using different methods, that for $4 < \kappa < 8$ the SLE$_\kappa$ curve intersected with the real line has Hausdorff dimension $2 - 8/\kappa$ a.s. Specifically, Alberts and Sheffield [2] establish an upper bound on the asymptotic probability of an SLE$_\kappa$ curve hitting two small intervals on the real line as the interval width goes to zero, whereas Schramm and Zhou [11] examine how close the chordal SLE$_\kappa$ curve gets to the real line asymptotically far away from its starting point. In fact, a combination of Lemma 2.1 and Proposition 2.3 from Schramm and Zhou [11] can be used to derive an alternate proof of our Theorem 1.1.

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