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FROM THE LIFSHITZ TAIL TO THE QUENCHED SURVIVAL ASYMPTOTICS IN THE TRAPPING PROBLEM

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Abstract
The survival problem for a diffusing particle moving among random traps is considered. We introduce a simple argument to derive the quenched asymptotics of the survival probability from the Lifshitz tail effect for the associated operator. In particular, the upper bound is proved in fairly general settings and is shown to be sharp in the case of the Brownian motion among Poissonian obstacles. As an application, we derive the quenched asymptotics for the Brownian motion among traps distributed according to a random perturbation of the lattice.

1 Introduction and main results

In this article, we consider a diffusing particle moving among random traps. The motion of the particle is given by a simple random walk or a Brownian motion and it is killed at a certain rate when it stays in a trap. Such a model naturally appears in chemical physics and also has some relations to the quantum physics in disordered media. We refer to the papers by Havlin and Ben-Avraham [9] and den Hollander and Weiss [4] for reviews on this model.
The mathematical description of the trapping model is given by the sub-Markov process with generator

$$H_\omega = -\kappa\Delta + V_\omega, \tag{1.1}$$

where Δ is the Laplacian on $L^2(\mathbb{R}^d)$ or $l^2(\mathbb{Z}^d)$ and $(V_\omega, \mathbb{P})$ a nonnegative, stationary, and ergodic random field. Heuristically, the height of V_ω corresponds to the rate of killing. Let us write $(\{X_t\}_{t\geq 0}, \{P_x\}_{x\in\mathbb{R}^d \text{ or } \mathbb{Z}^d})$ for the Markov process generated by $-\kappa\Delta$. A quantity of primary interest in the trapping model is the survival probability of the particle up to a fixed time t, which is expressed as

$$u_\omega(t,x) = E_x\left[\exp\left\{-\int_0^t V_\omega(X_s)\,ds\right\}\right]. \tag{1.2}$$

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From this expression, we can identify the survival probability as the Feynman-Kac representation of a solution of the initial value problem

$$\begin{aligned} &\partial_t u(t,x) = \kappa\Delta u(t,x) - V_\omega(x)u(t,x) \quad \text{for} \quad (t,x)\in(0,\infty)\times\mathbb{R}^d \text{ (or } \mathbb{Z}^d),\\ &u(0,\cdot)\equiv 1. \end{aligned} \tag{1.3}$$

Therefore, it is natural to expect that the long time asymptotics of the survival probability gives some information about the spectrum of H_ω around the ground state energy and vice versa. This idea has been made rigorous first by Fukushima [6], Nakao [12], and Pastur [13] (with the analysis of some concrete examples) in the following sense: from the annealed long time asymptotics of the survival probability, one can derive the decay rate of the integrated density of states around the ground state energy. Their arguments are based on the fact that the Laplace transform of the integrated density of states can be expressed as the annealed survival probability for the process conditioned to come back to the starting point at time t. Therefore, the above implication follows by an appropriate Tauberian theorem and, since there is the corresponding Abelian theorem (see e.g. Kasahara [10]), the converse is also true.

The aim of this article is to study a relation between the *quenched* asymptotics of $u_\omega(t,x)$ and the integrated density of states. Let us start by recalling the notion of the integrated density of states. To define it, we assume the following:

Assumption 1. In the continuous setting, V_ω belongs to the local Kato class $K_{d,\mathrm{loc}}$, that is,

$$\lim_{\epsilon\to 0}\sup_{|x|\le R}\int_{|y|\le\epsilon} g(x-y)V_\omega(y)\,dy = 0 \tag{1.4}$$

for each $R>0$, where $g(z)=1$ for $d=1$, $-\log|z|$ for $d=2$, and $|z|^{2-d}$ for $d\ge 3$.

Under the above assumption, the integrated density of states of H_ω can be defined as follows (see e.g. Chap. VI of [3]):

$$N^*(\lambda) = \lim_{R\to\infty}\frac{1}{(2R)^d}\mathbb{E}\big[\#\{k\in\mathbb{N};\lambda^*_{\omega,k}\big((-R,R)^d\big)\le\lambda\}\big], \quad *=\mathrm{D} \text{ or } \mathrm{N}, \tag{1.5}$$

where $\lambda^{\mathrm{D}}_{\omega,k}((-R,R)^d)$ (resp. $\lambda^{\mathrm{N}}_{\omega,k}((-R,R)^d)$) is the k-th smallest eigenvalue of H_ω in $(-R,R)^d$ with the Dirichlet (resp. Neumann) boundary condition. In fact, the above assumption is slightly more than necessary to ensure the existence of the integrated density of states but we need it to utilize a uniform bound for the semigroup e^{-tH_ω} in the proof.

Before stating the results, let us recall some notations and a fact about regularly varying functions from [16]. A function ϕ from $(0,\infty)$ to itself is said to be regularly varying with index $L>0$ if

$$\lim_{x\to\infty}\frac{\phi(\lambda x)}{\phi(x)} = \lambda^L \tag{1.6}$$

for each $\lambda\in(0,\infty)$. It is known that for a regularly varying function ϕ, there exists a function ψ satisfying

$$\lim_{x\to\infty}\frac{\psi\circ\phi(x)}{x} = \lim_{x\to\infty}\frac{\phi\circ\psi(x)}{x} = 1. \tag{1.7}$$

The function ψ is asymptotically unique—if ψ_1 and ψ_2 satisfy (1.7), then $\lim_{x\to\infty}\psi_1(x)/\psi_2(x) = 1$— and is called the asymptotic inverse of ϕ.

Now we state our first result.

Theorem 1.1. *Suppose that Assumption 1 holds and that there exists a regularly varying function ϕ with index $L > 0$ such that the integrated density of states N^{D} associated with the operator H_ω in* (1.1) *admits the upper bound*

$$N^{\mathrm{D}}(\lambda) \le \exp\left\{-\phi(1/\lambda)(1+o(1))\right\} \quad \text{as} \quad \lambda \to 0. \tag{1.8}$$

Then, for any fixed $x \in \mathbb{R}^d$ (or $\mathbb{Z}^d$),

$$\mathbb{P}\text{-a.s.} \quad u_\omega(t,x) \le \exp\left\{-t/\psi(d\log t)(1+o(1))\right\} \quad \text{as} \quad t \to \infty, \tag{1.9}$$

where ψ is the asymptotic inverse of ϕ.

The following assumptions are necessary only for the lower bound.

Assumption 2. (*Moment condition*) There exists $\alpha > 0$ such that

$$\mathbb{E}\left[\sup_{x\in[0,1)^d} \exp\{V_\omega(x)^\alpha\}\right] < \infty \tag{1.10}$$

in the continuous setting. In the discrete setting, the left-hand side is interpreted as $\mathbb{E}[\exp\{V_\omega(0)^\alpha\}]$.

Assumption 3. (*Short range correlation*) There exists $\beta > 0$ and $r_0 > 0$ such that for $\lambda > 0$ and boxes $A_k \subset \mathbb{R}^d$ or $\mathbb{Z}^d$ $(1 \le k \le n)$ with $\min_{k\neq l} \operatorname{dist}(A_k, A_l) > r \ge r_0$ and $\max_{1\le k\le n} \operatorname{diam}(A_k) < r$,

$$\left|\mathbb{P}\left(\bigcap_{1\le k\le n} E_k(\lambda)\right) - \mathbb{P}(E_1(\lambda))\mathbb{P}\left(\bigcap_{2\le k\le n} E_k(\lambda)\right)\right| < \exp\{-r^\beta\}, \tag{1.11}$$

where $E_k(\lambda) = \{\lambda^{\mathrm{N}}_{\omega,1}(A_k) \le \lambda\}$.

Now we are ready to state our second result.

Theorem 1.2. *Suppose that Assumptions 1– 3 hold and that there exists a regularly varying function ϕ with index $L > 0$ such that the integrated density of states N^{D} associated with the operator H_ω in* (1.1) *admits the lower bound*

$$N^{\mathrm{D}}(\lambda) \ge \exp\left\{-\phi(1/\lambda)(1+o(1))\right\} \quad \text{as} \quad \lambda \to 0. \tag{1.12}$$

Then, there exists a constant $c_1 > 1$ such that for any fixed $x \in \mathbb{R}^d$ (or $\mathbb{Z}^d$),

$$\mathbb{P}\text{-a.s.} \quad u_\omega(t,x) \ge \exp\left\{-c_1 t/\psi(d\log t)(1+o(1))\right\} \quad \text{as} \quad t \to \infty, \tag{1.13}$$

where ψ is the asymptotic inverse of ϕ.

Remark 1. The exponential behavior (1.8) and (1.12) of the integrated density of states is called the “Lifshitz tail effect” (cf. [11]) and is typical for the trapping Hamiltonian H_ω. The index L is called “Lifshitz exponent”. Using these terminologies, we can summarize our results as follows: if we have the Lifshitz tail effect with exponent $L > 0$, then $\log u_\omega(t,x)$ behaves like $-t/(\log t)^{1/L+o(1)}$.

In Section 4.2, we shall use the above general theorems to derive another new result. That is, the quenched asymptotics of the survival probability of the Brownian motion among traps distributed according to a randomly perturbed lattice. This model has recently been introduced by the author in [7], where the annealed asymptotics was discussed.

Finally we comment on the relation to early studies on the quenched asymptotics of $u_\omega(t,x)$. We first give historical remarks. The first result in this direction has been obtained for the Brownian motion among Poissonian traps by Sznitman [19] (see also [20]):

$$\mathbb{P}\text{-a.s.}\quad u_\omega(t,0) = \exp\left\{-ct/(\log t)^{2/d}(1+o(1))\right\} \quad \text{as} \quad t\to\infty, \tag{1.14}$$

with an explicit constant $c > 0$. The same asymptotics has also been proved for the discrete counterpart (the simple random walk among Bernoulli traps) by Antal [1]. These results are consistent to ours since in these cases, the Lifshitz exponent is known to be $d/2$ [12, 15]. Later, Biskup and König [2] considered the simple random walk among i.i.d. traps with more general distributions. A representative example in their framework is

$$\mathbb{P}(V_\omega(0) < v) \sim \exp\left\{-v^{-\gamma}\right\} \quad \text{as} \quad v\to 0 \tag{1.15}$$

for some $\gamma\in(0,\infty)$. For such a model, they proved the quenched asymptotics

$$\mathbb{P}\text{-a.s.}\quad u_\omega(t,0) = \exp\left\{-\chi t/(\log t)^{2/(d+2\gamma)}(1+o(1))\right\} \quad \text{as} \quad t\to\infty \tag{1.16}$$

with a constant $\chi > 0$ described by a certain variational problem. It is remarkable that they also discussed the annealed asymptotics and as a consequence, the Lifshitz tail effect with the Lifshitz exponent $(d+2\gamma)/2$ was proved. Hence the relation we mentioned in Remark 1 has already appeared in this special class.

Next, we comment on some technical points. The lower bound (Theorem 1.2) is a slight modification of that of Theorem 4.5.1 in p.196 of [20] and not genuinely new. We include it for the completeness and to use in an application given in Section 4.2. On the other hand, the upper bound (Theorem 1.1) contains some novelties. Besides the generality of the statement, our proof simplifies an existing argument. To be precise, in [2], the upper bound of quenched asymptotics is derived essentially from the annealed one. This is in the same spirit of ours since the annealed asymptotics and the Lifshitz tail effect have a direct relationship as mentioned before. However, they need a certain *localizing* procedure (see Lemma 4.6 in [2]) which we do not need. Such a *localizing* argument is also used, and in fact crucial, in the proof of the annealed asymptotics but we find that it is not necessary in the step from the annealed asymptotics to the quenched one. The arguments in [19, 1] on the other hand rely on the so-called "method of enlargement of obstacles". They have an advantage of avoiding any use of annealed results but they are quite complicated themselves. We will see in Section 4.1 that, assuming the Lifshitz tail effect in [12], our result indeed derives the correct upper bound of the quenched asymptotics for the Brownian motion among Poissonian obstacles.

2 Proof of the upper bound

We take $\kappa = 1/2$ and $x = 0$ in the proof. The extension to general κ and x are verbatim. Also, we give the proof only for the continuous setting. The proof of the discrete case follows by the same argument. We begin with the following general upper bound for $u_\omega(t,x)$ in terms of the principal eigenvalue.

Lemma 2.1. *Under Assumption 1, there exist constants $c_2, c_3 > 0$ such that*

$$u_\omega(t,0) \le c_2(1+(\lambda^{\mathrm{D}}_{\omega,1}((-t,t)^d)t)^{d/2})\exp\left\{-\lambda^{\mathrm{D}}_{\omega,1}((-t,t)^d)t\right\} + e^{-c_3 t}. \tag{2.1}$$

Proof. Let τ denote the exit time of the process from $(-t,t)^d$. Then, by the reflection principle, we have

$$\begin{aligned}u_\omega(t,0) &\le E_0\left[\exp\left\{-\int_0^t V_\omega(X_s)\,ds\right\};\tau>t\right]+P_0(\tau\le t)\\&\le E_0\left[\exp\left\{-\int_0^t V_\omega(X_s)\,ds\right\};\tau>t\right]+e^{-c_3 t}.\end{aligned}\tag{2.2}$$

Now, (2.1) follows immediately from (3.1.9) in p.93 of [20] under Assumption 1. □

Due to this lemma, it suffices for (1.9) to obtain the almost sure lower bound for the principal eigenvalue $\lambda^{\mathrm{D}}_{\omega,1}\big((-t,t)^d\big)$. We use the following inequality for the integrated density of states

$$\begin{aligned}N^{\mathrm{D}}(\lambda) &\ge \frac{1}{(2R)^d}\mathbb{E}\left[\#\{k\in\mathbb{N};\lambda^{\mathrm{D}}_{\omega,k}\big((-R,R)^d\big)\le\lambda\}\right]\\&\ge \frac{1}{(2R)^d}\mathbb{P}\big(\lambda^{\mathrm{D}}_{\omega,1}\big((-R,R)^d\big)\le\lambda\big),\end{aligned}\tag{2.3}$$

which holds for any $\lambda>0$ and $R>0$. The first inequality is an easy application of the so-called "Dirichlet–Neumann bracketing" and can be found in [3], (VI.15) in p.311. Now, fix $\epsilon>0$ arbitrarily and let $\lambda=(1-\epsilon)\psi(d\log t)^{-1}$ and $R=t$. Then it follows from (2.3) and (1.8) that

$$\begin{aligned}&\mathbb{P}\left(\lambda^{\mathrm{D}}_{\omega,1}\big((-t,t)^d\big)\le(1-\epsilon)\psi(d\log t)^{-1}\right)\\&\quad\le(2t)^d\exp\left\{-\phi((1-\epsilon)^{-1}\psi(d\log t))(1+o(1))\right\}\\&\quad=2^d t^{d-d/(1-\epsilon)^L(1+o(1))}\\&\quad\le t^{-\delta(\epsilon)}\end{aligned}\tag{2.4}$$

for some $\delta(\epsilon)>0$ when t is sufficiently large. This right-hand side is summable along the sequence $t_k=e^k$ and therefore Borel-Cantelli's lemma shows

$$\lambda^{\mathrm{D}}_{\omega,1}\big((-t_k,t_k)^d\big)\ge(1-\epsilon)\psi(d\log t_k)^{-1}\tag{2.5}$$

except for finitely many k, $\mathbb{P}$-almost surely. We can extend this bound for all large t as follows: since $\psi(d\log t)$ is slowly varying in t, we have

$$\begin{aligned}\lambda^{\mathrm{D}}_{\omega,1}\big((-t,t)^d\big)&\ge\lambda^{\mathrm{D}}_{\omega,1}\big((-t_k,t_k)^d\big)\\&\ge(1-\epsilon)\psi(d\log t_k)^{-1}\\&\ge(1-2\epsilon)\psi(d\log t)^{-1}\end{aligned}\tag{2.6}$$

for $t_{k-1}\le t\le t_k$ when k is sufficiently large. Combined with Lemma 2.1, this proves the upper bound (1.9).

3 Proof of the Lower bound

We take $\kappa=1/2$ and $x=0$ again. Also, we only consider the continuous case. As in the proof of the upper bound, the principal eigenvalue plays a key role. Let us write $\lambda^{\mathrm{N}}_k(U)$ for the k-th smallest eigenvalue of $-(1/2)\Delta$ in U with the Neumann boundary condition. Then we have another inequality for the integrated density of states

$$\begin{aligned}
N^{\mathrm{D}}(\lambda) &\le \frac{1}{(2R)^d}\mathbb{E}\left[\#\{k\in\mathbb{N};\lambda^{\mathrm{N}}_{\omega,k}\big((-R,R)^d\big)\le\lambda\}\right]\\
&\le \frac{1}{(2R)^d}\#\{k\in\mathbb{N};\lambda^{\mathrm{N}}_{k}((-R,R)^d)\le\lambda\}\mathbb{P}\big(\lambda^{\mathrm{N}}_{\omega,1}\big((-R,R)^d\big)\le\lambda\big)\\
&\le c_4\mathbb{P}\big(\lambda^{\mathrm{N}}_{\omega,1}\big((-R,R)^d\big)\le\lambda\big),
\end{aligned} \tag{3.1}$$

which holds for any $\lambda\in(0,1)$ and $R>0$. The first inequality can be found in [3] again, (VI.16) in p. 331, the second one follows from $\lambda^{\mathrm{N}}_k\le\lambda^{\mathrm{N}}_{k,\omega}$, and the third one is a consequence of the classical Weyl asymptotics for the free Laplacian, see e.g. Proposition 2 in Section XIII.15 of [14]. For arbitrary $\epsilon>0$, let $\lambda=(1+\epsilon)\psi(d\log t)^{-1}$. Then, using (3.1) and (1.12), we find

$$\begin{aligned}
\mathbb{P}\big(\lambda^{\mathrm{N}}_{\omega,1}\big((-R,R)^d\big)>\lambda\big) &\le 1-c_4^{-1}N^{\mathrm{D}}((1+\epsilon)\psi(d\log t)^{-1})\\
&\le 1-c_4^{-1}(2t)^{-d/(1+\epsilon)^L(1+o(1))}\\
&\le 1-t^{-d+\delta(\epsilon)}
\end{aligned} \tag{3.2}$$

for some $\delta(\epsilon)>0$ when t is sufficiently large.
Now we introduce some notations to proceed the proof. Let us fix a positive number

$$M>\frac{1}{\alpha}+\frac{2}{\beta}+\frac{1}{L} \tag{3.3}$$

and define

$$\mathscr{I}=\big(-t/(\log t)^M,t/(\log t)^M\big)^d\cap(\log t)^M\mathbb{Z}^d, \tag{3.4}$$

$$C_i=i+\big(0,\psi(d\log t)^{1/2}\big)^d\quad(i\in\mathscr{I}). \tag{3.5}$$

Note that $\min_{i\neq j}\mathrm{d}(C_i,C_j)>\mathrm{diam}(C_i)$ and both of them go to infinity as $t\to\infty$. Therefore, by using (3.2) and Assumption 3 recursively, we obtain

$$\begin{aligned}
&\mathbb{P}\big(\lambda^{\mathrm{N}}_{\omega,1}(C_i)>(1+\epsilon)\psi(d\log t)^{-1}\text{ for all }i\in\mathscr{I}\big)\\
&\quad\le\prod_{i\in\mathscr{I}}\mathbb{P}\big(\lambda^{\mathrm{N}}_{\omega,1}(C_i)>(1+\epsilon)\psi(d\log t)^{-1}\big)+\exp\{-(\log t)^2\}\\
&\quad\le(1-t^{-d+\delta(\epsilon)})^{t^d(\log t)^{-2dM}}+\exp\{-(\log t)^2\}\\
&\quad\le\exp\{-t^{\delta(\epsilon)}(\log t)^{-2dM}\}+\exp\{-(\log t)^2\}
\end{aligned} \tag{3.6}$$

for sufficiently large t. Since the right hand side is summable in $t\in\mathbb{N}$, Borel-Cantelli's lemma tells us that $\mathbb{P}$-almost surely,

$$\text{there exists } i\in\mathscr{I}\text{ such that }\lambda^{\mathrm{N}}_{\omega,1}(C_i)\le(1+\epsilon)\psi(d\log t)^{-1} \tag{3.7}$$

for all large $t\in\mathbb{N}$. The next lemma translates (3.7) to an upper bound for the Dirichlet eigenvalue:

Lemma 3.1. *There exists a constant $c_1>1$ such that $\mathbb{P}$-almost surely,*

$$\textit{there exists } i\in\mathscr{I}\textit{ such that }\lambda^{\mathrm{D}}_{\omega,1}(C_i)\le c_1\psi(d\log t)^{-1} \tag{3.8}$$

for all large t.

Proof. We choose C_i $(i \in \mathscr{I})$ for which $\lambda^{\mathrm{N}}_{\omega,1}(C_i) \le (1+\epsilon)\psi(d\log t)^{-1}$. This is possible for large $t \in \mathbb{N}$ by (3.7) and then it also holds for all large t with slightly larger ϵ by regularly varying property of ψ. Let ϕ_i^{N} denote the L^2-normalized nonnegative eigenfunction corresponding to $\lambda^{\mathrm{N}}_{\omega,1}(C_i)$ and $\partial_\epsilon C_i$ $(i \in \mathscr{I})$ the set

$$\{x \in C_i; \mathrm{d}(x, \partial C_i) < \epsilon\psi(d\log t)^{1/2}\}. \tag{3.9}$$

We further take a nonnegative function $\rho \in C^1_c(C_i)$ which satisfies

$$\rho = 1 \text{ on } C_i \setminus \partial_\epsilon C_i \quad \text{and} \quad \|\nabla\rho\|_\infty < 2\epsilon^{-1}\psi(d\log t)^{-1/2}. \tag{3.10}$$

Such a function can easily be constructed by a standard argument using mollifier. Substituting $\rho\phi_i^{\mathrm{N}}$ to the variational formula for the principal eigenvalue, we obtain

$$\lambda^{\mathrm{D}}_{\omega,1}(C_i) \le \frac{1}{\|\rho\phi_i^{\mathrm{N}}\|_2^2}\int_{C_i} |\nabla(\rho\phi_i^{\mathrm{N}})|^2(x) + V_\omega(x)(\rho\phi_i^{\mathrm{N}})^2(x)\,dx. \tag{3.11}$$

To bound the right hand side, we first use the uniform bound on eigenfunctions $\|\phi_i^{\mathrm{N}}\|_\infty \le c_5\lambda^{\mathrm{N}}_{\omega,1}(C_i)^{d/4}$ (see e.g. (3.1.55) in p.107 of [20]) to see

$$\|\rho\phi_i^{\mathrm{N}}\|_2^2 \ge \int_{C_i\setminus\partial_\epsilon C_i} \phi_i^{\mathrm{N}}(x)^2\,dx \ge 1 - c_6\epsilon. \tag{3.12}$$

Next, it is clear from (3.10) and the above uniform bound that

$$\begin{aligned}
&\int_{C_i} |\nabla(\rho\phi_i^{\mathrm{N}})|^2(x) + V_\omega(x)(\rho\phi_i^{\mathrm{N}})^2(x)\,dx\\
&\quad\le 2\int_{C_i} |\nabla\phi_i^{\mathrm{N}}|^2(x) + V_\omega(x)\phi_i^{\mathrm{N}}(x)^2\,dx + 2\int_{C_i} |\nabla\rho|^2(x)\phi_i^{\mathrm{N}}(x)^2\,dx\\
&\quad\le (2 + 8c_6\epsilon^{-1})\psi(d\log t)^{-1}.
\end{aligned} \tag{3.13}$$

Taking $\epsilon = (2c_6)^{-1}$ and plugging these bounds into (3.11), the result follows. □

We also need the following almost sure upper bound.

Lemma 3.2. *Under Assumption 2, we have $\mathbb{P}$-almost surely,*

$$\sup_{x\in(-t,t)^d} V_\omega(x) \le (3d\log t)^{1/\alpha} \tag{3.14}$$

for sufficiently large t.

Proof. By Chebyshev's inequality,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{x\in(-2t,2t)^d} V_\omega(x) > (3d\log t)^{1/\alpha}\right)\\
&\quad\le (4t)^d\,\mathbb{P}\left(\sup_{x\in[0,1)^d} V_\omega(x) > (3d\log t)^{1/\alpha}\right)\\
&\quad\le 4^d t^{-2d}\,\mathbb{E}\left[\sup_{x\in[0,1)^d} \exp\{V_\omega(x)^\alpha\}\right].
\end{aligned} \tag{3.15}$$

Since the last expression is summable in $t \in \mathbb{N}$, the claim follows by Borel-Cantelli's lemma and monotonicity of $\sup_{x\in(-t,t)^d} V_\omega(x)$ in t. □

Now, we can finish the proof of the lower bound. We pick ω for which Lemma 3.1 and Lemma 3.2 holds. Then we can find a box C_i $(i \in \mathscr{I})$ satisfying

$$\lambda_{\omega,1}^{\mathrm{D}}(C_i) \le c_1 \psi(d \log t)^{-1} \tag{3.16}$$

for sufficiently large t. Let ϕ_i^{D} denote L^2-normalized nonnegative eigenfunction associated with $\lambda_{\omega,1}^{\mathrm{D}}(C_i)$. It is easy to see that there exists a box $q + [0,1]^d \subset C_i$ $(q \in \mathbb{Z}^d)$ such that

$$\|\phi_i^{\mathrm{D}}\|_\infty \int_{q+[0,1]^d} \phi_i^{\mathrm{D}}(x) dx \ge \int_{q+[0,1]^d} \phi_i^{\mathrm{D}}(x)^2 dx \ge \frac{1}{2} \psi(d \log t)^{-d}. \tag{3.17}$$

We also know the following uniform upper bound:

$$\|\phi_i^{\mathrm{D}}\|_\infty \le c_5 \lambda_{\omega,1}^{\mathrm{D}}(C_i)^{d/4} \tag{3.18}$$

from (3.1.55) in [20]. Let us recall that the semigroup generated by H_ω has the kernel $p_\omega(s,x,y)$ under Assumption 1 (see Theorem B.7.1 in [17]). We can bound this kernel from below by using the Dirichlet heat kernel $p_{(-t,t)^d}(s,x,y)$ in $(-t,t)^d$ as follows:

$$\begin{aligned} p_\omega(s,0,y) &\ge \exp\Big\{-s \sup_{x \in (-t,t)^d} V_\omega(x)\Big\} p_{(-t,t)^d}(s,0,y) \\ &\ge c_7 s^{-d/2} \exp\{-s(3d \log t)^{1/\alpha} - c_8 |y|^2/s\} \quad \text{if} \quad |y| < t/2, \end{aligned} \tag{3.19}$$

where the second inequality follows by Lemma 3.2 and a Gaussian lower bound for the Dirichlet heat kernel in [21]. Taking $s = t/(\log t)^M$ and noting that $|q| < 2s$, we arrive at

$$\inf_{y \in q+[0,1]^d} p_\omega(s,0,y) \ge \exp\{-c_8 s/2\} \tag{3.20}$$

for sufficiently large t.

Plugging (3.16)–(3.20) into an obvious inequality, we arrive at

$$\begin{aligned} u_\omega(t,0) &= \int_{\mathbb{R}^d} p_\omega(t,0,x) dx \\ &\ge \int_{\mathbb{R}^d} \int_{q+[0,1]^d} p_\omega(s,0,y) p_\omega(t-s,y,x) \frac{\phi_i^{\mathrm{D}}(x)}{\|\phi_i^{\mathrm{D}}\|_\infty} dy dx \\ &\ge \frac{1}{\|\phi_i^{\mathrm{D}}\|_\infty} \exp\{-\lambda_{\omega,1}^{\mathrm{D}}(C_i)t - c_8 s/2\} \int_{q+[0,1]^d} \phi_i^{\mathrm{D}}(x) dx \\ &\ge c_9 \psi(\log t)^{-3d/2} \exp\{-c_1 t/\psi(d \log t) - c_8 s/2\}, \end{aligned} \tag{3.21}$$

where in the third line, we have replaced p_ω by the kernel of the semigroup generated by H_ω with the Dirichlet boundary condition outside C_i. This completes the proof of the lower bound of Theorem 1.2 since $s = t/(\log t)^M$ was chosen to be $o(t/\psi(\log t))$.

4 Examples

We apply our results to two models in this section. The first is the Brownian motion among Poissonian obstacles, where we see that our result recovers the correct upper bound. The second is the Brownian motion among perturbed lattice traps introduced in [7], for which the quenched result is new.

4.1 Poissonian obstacles

Let us consider the standard Brownian motion ($\kappa = 1/2$) killed by the random potential of the form

$$V_\omega(x) = \sum_i W(x - \omega_i), \tag{4.1}$$

where $(\omega = \sum_i \delta_{\omega_i}, \mathbb{P}_\nu)$ is a Poisson point process with intensity $\nu > 0$ and W is a nonnegative, bounded, and compactly supported function. As is mentioned in Section 1, Sznitman proved in [19] the quenched asymptotics for this model:

$$\mathbb{P}_\nu\text{-a.s.}\quad u_\omega(t,0) = \exp\left\{-c(d,\nu)t/(\log t)^{2/d}(1+o(1))\right\} \quad \text{as} \quad t \to \infty, \tag{4.2}$$

where $c(d,\nu) = \lambda_d(\nu\omega_d/d)^{2/d}$ with λ_d denoting the principal Dirichlet eigenvalue of $-1/2\Delta$ in $B(0,1)$ and $\omega_d = |B(0,1)|$.
We can recover the upper bound by using classical Donsker-Varadhan's result [5] and Theorem 1.1. Indeed, the above potential clearly satisfies Assumption 1 and the asymptotics of the integrated density of states

$$\log N^{\mathrm{D}}(\lambda) \sim -\nu\omega_d \lambda_d^{d/2} \lambda^{-d/2} \quad \text{as} \quad \lambda \to 0 \tag{4.3}$$

has been derived by Nakao [12] by applying an exponential Tauberian theorem to Donsker-Varadhan's asymptotics

$$\mathbb{E}[u_\omega(t,0)] = \exp\left\{-\tilde{c}(d,\nu)t^{\frac{d}{d+2}}(1+o(1))\right\} \quad \text{as} \quad t \to \infty \tag{4.4}$$

with

$$\tilde{c}(d,\nu) = \frac{d+2}{2}(\nu\omega_d)^{\frac{2}{d+2}}\left(\frac{2\lambda_d}{d}\right)^{\frac{d}{d+2}}. \tag{4.5}$$

Now an easy computation shows that the asymptotic inverse of the right hand side of (4.3) is

$$\psi(\lambda) = \lambda_d^{-1}(\nu\omega_d)^{-2/d}\lambda^{2/d} \tag{4.6}$$

and then Theorem 1.1 proves the upper bound in (4.2).

Remark 2. In this case, the lower bound given by Theorem 1.2 is not sharp as is obvious from the statement. (In the proof, we lose the precision in Lemma 3.1.) However, the lower bound can be complemented by a rather direct and simple argument in the Poissonian soft obstacles case, see e.g. [19]. So our argument replaces the harder part.

4.2 Perturbed lattice traps

In this subsection, we use our results to derive the quenched asymptotics for the model introduced in [7]. We consider the standard Brownian motion ($\kappa = 1/2$) killed by the potential of the form

$$V_\omega(x) = \sum_{q\in\mathbb{Z}^d} W(x - q - \omega_q), \tag{4.7}$$

where $(\{\omega_q\}_{q\in\mathbb{Z}^d}, \mathbb{P}_\theta)$ ($\theta > 0$) is a collection of independent and identically distributed random vectors with density

$$\mathbb{P}_\theta(\omega_q \in dx) = N(d,\theta)\exp\{-|x|^\theta\}dx \tag{4.8}$$

and W is a nonnegative, bounded, and compactly supported function. The author has derived the annealed asymptotics for this model in [7] and also proved the following Lifshitz tail effect as a corollary:

$$\log N^{\mathrm{D}}(\lambda) \asymp_{\lambda\to 0} \begin{cases} -\lambda^{-1-\frac{\theta}{2}}\left(\log\frac{1}{\lambda}\right)^{-\frac{\theta}{2}} & (d=2), \\ -\lambda^{-\frac{d}{2}-\frac{\theta}{d}} & (d\geq 3), \end{cases}$$

where $f(x) \asymp_{x\to *} g(x)$ means $0 < \liminf_{x\to *} f(x)/g(x) \leq \limsup_{x\to *} f(x)/g(x) < \infty$.
We can prove the quenched asymptotics from this result.

Theorem 4.1. *For any $\theta > 0$ and $x \in \mathbb{R}^d$, we have*

$$\log u_\xi(t,x) \asymp_{t\to\infty} \begin{cases} -t\,(\log t)^{-\frac{2}{2+\theta}}(\log\log t)^{-\frac{\theta}{2+\theta}} & (d=2), \\ -t\,(\log t)^{-\frac{2d}{d^2+2\theta}} & (d\geq 3), \end{cases} \tag{4.9}$$

with $\mathbb{P}_\theta$-probability one.

Proof. The Assumption 1 is clearly satisfied since V_ω is locally bounded almost surely. Hence the upper bound readily follows by computing the asymptotic inverse of (4.9) and using Theorem 1.1. To use Theorem 1.2, we have to verify Assumptions 2 and 3. The former is rather easy and can be found in Lemma 11 in [8]. The latter is verified as follows: we first fix $r_0 > 0$ sufficiently large so that $\operatorname{supp} W \subset B(0, r_0/4)$. For $r > r_0$ and boxes $\{A_k\}_{1\leq k\leq n}$ as in Assumption 3, let us define events

$$E_1 \stackrel{\text{def}}{=} \{\text{for all } q \in \mathbb{Z}^d \text{ with } \mathrm{d}(q,A_1) \leq r/2, \mathrm{d}(q+\omega_q, A_1) \leq 3r/4\}, \tag{4.10}$$

$$E_2 \stackrel{\text{def}}{=} \{\text{for all } q \in \mathbb{Z}^d \text{ with } \mathrm{d}(q,A_1) \geq r/2, \mathrm{d}(q+\omega_q, A_1) \geq r/4\}. \tag{4.11}$$

Then, $\lambda^{\mathrm{N}}_{\omega,1}(A_1)$ and $\{\lambda^{\mathrm{N}}_{\omega,1}(A_k)\}_{2\leq k\leq n}$ are mutually independent on $E_1 \cap E_2$ thanks to our choice of r_0. Therefore, the left hand side of (1.11) is bounded by $\mathbb{P}_\theta(E_1^c) + \mathbb{P}_\theta(E_2^c)$. Let us denote the s-neighborhood of A_1 by $N_s(A_1)$. The first term is estimated as

$$\begin{aligned} \mathbb{P}_\theta(E_1^c) &\leq \mathbb{P}_\theta\left(|\omega_q| \geq r/4 \text{ for some } q \in \mathbb{Z}^d \cap N_{r/2}(A_1)\right) \\ &\leq N(d,\theta)\#\{q \in \mathbb{Z}^d \cap N_{r/2}(A_1)\} \int_{|x|\geq r/4} \exp\{-|x|^\theta\}dx \\ &\leq N(d,\theta) r^d \exp\{-(r/8)^\theta\} \end{aligned} \tag{4.12}$$

for large r, where we have used $\mathrm{diam}(A_1) < r$ in the last line. Next, we bound the second term $\mathbb{P}_\theta(E_2^c)$. Using the distribution of ω_q, we have

$$\begin{aligned} \mathbb{P}_\theta(E_2^c) &= \mathbb{P}_\theta\left(q+\omega_q \in N_{r/4}(A_1) \text{ for some } q \in \mathbb{Z}^d \setminus N_{r/2}(A_1)\right) \\ &\leq N(d,\theta) \sum_{q\in\mathbb{Z}^d\setminus N_{r/2}(A_1)} \int_{N_{r/4}(A_1)} \exp\{-|x-q|^\theta\}dx \\ &\leq N(d,\theta) r^d \sum_{q\in\mathbb{Z}^d\setminus N_{r/2}(A_1)} \exp\{-\,\mathrm{d}(q, N_{r/4}(A_1))^\theta\}. \end{aligned} \tag{4.13}$$

We can assume by shift invariance that A_1 is centered at the origin. We divide the sum into two parts $\{|q| \le r\}$ and $\{|q| > r\}$. The former part of the sum is bounded by

$$\begin{aligned} &\#\{q \in \mathbb{Z}^d \cap B(0,r)\} \sup_{q \in \mathbb{Z}^d \setminus N_{r/2}(A_1)} \exp\{-\mathrm{d}(q, N_{r/4}(A_1))^\theta\} \\ &\quad \le c_{10} r^d \exp\{-(r/4)^\theta\}. \end{aligned} \tag{4.14}$$

For the latter part, we use the fact that $N_{r/4}(A_1) \subset B(0, 3r/4)$, which follows from the assumption $\mathrm{diam}(A_1) < r$. By using this fact, we find

$$\mathrm{d}(q, N_{r/4}(A_1)) \ge |q| - 3r/4 > |q|/4 \quad \text{for} \quad |q| > r \tag{4.15}$$

and therefore

$$\sum_{q \in \mathbb{Z}^d \setminus N_{r/2}(A_1),\, |q| > r} \exp\{-\mathrm{d}(q, N_{r/4}(A_1))^\theta\} \le \sum_{q \in \mathbb{Z}^d,\, |q| > r} \exp\{-|q/4|^\theta\}. \tag{4.16}$$

It is not difficult to see that this right hand side is bounded by $\exp\{-(r/8)^\theta\}$ for sufficiently large r. Combining all the estimates, we arrive at

$$\mathbb{P}_\theta(E_1^c) + \mathbb{P}_\theta(E_2^c) \le N(d,\theta) r^d \left(2 + c_{10} r^d\right) \exp\{-(r/8)^\theta\} \tag{4.17}$$

for large r, which verifies Assumption 3. □

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