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ON SUBORDINATORS, SELF-SIMILAR MARKOV PROCESSES AND SOME FACTORIZATIONS OF THE EXPONENTIAL VARIABLE

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Abstract

Let ξ be a subordinator with Laplace exponent Φ, $I = \int_0^\infty \exp(-\xi_s)ds$ the so-called exponential functional, and X (respectively, $\hat{X}$) the self-similar Markov process obtained from ξ (respectively, from $\hat{\xi} = -\xi$) by Lamperti's transformation. We establish the existence of a unique probability measure ρ on $]0,\infty[$ with k-th moment given for every $k \in \mathbb{N}$ by the product $\Phi(1)\cdots\Phi(k)$, and which bears some remarkable connections with the preceding variables. In particular we show that if R is an independent random variable with law ρ then IR is a standard exponential variable, that the function $t \to \mathbb{E}(1/X_t)$ coincides with the Laplace transform of ρ, and that ρ is the 1-invariant distribution of the sub-markovian process $\hat{X}$. A number of known factorizations of an exponential variable are shown to be of the preceding form IR for various subordinators ξ.

1 Introduction and main results

Let $\xi = (\xi_t, t \geq 0)$ be a subordinator (that is an increasing Lévy process) started from 0; the degenerate case when $\xi \equiv 0$ being implicitly excluded. Its distribution is characterized via the Laplace transform

$$\mathbb{E}(\exp(-q\xi_t)) = \exp(-t\Phi(q)), \qquad t, q \geq 0,$$

where $\Phi : [0,\infty[\to [0,\infty[$ denotes the so-called Laplace exponent (sometimes also referred to as the Bernstein function) of ξ. In turn, Φ is given by the celebrated Lévy-Khintchine formula; we refer to [3] for background.

This note is motivated by several recent works [4, 5, 6, 7, 8] related to the so-called *exponential functional*

$$I = \int_0^\infty \exp(-\xi_t)dt\,.$$

This random variable plays an important part in mathematical finance as well as in the study of the self-similar Markov processes obtained from ξ by the classical transformation of Lamperti [12]. Our purpose here is to point out the role of a remarkable probability measure which may be associated to I in this framework.
To start with, recall that the distribution of I is determined by its entire moments, which are given in terms of the Laplace exponent Φ by the identity

$$\mathbb{E}\left(I^k\right) = \frac{k!}{\Phi(1)\cdots\Phi(k)}\,, \qquad \text{for } k=1,2,\ldots \tag{1}$$

See for instance [6, 7]. We may now state our first result.

Proposition 1 *There exists a unique probability measure $\rho = \rho_\Phi$ on $[0,\infty[$ which is determined by its entire moments*

$$\int_{[0,\infty[} x^k \rho(dx) = \Phi(1)\cdots\Phi(k) \qquad \textit{for } k=1,2,\ldots$$

In particular, if R is a random variable with law ρ that is independent of I, then we have the identity in distribution

$$IR \stackrel{\mathcal{L}}{=} \mathbf{e}\,,$$

where $\mathbf{e}$ denotes a standard exponential variable.

In general, we do not know any example of a random variable naturally related to ξ with distribution ρ. We shall not establish Proposition 1 by checking e.g. Stieltjes' moments condition, but we shall rather use properties of a self-similar Markov process obtained from ξ by Lamperti's transformation. Along the way, our approach will unravel some interesting features of the latter.
Let us now recall Lamperti's transformation. We first define implicitly a time-change $\tau(t)$ for $t \geq 0$ by the identity

$$t = \int_0^{\tau(t)} \exp(\xi_s)ds\,,$$

and then we introduce

$$X_t = \exp\left(\xi_{\tau(t)}\right)\,, \qquad t \geq 0\,.$$

The process $X = (X_t, t \geq 0)$ is a strong Markov process started from $X_0 = 1$ (recall that the subordinator ξ starts from 0), which enjoys the scaling property. Specifically, if for $x > 0$ we write $\mathbb{P}_x$ for the distribution of the process $\left(xX_{t/x}, t \geq 0\right)$, then $\mathbb{P}_x$ coincides with the law of the process X started from x (that is when the subordinator ξ is replaced by $\xi + \log x$ in Lamperti's construction). Of course, we keep using the notation $\mathbb{P} = \mathbb{P}_1$ in the sequel.
As a first connection between the probability measure ρ and the self-similar Markov process X, we point out that the expectation of $1/X_t$ coincides with the Laplace transform of ρ in the variable t.

Proposition 2 *For every $t \geq 0$, it holds that*

$$\mathbb{E}(1/X_t) = \int_{[0,\infty[} \mathrm{e}^{-tx} \rho(dx)\,.$$

More generally, we shall see that for any $p > 0$, the map $t \to \mathbb{E}(X_t^{-p})$ can be identified as the Laplace transform of a probability measure ρ_p whose entire moments can be expressed in terms of the Laplace exponent Φ; see the forthcoming equations (2) and (3) in section 2.
As an immediate consequence of Proposition 2, we have the following.

Corollary 3 (i) *Let $\mathtt{d} \in [0, \infty[$ denote the drift coefficient of ξ, viz.*

$$\mathtt{d} = \lim_{q\to\infty} \frac{\Phi(q)}{q}\,.$$

Then for every $t < 1/\mathtt{d}$, $\mathbb{E}(\mathrm{e}^{tR}) < \infty$ and the series

$$\sum_{k=0}^{\infty} \frac{\Phi(1)\cdots\Phi(k)}{k!} (-t)^k$$

is absolutely convergent (with the convention that the term corresponding to $k = 0$ equals 1) and its sums coincides with $\mathbb{E}(1/X_t)$.
(ii) *The law ρ has no mass at 0.*

In order to present a further relation involving the law ρ, we introduce a second self-similar Markov process, denoted by $\hat{X}$. Formally, it is obtained by replacing the subordinator ξ by its dual $\hat{\xi} = -\xi$ in Lamperti's construction. More precisely, we first define implicitly $\hat{\tau}(t)$ for every $t < I$ by

$$t = \int_0^{\hat{\tau}(t)} \exp(-\xi_s) ds\,,$$

and by $\hat{\tau}(t) = \infty$ for every $t \geq I$. Then we introduce the time-changed process

$$\hat{X}(t) = \exp\left(-\xi_{\hat{\tau}(t)}\right)\,, \qquad t \geq 0\,,$$

with the convention $\mathrm{e}^{-\infty} = 0$, so that 0 is a cemetery point which $\hat{X}$ reaches at time I. We may think of $\hat{X}$ as the dual of X with respect to the Lebesgue measure; see [5]. It is again a strong Markov process, and its semigroup will be denoted by $\hat{P}_t$, i.e.

$$\hat{P}_t f(y) = \mathbb{E}\left(f(\hat{X}_t) \mid \hat{X}_0 = y\right)$$

for every bounded continuous function $f :]0, \infty[\to \mathbb{R}$ with the convention that $f(0) = 0$ (since 0 is a cemetery point). We stress that this semigroup is only *sub*-markovian.

Proposition 4 *The law ρ is 1-invariant for the dual process $\hat{X}$, that is*

$$\int_{[0,\infty[} \hat{P}_t f(y) \rho(dy) = \mathrm{e}^{-t} \int_{[0,\infty[} f(y) \rho(dy)$$

for every $t \geq 0$ and every bounded continuous function $f :]0, \infty[\to \mathbb{R}$.

Of course, by the duality between X and $\hat{X}$ (cf. [5]), if ρ is absolutely continuous with respect to the Lebesgue measure on $]0,\infty[$, say with density r, then we may find a version of r which is an 1-harmonic function for X, in the sense that the process $(\mathrm{e}^t r(X_t), t \geq 0)$ is a martingale. Furthermore, it can be checked that ρ is absolutely continuous if and only if the same holds for the potential measure of ξ.
The rest of this note is organized as follows. The proofs of the results stated above are presented in the next section. In the final section, we identify several well-known factorizations of the exponential law with the one based on the exponential functional of a subordinator that appears in Proposition 1. Beta and Gamma variables play an important role in these case-studies.

2 Proofs of the main results

The key to our analysis is provided by the following elementary identity.

Lemma 5 *For every $t \geq 0$ and $p > 0$, the variable*

$$X_t^p \int_t^\infty \frac{ds}{X_s^{p+1}}$$

is independent from $\mathcal{F}_t := \sigma\{X_s, 0 \leq s \leq t\}$ and is distributed as

$$\int_0^\infty \exp(-p\xi_s)ds\,.$$

As a consequence, we have

$$\mathbb{E}\left(\int_t^\infty \frac{ds}{X_s^{p+1}}\right) = \frac{\mathbb{E}(X_t^{-p})}{\Phi(p)}\,.$$

Proof: Recall that for every $x > 0$, $\mathbb{P}_x$ denotes the distribution of the self-similar Markov process X started from x (so in particular $\mathbb{P}_1 = \mathbb{P}$). As a consequence of the Markov property at time t, we need only show that under $\mathbb{P}_x$, the variable

$$x^p \int_0^\infty \frac{ds}{X_s^{p+1}}$$

is distributed as $\int_0^\infty \exp(-p\xi_s)ds$, and by self-similarity, we may focus on the case $x = 1$. Then, the change of variables

$$t = \tau(s) \quad , \quad s = \int_0^t \exp(\xi_u)du$$

yields

$$\begin{aligned}\int_0^\infty X_s^{-(p+1)}\,ds &= \int_0^\infty \exp\left(-(p+1)\xi_{\tau(s)}\right)ds \\ &= \int_0^\infty \exp\left(-(p+1)\xi_t\right)\mathrm{e}^{\xi_t}\,dt \\ &= \int_0^\infty \exp\left(-p\xi_t\right)\,dt\,,\end{aligned}$$

which establishes the desired identity in law. □

Remark. An interesting by-product of the argument above is that the process

$$X_t^{-p} + \Phi(p) \int_0^t \frac{ds}{X_s^{p+1}}, \qquad t \geq 0$$

is a uniformly integrable martingale; more precisely its terminal value is

$$\Phi(p) \int_0^\infty \frac{ds}{X_s^{p+1}}.$$

We are now able to tackle the proof of the first two propositions.

Proof of Propositions 1 and 2: It follows from Lemma 5 that

$$\frac{\partial \mathbb{E}(X_t^{-p})}{\partial t} = -\Phi(p)\mathbb{E}\left(X_t^{-(p+1)}\right).$$

By iteration, we get that the function $t \to \mathbb{E}(X_t^{-p})$ is completely monotone, and takes value 1 for $t = 0$. Thus by Bernstein's theorem, it coincides with the Laplace transform of some probability measure on $[0, \infty[$ which we shall denote by ρ_p in the sequel, i.e.

$$\mathbb{E}\left(X_t^{-p}\right) = \int_{[0,\infty[} \mathrm{e}^{-tx} \rho_p(dx), \qquad t \geq 0. \tag{2}$$

The entire moments of ρ_p are given by the iterated derivatives of its Laplace transform at $t = 0$, and we get

$$\int_{[0,\infty[} x^k \rho_p(dx) = \Phi(p) \cdots \Phi(p + (k-1)), \qquad \text{for } k = 1, 2, \ldots \tag{3}$$

Specifying this for $p = 1$ (and the notation $\rho = \rho_1$) completes the proof of Propositions 1 and 2, as the second assertion in Proposition 1 follows immediately from the first and the formula (1) for the entire moments of the exponential functional. □

Corollary 3 is now obvious. More precisely the first part is an immediate combination of Propositions 1 and 2. The second follows from Proposition 2 and the fact that $1/X_t$ decreases to 0 as $t \to \infty$.
We shall now provide two proofs for Proposition 4, based on two different aspects of Proposition 1.

First proof of Proposition 4: Denote by $\hat{\mathbb{P}}_\rho$ the distribution of the dual self-similar Markov process $\hat{X}$ with initial law ρ, and by ζ its lifetime (i.e. the hitting time of the cemetery point 0). Recall that the exponential functional I has the same law as ζ under $\hat{\mathbb{P}}_1$. By self-similarity, the distribution of ζ under $\hat{\mathbb{P}}_\rho$ is thus the same as that of RI, where R is a random variable with law ρ which is independent of the exponential functional I. We know from Proposition 1 that the latter is a standard exponential variable. In other words, we have for every $s, t \geq 0$

$$\mathrm{e}^{-(t+s)} = \hat{\mathbb{P}}_\rho(\zeta > s + t).$$

Applying the Markov property (for the dual process $\hat{X}$) at time t, and then the self-similarity property, we may express the right-hand side as

$$\int_{]0,\infty[} \hat{\mathbb{P}}_x(\zeta > s)\hat{\mathbb{P}}_\rho(\hat{X}_t \in dx) \;=\; \int_{]0,\infty[} \mathbb{P}(xI > s)\hat{\mathbb{P}}_\rho(\hat{X}_t \in dx)\,.$$

Putting the pieces together, we see that if we define

$$\rho'(dx) \;:=\; \mathrm{e}^t\hat{\mathbb{P}}_\rho(\hat{X}_t \in dx)\,,$$

then ρ' is a finite measure on $]0,\infty[$ which solves the equation

$$\int_{]0,\infty[} \mathbb{P}(xI > s)\rho'(dx) \;=\; \mathrm{e}^{-s}\,, \qquad s \geq 0\,.$$

First, specifying this for $s = 0$, we get that ρ' must be a probability measure. The probabilistic interpretation of the equation is now clear. Introducing a random variable R' with law ρ' which is independent of I, the product IR' follows the standard exponential law. In particular, we can calculate the entire moments of ρ', and we get that they coincide with those of ρ. We conclude that $\hat{\mathbb{P}}_\rho(\hat{X}_t \in dx) = \mathrm{e}^{-t}\rho(dx)$. $\square$

Second proof of Proposition 4: We shall merely sketch the main line, and leave technical details to the interested reader.
Let us denote the drift coefficient of the subordinator ξ by $\mathtt{d} \geq 0$, and its Lévy measure by Π, so the Lévy-Khintchine formula reads

$$\Phi(q) \;=\; q\mathtt{d} + \int_{]0,\infty]} (1 - \mathrm{e}^{-qx})\Pi(dx)\,, \qquad q \geq 0\,,$$

(as usual, the mass of the Lévy measure at ∞ has to be viewed as the killing rate of the subordinator). It is well-known that the infinitesimal generator $\mathcal{G}$ of ξ is given by

$$\mathcal{G}f(x) \;=\; f'(x)\mathtt{d} + \int_{]0,\infty]} (f(x+y) - f(x))\Pi(dy)\,, \qquad x \in \mathbb{R},$$

where f denotes a generic function of class $\mathcal{C}^1$ on $\mathbb{R}$ with bounded derivative and limit 0 at ∞. A classical result on the effect of time-substitution on infinitesimal generators of Markov processes (see e.g. Section III.38 in Williams [17]) entails that the infinitesimal generator $\hat{G}$ of the dual self-similar Markov process $\hat{X}$ is given by

$$\hat{G}f(x) \;=\; -f'(x)\mathtt{d} + \frac{1}{x}\int_{]0,\infty]} (f(x\mathrm{e}^{-y}) - f(x))\Pi(dy)\,, \qquad x > 0\,,$$

say at least when now f is a function of class $\mathcal{C}^1$ on $]0,\infty[$ with compact support.
If we define for every integer $k \geq 0$ the power function $f_k(x) = x^k$, we thus obtain formally from the Lévy-Khintchine formula that for $k \geq 1$

$$\hat{G}f_k(x) \;=\; -kx^{k-1}\mathtt{d} + x^{k-1}\int_{]0,\infty]} (e^{-ky} - 1)\Pi(dy) \;=\; -\Phi(k)f_{k-1}(x)\,.$$

Integrating this identity with respect to the probability measure ρ, we get (recall that ρ has been defined via its entire moments)

$$\begin{aligned}\int_{]0,\infty]} \hat{G} f_k(x)\rho(dx) &= -\Phi(k)\int_{]0,\infty]} f_{k-1}(x)\rho(dx) \\ &= -\Phi(1)\cdots\Phi(k) \\ &= -\int_{]0,\infty]} f_k(x)\rho(dx)\,.\end{aligned}$$

It follows that, in standard Markovian notation,

$$\rho\hat{G} + \rho \,=\, 0\,,$$

that is precisely that ρ is a 1-invariant distribution for $\hat{X}$. □

3 Examples

A part of the statement of Proposition 1 is the important fact that $\mathbf{e} \stackrel{\mathcal{L}}{=} IR$, where on the left-hand side, $\mathbf{e}$ denotes a standard exponential variable and on the right-hand side I and R are independent. This obviously invites to look for factorizations of an exponential variable in two independent factors, and to study whether one, or each factor may be obtained (in distribution) as the exponential functional I. In this direction, we recall that Gjessing and Paulsen [9] have determined explicitly the law of the latter in several special cases.
There are at least two quite classical such factorizations. First, for every fixed $a > 0$, if $\beta_{1,a}$ denotes a beta variable with parameters 1 and a, and γ_p an independent gamma variable with parameter p, i.e.

$$\begin{aligned}\mathbb{P}\left(\beta_{1,a} \in du\right) &= a(1-u)^{a-1}du\,, \qquad 0 < u < 1\,; \\ \mathbb{P}\left(\gamma_p \in dt\right) &= t^{p-1}\mathrm{e}^{-t}\frac{dt}{\Gamma(p)}\,, \qquad t \geq 0\,,\end{aligned}$$

then it is well-known that

$$\mathbf{e} \stackrel{\mathcal{L}}{=} \beta_{1,a}\gamma_{a+1}\,. \tag{4}$$

Second, for every $\alpha \in]0,1[$, if τ_α denotes an independent α-stable variable, i.e.

$$\mathbb{E}(\exp\left(-\lambda\tau_\alpha\right)) \,=\, \exp\left(-\lambda^\alpha\right)\,, \qquad \lambda \geq 0\,,$$

then one has

$$\mathbf{e} \stackrel{\mathcal{L}}{=} \mathbf{e}^\alpha\,\tau_\alpha^{-\alpha}\,. \tag{5}$$

Recall that (4) is a particular case of the *beta-gamma algebra*

$$\gamma_p \stackrel{\mathcal{L}}{=} \beta(p,q)\gamma_{p+q}\,, \qquad p, q > 0\,, \tag{6}$$

with obvious notation and hypothesis; whereas (5), which has been discussed in particular in Shanbhag and Sreehari [15, 16], exhibits the self-decomposability of $\log \mathbf{e}$. More precisely, one has the identity

$$\log \mathbf{e} \stackrel{\mathcal{L}}{=} \alpha \log \mathbf{e} + \alpha \log 1/\tau_\alpha\,,$$

which may be explained as follows, in terms of the local time at 0, $(\ell_t, t \geq 0)$, of the Bessel process with dimension $d = 2(1-\alpha)$ (see Barlow, Pitman and Yor [1] for details): if T is an independent exponential time, one has

$$\ell_T \stackrel{\mathcal{L}}{=} T^\alpha \, \ell_1 \stackrel{\mathcal{L}}{=} T^\alpha \tau_\alpha^{-\alpha} \,,$$

as a consequence of the scaling properties of $(\ell_t, t \geq 0)$. Moreover it is a classical result that ℓ_T is exponentially distributed.
We first consider (5), using the fact that the entire moments of $\mathbf{e}^\alpha$ are given by

$$\mathbb{E}\left(\mathbf{e}^{\alpha k}\right) \;=\; \Gamma(\alpha k + 1) \,, \qquad k \in \mathbb{N} .$$

We thus see that if we take

$$\Phi(k) \;=\; \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha(k-1)+1)} \,,$$

then $\mathbb{E}\left(\mathbf{e}^{\alpha k}\right) = \Phi(1) \cdots \Phi(k)$. It is easily checked that Φ is indeed the Laplace exponent of a subordinator; more precisely straightforward calculations yield

$$\begin{aligned} \Phi(k) &= \frac{\alpha k}{\Gamma(1-\alpha)} \beta(\alpha k, 1-\alpha) \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^\infty (1 - \mathrm{e}^{-uk}) \frac{\mathrm{e}^{-u/\alpha}}{(1-\mathrm{e}^{-u/\alpha})^{\alpha+1}} \, du \,. \end{aligned}$$

In other words, Φ is the Laplace exponent of the subordinator ξ with zero drift and Lévy measure

$$\Pi(du) \;=\; \frac{\mathrm{e}^{-u/\alpha}}{\Gamma(1-\alpha)(1-\mathrm{e}^{-u/\alpha})^{\alpha+1}} \, du \,, \qquad u > 0 \,,$$

and the exponential functional corresponding to this subordinator has thus the same distribution as $\tau_\alpha^{-\alpha}$.
Alternatively, we also have

$$\mathbb{E}\left(\mathbf{e}^{\alpha k}\right) \;=\; \Gamma(\alpha k + 1) \;=\; \frac{k!}{\Phi(1) \cdots \Phi(k)}$$

where now

$$\Phi(k) \;=\; k \frac{\Gamma(\alpha(k-1)+1)}{\Gamma(\alpha k + 1)} \;=\; \frac{k}{\Gamma(\alpha)} \beta\left(\alpha(k-1)+1, \alpha\right) \,.$$

Again it is easy to check that Φ is indeed given by a Lévy-Khintchine formula; more precisely

$$\Phi(k) \;=\; \frac{(1-\alpha)^2}{\alpha \Gamma(\alpha+1)} \int_0^\infty (1 - \mathrm{e}^{-uk}) \frac{\mathrm{e}^{u/\alpha}}{(\mathrm{e}^{u/\alpha} - 1)^{2-\alpha}} \, du \,.$$

In words, the corresponding subordinator ξ has no drift and Lévy measure

$$\Pi(du) \;=\; \frac{(1-\alpha)^2 \mathrm{e}^{u/\alpha}}{\alpha \Gamma(\alpha+1)(\mathrm{e}^{u/\alpha} - 1)^{2-\alpha}} \, du \,, \qquad u > 0 \,.$$

In this direction, it is interesting to point out that this subordinator can be identified as the inverse local time at 0 of the so-called Ornstein-Uhlenbeck process with dimension $\delta = 2\alpha$ and

parameter $\mu = \frac{1}{2\alpha}$; cf. formula (16) on page 276 in Pitman and Yor [13] (see also [10, 14] for related works).
Next, we turn our attention to (4). In this direction, we first mention that Jedidi [11] pointed out recently that more generally, for every $\alpha \in]0,1]$ and $s \geq \alpha$, there is a factorization of the exponential law in the form

$$\mathbf{e} \stackrel{\mathcal{L}}{=} \gamma_s^\alpha \, J_s^{(\alpha)}, \tag{7}$$

where $J_s^{(\alpha)}$ denotes a certain random variable, independent of γ_s. One recovers (4) and (5) by specifying (7) respectively for $\alpha = 1$ and $s = 1$, so (7) unifies the preceding.
Let us now discuss the factorization (7) in the framework of this note, and more precisely, let us check that γ_s^α can be represented in the form R. The entire moments of ${\gamma_s}^\alpha$ are given by

$$\mathbb{E}\left({\gamma_s}^{\alpha k}\right) = \frac{\Gamma(\alpha k + s)}{\Gamma(s)}, \qquad k \in \mathbb{N}.$$

We may express this quantity as the product $\Phi(1)\cdots\Phi(k)$ for

$$\Phi(k) = \frac{\Gamma(\alpha k + s)}{\Gamma(\alpha(k-1)+s)},$$

and we have to check that Φ is the Laplace exponent of some subordinator. One gets

$$\Phi(k) = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty (1 - \mathrm{e}^{-uk\alpha})\, d_u \left(\frac{-\mathrm{e}^{u(1-s)}}{(\mathrm{e}^u - 1)^{1-\alpha}} \right).$$

One may verify that the function $u \to -\mathrm{e}^{u(1-s)}/(\mathrm{e}^u - 1)^{1-\alpha}$ increases by using the change of variables $x = \mathrm{e}^u$, and this yields a Lévy-Khintchine formula.
In fact, Jedidi's observation takes even a more general form which we now discuss.

Lemma 6 (i) *Let $\alpha \in]0,1]$ and $s, t > 0$ such that $t \leq s/\alpha$. Then there is the factorization*

$$\gamma_t \stackrel{\mathcal{L}}{=} \gamma_s^\alpha \; J_{s,t}^{(\alpha)}, \tag{8}$$

where $J_{s,t}^{(\alpha)}$ denotes a certain random variable which is independent of γ_s, and whose law is characterized by:

$$\mathbb{E}\left[\left(J_{s,t}^{(\alpha)}\right)^p\right] = \frac{\Gamma(t+p)\Gamma(s)}{\Gamma(t)\Gamma(s+\alpha p)}, \qquad \textit{for every } p \geq 0. \tag{9}$$

(ii) *For $t < s/\alpha$, there is the identity*

$$J_{s,t}^{(\alpha)} \stackrel{\mathcal{L}}{=} \beta(t, s/\alpha - t) \; J_{s,s/\alpha}^{(\alpha)}, \tag{10}$$

where on the right-hand side, the variables are assumed independent.
(iii) *The law of $J_{s,s/\alpha}^{(\alpha)}$ may be described as follows:*

$$\mathbb{P}\left(J_{s,s/\alpha}^{(\alpha)} \in dj\right) = \mathbb{E}\left[\frac{C_{s,\alpha}}{(\tau_\alpha)^s}; \frac{1}{(\tau_\alpha)^\alpha} \in dj\right], \tag{11}$$

with $C_{s,\alpha} = \Gamma(s+1)/\Gamma(1+s/\alpha) = \alpha\Gamma(s)/\Gamma(s/\alpha)$.

Proof: We prove the lemma in the reverse order of its statement: we take (11) as a definition of the law of $J^{(\alpha)}_{s,s/\alpha}$; then we define the law of $J^{(\alpha)}_{s,t}$ from (10); it is easily checked that (9) holds (see below), and hence also (8) by identification of moments.
We now provide some details for the proof of (9) for $t = s/\alpha$. The negative moments of τ_α are given by

$$\mathbb{E}\left(\tau_\alpha^{-m}\right) = \Gamma(1+m/\alpha)/\Gamma(1+m), \qquad m \geq 0.$$

Hence, from (11), we get

$$\begin{aligned}\mathbb{E}\left[\left(J^{(\alpha)}_{s,s/\alpha}\right)^p\right] &= C_{s,\alpha}\mathbb{E}\left(\tau_\alpha^{-(s+\alpha p)}\right) \\ &= \frac{\Gamma(s+1)\Gamma(p+1+s/\alpha)}{\Gamma(1+s/\alpha)\Gamma(s+sp+1)},\end{aligned}$$

which, using the functional equation for Γ simplifies to

$$\mathbb{E}\left[\left(J^{(\alpha)}_{s,s/\alpha}\right)^p\right] = \frac{\Gamma(s)\Gamma(p+s/\alpha)}{\Gamma(s/\alpha)\Gamma(s+sp)}.$$

This is precisely (9) in the particular case $t = s/\alpha$. □

We now consider the case $1 \leq t \leq s/\alpha$, and deduce from (8) and the beta-gamma algebra relation (6) that

$$\mathbf{e} \stackrel{\mathcal{L}}{=} \theta^{(\alpha)}_{s,t}\, J^{(\alpha)}_{s,t},$$

where, on the right-hand side, $\theta^{(\alpha)}_{s,t} = \beta(1,t-1)\gamma_s^\alpha$, and $\beta(1,t-1)$, γ_s, and $J^{(\alpha)}_{s,t}$ are independent. In the rest of this section, we shall show that $J^{(\alpha)}_{s,t}$ may be represented in the form R, i.e. there exists a Laplace exponent $\Phi = \Phi^{(\alpha)}_{s,t}$ of a subordinator such that

$$\mathbb{E}\left(\left(J^{(\alpha)}_{s,t}\right)^k\right) = \frac{\Gamma(t+k)\Gamma(s)}{\Gamma(t)\Gamma(s+\alpha k)} = \Phi(1)\cdots\Phi(k), \qquad \text{for } k = 1, \dots. \tag{12}$$

From (12), we deduce

$$\Phi(k) = (t+k-1)\frac{\Gamma(s+\alpha(k-1))}{\Gamma(s+\alpha k)} = \frac{t+k-1}{\Gamma(\alpha)} B(\alpha, (s-\alpha)+\alpha k).$$

Thus, we look for a Lévy measure $\nu(dy)$ on $]0,\infty[$ such that

$$k\int_0^\infty dy \mathrm{e}^{-ky}\overline{\nu}(y) = \frac{1}{\Gamma(\alpha)}\left((t-1)+k\right)\int_0^1 du u^{(s-\alpha)+\alpha k-1}(1-u)^{\alpha-1},$$

with the notation $\overline{\nu}(y) = \nu(]y,\infty[)$. Hence, making the changes of variables $u = \mathrm{e}^{-v}$ and then $x = \alpha v$, we obtain

$$\begin{aligned}\int_0^\infty dy \mathrm{e}^{-ky}\overline{\nu}(y) &= \frac{1}{\Gamma(\alpha)}\left(\frac{t-1}{k}+1\right)\int_0^\infty dv \mathrm{e}^{-v(s-\alpha)}\mathrm{e}^{-v\alpha k}(1-\mathrm{e}^{-v})^{\alpha-1} \\ &= \frac{1}{\Gamma(\alpha)}\left(\frac{t-1}{k}+1\right)\int_0^\infty dx\, q(x)\mathrm{e}^{-kx},\end{aligned} \tag{13}$$

with $q(x) = \alpha^{-1} e^{(1-s/\alpha)x} \left(1 - e^{-x/\alpha}\right)^{\alpha-1}$. We deduce from the equality (13) after elementary calculations that

$$\begin{aligned} d\nu(x) &= -\left((t-1)q(x) + q'(x)\right) \frac{dx}{\Gamma(\alpha)} \\ &= e^{(1-s/\alpha)x} \left(1 - e^{-x/\alpha}\right)^{\alpha-2} \left(\left(\frac{s}{\alpha} - t\right)\left(1 - e^{-x/\alpha}\right) + \left(\frac{1}{\alpha} - 1\right) e^{-x/\alpha} \right) \frac{dx}{\Gamma(\alpha+1)} . \end{aligned}$$

Remark. In some of the cases discussed above, the distribution ρ can be viewed as the law of the exponential functional associated to a different subordinator. To conclude this note, we point out that this feature holds under a fairly common hypothesis. Specifically, let $\Phi^{(1)}$ and $\Phi^{(2)}$ be two Laplace exponents of subordinators, say $\xi^{(1)}$ and $\xi^{(2)}$, and suppose that the following factorization holds:

$$\Phi^{(1)}(q)\Phi^{(2)}(q) = q\,, \qquad q \geq 0\,. \tag{14}$$

Then, we have for every $k \in \mathbb{N}$ that

$$\Phi^{(1)}(1) \cdots \Phi^{(1)}(k) = \frac{k!}{\Phi^{(2)}(1) \cdots \Phi^{(2)}(k)}\,,$$

so in the obvious notation, $\rho^{(1)}$ is the law of the exponential functional $I^{(2)} = \int_0^\infty \exp(-\xi_t^{(2)})dt$. Plainly, this occurs for instance for pairs of stable subordinators with respective indices $\alpha^{(1)} = \alpha \in]0,1[$ and $\alpha^{(2)} = 1-\alpha$. More generally, the factorization (14) arises naturally in fluctuation theory for Lévy processes, see Equation (VI.3) on page 166 in [2]. Conversely, if $\rho^{(1)}$ is the distribution of the exponential functional of some subordinator $\xi^{(2)}$, then (14) holds.

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