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AN EXTREME-VALUE ANALYSIS OF THE LIL FOR BROWNIAN MOTION

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Abstract
We use excursion theory and the ergodic theorem to present an extreme-value analysis of the classical law of the iterated logarithm (LIL) for Brownian motion. A simplified version of our method also proves, in a paragraph, the classical theorem of Darling and Erdős (1956).

1 Introduction

Let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion. The law of the iterated logarithm (LIL) of Khintchine (1933) states that $\limsup_{t\to\infty}(2t\ln\ln t)^{-1/2}B(t)=1$ a.s. Equivalently,

$$\text{With probability one,}\quad \sup_{s\geq t}\frac{B(s)}{\sqrt{2s\ln\ln s}}\to 1 \quad \text{as } t\to\infty\,. \tag{1.1}$$

The goal of this note is to determine the rate at which this convergence occurs.
We consider the extreme-value distribution function (Resnick, 1987, p. 38),

$$\Lambda(x) := \exp\left(-e^{-x}\right) \qquad {}^{\forall}x\in\mathbf{R}. \tag{1.2}$$

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Also, let $Lx := L(x) := \ln(x)$, and define, iteratively, $L_{k+1}x := L_{k+1}(x) := L(L_k x)$ for $k \geq 1$. Then, our main result can be described as follows:

Theorem 1.1. *For all* $x \in \mathbf{R}$,

$$\lim_{t\to\infty} \mathrm{P}\left\{2L_2 t\left(\sup_{s\geq t} \frac{B(s)}{\sqrt{2sL_2 s}} - 1\right) - \frac{3}{2}L_3 t + L_4 t + L\left(\frac{3}{\sqrt{2}}\right) \leq x\right\} = \Lambda(x), \tag{1.3}$$

$$\lim_{t\to\infty} \mathrm{P}\left\{2L_2 t\left(\sup_{s\geq t} \frac{|B(s)|}{\sqrt{2sL_2 s}} - 1\right) - \frac{3}{2}L_3 t + L_4 t + L\left(\frac{3}{2\sqrt{2}}\right) \leq x\right\} = \Lambda(x). \tag{1.4}$$

An anonymous referee has kindly pointed out that our Theorem 1.1 is similar to the classical result of Darling and Erdős (1956). We will show that, in a sense, this is so: While the Darling–Erdős theorem does not seem to imply Theorem 1.1. a simplified version of our proof of Theorem 1.1 also proves the Darling–Erdős theorem. See Section 5 below for details.
Theorem 1.1 is accompanied by the following strong law:

Theorem 1.2. *With probability one,*

$$\lim_{t\to\infty} \frac{L_2 t}{L_3 t}\left(\sup_{s\geq t} \frac{B(s)}{\sqrt{2sL_2 s}} - 1\right) = \frac{3}{4}. \tag{1.5}$$

This should be compared with the following consequence of the theorem of Erdős (1942):

$$\limsup_{t\to\infty} \frac{L_2 t}{L_3 t}\left(\sup_{s\geq t} \frac{B(s)}{\sqrt{2sL_2 s}} - 1\right) = \frac{3}{4} \qquad \text{a.s.} \tag{1.6}$$

[Erdős's theorem is stated for Bernoulli walks, but applies equally well—and for precisely the same reasons—to Brownian motion. For the most general result along these lines see Feller (1946); see also Einmahl (1989) where a gap in Feller's proof is bridged.]
Theorem 1.1 is derived by analyzing the excursions of the Ornstein–Uhlenbeck process,

$$X(t) = e^{-t/2} B(e^t) \qquad t \geq 0. \tag{1.7}$$

Our method is influenced by the ideas of Motoo (1959), although it has some new features as well. Motoo's method has been used also in other similar contexts as well. See, for instance, the works of Anderson (1970), Berman (1964; 1986; 1988), Bertoin (1998), Breiman (1968), Rootzén (1988), and Serfozo (1980). For other results related to the general theme of this paper see Fill (1983), Sen and Wichura (1984), and Wichura (1973).

Acknowledgement. An anonymous referee kindly suggested that we consider the connection to the Darling–Erdős theorem. He/she also pointed out the reference Einmahl (1989). These remarks have improved the presentation of the paper, and put it in more proper historical context. We thank this referee heartily.

2 Proof of Theorem 1.1

An application of Itô's formula shows us that the process X satisfies the s.d.e.,

$$X(t) = X(0) + \int_1^{\exp(t)} \frac{1}{\sqrt{s}}\, dB(s) - \frac{1}{2}\int_0^t X(s)\, ds. \tag{2.1}$$

The stochastic integral in (2.1) has quadratic variation $\int_1^{\exp(t)} s^{-1}\, ds = t$. Therefore, this stochastic integral defines a Brownian motion. Call the said Brownian motion W to see that X satisfies the s.d.e.,

$$dX = dW - \frac{1}{2} X\, dt. \tag{2.2}$$

In particular, the quadratic variation of X at time t is t. This means that the semi-martingale local times of X are occupation densities (Revuz and Yor, 1999, Chapter VI). In particular, if $\{\ell_t^0(X)\}_{t\geq 0}$ denotes the local time of X at zero, then

$$\ell_t^0(X) = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{\{|X(s)|\leq\varepsilon\}}\, ds \qquad \text{a.s. and in } L^p(\mathrm{P}) \tag{2.3}$$

See Revuz and Yor (1999, Corollary 1.6, p. 224).
Let $\{\tau(t)\}_{t\geq 0}$ denote the right-continuous inverse-process to $\ell^0(X)$. By the ergodic theorem, $\tau(t)/t$ converges a.s. as t tends to ∞. In fact,

$$\lim_{t\to\infty} \frac{\tau(t)}{t} = \sqrt{2\pi} \qquad \text{a.s.} \tag{2.4}$$

To compute the constant $\sqrt{2\pi}$, we note that by monotonicity, $\tau(t)/t \sim t/\ell_t^0(X)$ a.s. But another application of the ergodic theorem implies that $\ell_t^0(X) \sim \mathrm{E}[\ell_t^0(X)]$ a.s. The assertion (2.4) then follows from the fact that $\mathrm{E}[\ell_t^0(X)] = t/\sqrt{2\pi}$; see (2.3).
Define

$$\varepsilon_t := \sup_{s\geq\tau(t)} \frac{X(s)}{\sqrt{2Ls}} \qquad \forall t \geq e. \tag{2.5}$$

Lemma 2.1. *Almost surely,*

$$\left| \varepsilon_n - \sup_{j\geq n} \frac{M_j}{\sqrt{2Lj}} \right| = O\left(\frac{1}{Ln} \cdot \sqrt{\frac{L_2 n}{n}} \right) \qquad (n\to\infty), \tag{2.6}$$

where

$$M_j := \sup_{s\in[\tau(j),\tau(j+1)]} X(s) \qquad \forall j \geq 1. \tag{2.7}$$

Proof. According to (2.4),

$$\sup_{s\in[\tau(j),\tau(j+1)]} \left| \frac{1}{\sqrt{\ln\tau(j)}} - \frac{1}{\sqrt{\ln s}} \right| \sim \frac{1}{2Lj} \cdot \sqrt{\frac{L_2 j}{jLj}} \qquad (j\to\infty). \tag{2.8}$$

On the other hand, according to (1.1) and (2.4), almost surely,

$$M_j = O\left(\sqrt{\ln\tau(j+1)}\right) = O\left(\sqrt{Lj}\right) \qquad (j\to\infty). \tag{2.9}$$

The lemma follows from a little algebra. □

Lemma 2.1, and monotonicity, together prove that Theorem 1.1 is equivalent to the following: For all $x\in\mathbf{R}$,

$$\lim_{n\to\infty} \mathrm{P}\left\{ 2Ln\left(\sup_{j\geq n} \frac{M_j}{\sqrt{2Lj}} - 1 \right) - \frac{3}{2} L_2 n + L_3 n + L\left(\frac{3}{\sqrt{2}} \right) \leq x \right\} = \Lambda(x). \tag{2.10}$$

We can derive this because: (i) By the strong Markov property of the OU process X, $\{M_j\}_{j=1}^{\infty}$ is an i.i.d. sequence; and (ii) the distribution of M_1 can be found by a combining a little bit of stochastic calculus with an iota of excursion theory. In fact, one has a slightly more general result for Itô diffusions (i.e., diffusions that solve smooth s.d.e.'s) at no extra cost.

Proposition 2.2. *Assume that $\sigma, a \in C^\infty(\mathbf{R})$, σ is bounded away from zero, and $\{W_t\}_{t\geq 0}$ is a Brownian motion. Let $\{Z_t\}_{t\geq 0}$ denote the regular Itô diffusion on $(-\infty,\infty)$ which solves the s.d.e.,*

$$dZ_t = \sigma(Z_t)\, dW_t + a(Z_t) dt. \tag{2.11}$$

Write $\{\theta_t\}_{t\geq 0}$ for the inverse local-time of $\{Z_t\}_{t\geq 0}$ at zero, and define f to be a scale function for Z. Then for all $\lambda > 0$,

$$\mathrm{P}\left(\sup_{t\in[0,\theta_1]} Z_t \leq \lambda \;\middle|\; Z_0 = 0 \right) = \exp\left(-\frac{f'(0)}{2\{f(\lambda) - f(0)\}} \right). \tag{2.12}$$

Proof. The scale function of a diffusion is defined only up to an affine transformation. Therefore, we can assume, without loss of generality, that $f'(0) = 1$ and $f(0) = 0$; else, we choose the scale function $x \mapsto \{f(x) - f(0)\}/f'(0)$ instead. Explicitly, $\{Z_t\}_{t\geq 0}$ has the scale function (Revuz and Yor, 1999, Exercise VII.3.20)

$$f(x) = \int_0^x \exp\left(-2\int_0^y \frac{a(u)}{\sigma^2(u)} du \right) dy. \tag{2.13}$$

Owing to Itô's formula, $N_t := f(Z_t)$ satisfies

$$dN_t = f'(Z_t)\, \sigma(Z_t)\, dW_t = f'\left(f^{-1}(N_t)\right)\, \sigma\left(f^{-1}(N_t)\right)\, dW_t\,, \tag{2.14}$$

and so N is a local martingale. According to the Dambis, Dubins, Schwartz representation theorem (Revuz and Yor, 1999, Theorem V.1.6, p. 181), there exists a Brownian motion $\{b(t)\}_{t\geq 0}$ such that

$$\begin{aligned} &N_t = b(\alpha_t)\,, \quad \text{where} \\ &\alpha_t = \alpha(t) = \langle N\rangle_t = \int_0^t \left[f'\left(f^{-1}(N_r)\right)\right]^2 \sigma^2\left(f^{-1}(N_r)\right)\, dr \qquad \forall t \geq 0. \end{aligned} \tag{2.15}$$

The process N is manifestly a diffusion; therefore, it has continuous local-time processes $\{\ell_t^x(N)\}_{t\geq 0, x\in\mathbf{R}}$ which satisfy the occupation density formula (Revuz and Yor, 1999, Corollary VI.1.6, p. 224 and time-change). By (2.13), $f' > 0$, and because σ is bounded away from zero, $\sigma^2 f' > 0$. Therefore, the inverse process $\{\alpha^{-1}(t)\}_{t\geq 0}$ exists a.s., and is uniquely defined by $\alpha(\alpha^{-1}(t)) = t$ for all $t \geq 0$.

Let $\{\ell_t^x(b)\}_{t\geq 0, x\in\mathbf{R}}$ denote the local-time processes of the Brownian motion b. It is well known (Rogers and Williams, 2000, Theorem V.49.1) that

$$\ell_t^0(N) = \ell_{\alpha(t)}^0(b) \qquad \forall t \geq 0. \tag{2.16}$$

By (2.11) and (2.14), $d\langle Z\rangle_t = \left[f'\left(f^{-1}(N_t)\right)\right]^{-2} d\langle N\rangle_t$. Thus, if $L(Z)$ denotes the local times of Z, then almost surely,

$$\begin{aligned}\int_{-\infty}^{\infty} h(x)L_t^x(Z)\,dx &= \int_0^t h(Z_r)\,d\langle Z\rangle_r = \int_0^t \frac{h\left(f^{-1}(N_r)\right)}{\left[f'\left(f^{-1}(N_r)\right)\right]^2}\,d\langle N\rangle_r \\ &= \int_{-\infty}^{\infty} h\left(f^{-1}(y)\right)\frac{\ell_t^y(N)}{f'\left(f^{-1}(y)\right)}\,d\left[f^{-1}(y)\right] \\ &= \int_{-\infty}^{\infty} h(x)\frac{\ell_t^{f(x)}(N)}{f'(x)}\,dx \qquad [x := f^{-1}(y)].\end{aligned} \tag{2.17}$$

This proves that $f'(x)L_t^x(Z) = \ell_t^{f(x)}(N)$ a.s. In particular, $L_t^0(Z) = \ell_t^0(N)$ for all $t \geq 0$, a.s. It follows from (2.16) that a.s.,

$$L_t^0(Z) = \ell_{\alpha(t)}^0(b) \qquad {}^{\forall}t \geq 0. \tag{2.18}$$

Define $\varphi_t := \inf\left\{s > 0 :\ \ell_s^0(b) > t\right\}$ to be the inverse local time of the Brownian motion b. According to (2.18), $\varphi_t = \alpha(\theta_t)$ for all $t \geq 0$, a.s. Thus,

$$\begin{aligned}\mathrm{P}\left(\sup_{s\in[0,\theta_1]} Z_s \leq \lambda \;\middle|\; Z_0 = 0\right) &= \mathrm{P}\left(\sup_{s\in[0,\theta_1]} N_s \leq f(\lambda) \;\middle|\; N_0 = 0\right) \\ &= \mathrm{P}\left(\sup_{s\in[0,\varphi_1]} b_s \leq f(\lambda) \;\middle|\; b_0 = 0\right).\end{aligned} \tag{2.19}$$

The last identity follows from (2.15) and the fact that α and α^{-1} are both continuous and strictly increasing a.s.
Define $\mathcal{N}_\beta$ to be the total number of excursion of the Brownian motion b that exceed β by local-time 1. Then,

$$\begin{aligned}\mathrm{P}\left(\sup_{s\in[0,\varphi_1]} b_s \leq f(\lambda) \;\middle|\; b_0 = 0\right) &= \mathrm{P}\left(\mathcal{N}_{f(\lambda)} = 0 \;\middle|\; b_0 = 0\right) \\ &= \exp\left\{-\mathrm{E}\left[\mathcal{N}_{f(\lambda)} \;\middle|\; b_0 = 0\right]\right\},\end{aligned} \tag{2.20}$$

because $\mathcal{N}_\beta$ is a Poisson random variable (Itô, 1970). According to Proposition 3.6 of Revuz and Yor (1999, p. 492), $\mathrm{E}[\mathcal{N}_\beta \,|\, b_0 = 0] = (2\beta)^{-1}$ for all $\beta > 0$. See also Revuz and Yor (1999, Exercise XII.4.11). The result follows. □

Remark 2.3. Also, the following equality holds:

$$\mathrm{P}\left(\sup_{t\in[0,\theta_1]} |Z_t| \leq \lambda \;\middle|\; Z_0 = 0\right) = \exp\left(-\frac{f'(0)}{f(\lambda) - f(0)}\right). \tag{2.21}$$

This follows as above after noting that $f(-x) = -f(x)$, and that $\mathrm{E}[\mathcal{N}'_\beta \,|\, b_0 = 0] = \beta^{-1}$, where $\mathcal{N}'_\beta$ denotes the number of excursions of the Brownian motion b that exceed β in absolute value by local-time 1.

Proof of Theorem 1.1. If we apply the preceding computation to the diffusion X itself, then we find that $\mathrm{P}\{M_1 \leq \lambda\} = \exp\{-1/(2S(\lambda))\}$, where S is the scale function of X which satisfies

$S'(0) = 1$ and $S(0) = 0$. According to (2.2) and (2.13), $S(x) = \int_0^x \exp(y^2/2)\,dy$, whence it follows that for all $\lambda > 0$,

$$\mathrm{P}\{M_1 \le \lambda\} = \exp\left(-\frac{1}{2\int_0^\lambda \exp(y^2/2)\,dy}\right) = \exp\left(-\frac{\lambda + \delta(\lambda)}{2\exp(\lambda^2/2)}\right), \tag{2.22}$$

where $\delta(\lambda) = o(\lambda)$ as $\lambda \to \infty$.
Let $\{\beta_n(x)\}_{n=1}^\infty$ be a sequence which, for x fixed, satisfies $\beta_n(x) \to \infty$ as $n \to \infty$. We assume, in addition, that $\alpha_n(x) := \beta_n(x)/Ln$ goes to zero as $n \to \infty$. We will suppress x in the notation and write α_n and β_n for $\alpha_n(x)$ and $\beta_n(x)$, respectively.
A little calculus shows that if $\alpha_n > 0$, then

$$\begin{aligned}
\mathrm{P}\left\{\sup_{j\ge n} \frac{M_j}{\sqrt{2Lj}} \le 1 + \frac{1}{2}\alpha_n\right\} &= \prod_{j=n}^{\infty} \exp\left(-\frac{1}{2S\left((1+\frac{1}{2}\alpha_n)\sqrt{2Lj}\right)}\right) \\
&= \exp\left(-\frac{[1+o(1)](1+\frac{1}{2}\alpha_n)}{\sqrt{2}} \sum_{j=n}^{\infty} \frac{\sqrt{Lj}}{j^{(1+\frac{1}{2}\alpha_n)^2}}\right) \\
&= \exp\left(-\frac{[1+o(1)]\left(1+\frac{1}{2}\alpha_n\right)}{\sqrt{2}} \mathcal{I}_n\right) \\
&= \exp\left(-\frac{[1+o(1)]\left(1+\frac{1}{2}\alpha_n\right)\sqrt{Ln}}{\sqrt{2}\alpha_n\left(1+\frac{1}{4}\alpha_n\right)\alpha_n n^{\alpha_n(1+\alpha_n/4)}}\right) \\
&= \exp\left(-\frac{[1+o(1)]q_n(x)(Ln)^{3/2}e^{-\beta_n}}{\sqrt{2}\beta_n}\right).
\end{aligned} \tag{2.23}$$

Here,

$$q_n(x) := \left(\frac{2Ln + \beta_n}{2Ln + \frac{1}{2}\beta_n}\right)\exp\left(-\frac{\beta_n^2}{4Ln}\right), \text{ and } \mathcal{I}_n := \int_n^\infty \frac{\sqrt{Lz}}{z^{(1+\frac{1}{2}\alpha_n)^2}}\,dz. \tag{2.24}$$

If $\alpha_n \le 0$, then the probability on the right-hand side of (2.23) is 0. Define

$$\varphi_n := \varphi_n(x) := \frac{3}{2}L_2 n - L_3 n - \ln\left(3/\sqrt{2}\right) + x, \tag{2.25}$$

and set $\beta_n := \varphi_n$ in (2.23). This yields

$$\ln \mathrm{P}\left\{\sup_{j\ge n}\frac{M_j}{\sqrt{2Lj}} \le 1 + \frac{\varphi_n}{2Ln}\right\} \sim \begin{cases} -c_n e^{-x} & \text{if } \varphi_n > 0, \\ 0 & \text{if } \varphi_n \le 0, \end{cases} \tag{2.26}$$

where $c_n := c_n(x)$ is defined as

$$c_n := \left(\frac{2Ln + \varphi_n}{2Ln + \frac{1}{2}\varphi_n}\right)\exp\left(-\frac{\varphi_n^2}{4Ln}\right)\left[1 + \frac{-L_3 n - \ln(3/\sqrt{2}) + x}{\frac{3}{2}L_2 n}\right]^{-1}. \tag{2.27}$$

Note that the rate of convergence in (2.26) is independent of x. If $x \in \mathbf{R}$ is fixed, then by letting $n \to \infty$ in (2.26) we find that

$$\lim_{n\to\infty} \mathrm{P}\left\{2Ln\left(\sup_{j\ge n}\frac{M_j}{\sqrt{2Lj}} - 1\right) \le \varphi_n\right\} = \Lambda(x). \tag{2.28}$$

This proves (2.10), whence equation (1.3) of Theorem 1.1 follows.
By using (2.21), we obtain also that as $n \to \infty$,

$$\ln \mathrm{P}\left\{\sup_{j\geq n} \frac{|M_j|}{\sqrt{2Lj}} \leq 1 + \frac{\alpha_n}{2}\right\} \sim -\sqrt{2}\beta_n^{-1} e^{-\beta_n} (Ln)^{3/2} . \tag{2.29}$$

Because $\beta_n = \varphi_n = \frac{3}{2}L_2 n - L_3 n - \ln\left(\frac{3}{2\sqrt{2}}\right) + x$, (1.4) follows. □

3 Proof of Theorem 1.2

In light of (1.6) it suffices to prove that

$$\liminf_{t\to\infty} \frac{L_2 t}{L_3 t}\left(\sup_{s\geq t} \frac{B(s)}{\sqrt{2sL_2 s}} - 1\right) \geq \frac{3}{4} \qquad \text{a.s.} \tag{3.1}$$

We aim to prove that

$$\sup_{j\geq n} \frac{M_j}{\sqrt{2Lj}} > \sqrt{1 + c\frac{L_2 n}{Ln}} \quad \text{eventually a.s. if } c < \frac{3}{2}. \tag{3.2}$$

Theorem 1.2 follows from this by the similar reasons that yielded Theorem 1.1 from (2.10). But (3.2) follows from (2.23):

$$\mathrm{P}\left\{\sup_{j\geq n} \frac{M_j}{\sqrt{2Lj}} \leq \sqrt{1 + c\frac{L_2 n}{Ln}}\right\} = \exp\left\{-\frac{1+o(1)}{2c\sqrt{2}} \cdot \frac{(Ln)^{(3/2)-c}}{L_2 n}\right\}. \tag{3.3}$$

Replace n by ρ^n where $\rho > 1$ is fixed. We find that if $c < (3/2)$ then the probabilities sum in n. Thus, by the Borel–Cantelli lemma, for all $\rho > 1$ and $c < (3/2)$ fixed,

$$\sup_{j\geq \rho^n} \frac{M_j}{\sqrt{2Lj}} > \sqrt{1 + c\frac{L_2(\rho^n)}{L(\rho^n)}} \quad \text{eventually a.s.} \tag{3.4}$$

Equation (3.2) follows from this and monotonicity.

4 An Expectation Bound

We can use our results to improve on the bounds of Dobric and Marano (2003) for the rate of convergence of $\mathrm{E}[\sup_{s\geq t} B_s (2sL_2 s)^{-1/2}]$ to 1.

Proposition 4.1. *As $t \to \infty$,*

$$\mathrm{E}\left[\sup_{s\geq t} \frac{B(s)}{\sqrt{2sL_2 s}}\right] = 1 + \frac{3}{4}\frac{L_3 t}{L_2 t} - \frac{1}{2}\frac{L_4 t}{L_2 t} + \frac{1}{2}\frac{\gamma - \ln\left(3/\sqrt{2}\right)}{L_2 t} + o\left(\frac{1}{L_2 t}\right), \tag{4.1}$$

where $\gamma \approx 0.5772$ denotes Euler's constant.

Proof. Define

$$U_n := 2Ln\left(\sup_{j\ge n}\frac{M_j}{\sqrt{2Lj}}-1\right)-\frac{3}{2}L_2n+L_3n+\ln\left(3/\sqrt{2}\right). \tag{4.2}$$

We have shown that U_n converges weakly to Λ. We now establish that $\sup_n \mathrm{E}\left[U_n^2\right]<\infty$. This implies uniform integrability, whence we can deduce that $\mathrm{E}[U_n]\to\int x\,d\Lambda(x)$.
Let $\varphi_n(x)$ be as defined in (2.25), and note that $x_n^\star := (3/2)L_2n - L_3n - \ln(3/\sqrt{2})$ solves $\varphi_n(-x_n^\star)=0$.
We recall the definition of $c_n(x)$ in (2.27) and rewrite (2.26) to find that for all $x>-x_n^\star$,

$$\ln \mathrm{P}\left\{U_n\le x\right\} = -c_n(x)(1+o(1))e^{-x}. \tag{4.3}$$

Consequently, for n large enough,

$$\int_0^\infty x\mathrm{P}\left\{U_n\le -x\right\} \le \int_0^{x_n^\star} x\exp\left(-\frac{1}{2}c_n(-x)e^x\right)dx. \tag{4.4}$$

For n sufficiently large and $0\le x<x_n^\star$, $c_n(-x)\ge e^{-2}/(3/2)\ge(1/12)$. Thus, for n sufficiently large,

$$\int_0^\infty x\mathrm{P}\left\{U_n\le -x\right\} \le \int_0^{x_n^\star} xe^{-\frac{1}{24}e^x}dx \le \int_0^\infty xe^{-\frac{1}{24}e^x}dx<\infty. \tag{4.5}$$

Also for n sufficiently large,

$$\int_0^\infty x\mathrm{P}\left\{U_n>x\right\} \le \int_0^\infty x\left(1-e^{-\frac{3}{2}c_n(x)e^{-x}}\right)dx \le \int_0^\infty \frac{3}{2}xc_n(x)e^{-x}\,dx. \tag{4.6}$$

We can get the easy bound $c_n(x)\le(3/2+x)/(1/4)=(6+4x)$, valid for $x>0$. This, in turn, yields the following:

$$\int_0^\infty x\mathrm{P}\{U_n>x\}\,dx \le \int_0^\infty \frac{3}{2}x\left(6+4x\right)e^{-x}\,dx<\infty. \tag{4.7}$$

The preceding, (4.5), and integration by parts, together prove that $\sup_n \mathrm{E}[U_n^2]<\infty$; this proves that $\{U_n\}_{n=1}^\infty$ is uniformly integrable. From Lemma 2.1, it follows that as $t\to\infty$,

$$2L_2t\left(\mathrm{E}\left[\sup_{s\ge t}\frac{B_s}{\sqrt{2sL_2s}}\right]-1\right)-\frac{3}{2}L_3t+L_4t+\ln\left(\frac{3}{\sqrt{2}}\right)\to\int_{-\infty}^\infty x\,d\Lambda(x). \tag{4.8}$$

It remains to prove that $\int_{\mathbf{R}} x\,d\Lambda(x)=\gamma$; but $\int_{\mathbf{R}} x\,d[e^{-e^{-x}}]$ is manifestly equal to

$$-\int_0^\infty L(t)e^{-t}\,dt = -\frac{d}{dz}\int_0^\infty t^{z-1}e^{-t}\,dt\bigg|_{z=1} = -\frac{\Gamma'(z)}{\Gamma(z)}\bigg|_{z=1}=\gamma. \tag{4.9}$$

Confer with Davis (1965, Eq. 6.3.2) for the final identity. □

5 Miscellany

This final section is concerned with some remarks about random walks. Throughout let $X_1,X_2,\ldots$ be i.i.d. random variables with

$$\mathrm{E}[X_1]=0,\ \mathrm{E}[X_1^2]=1,\quad\text{and}\quad \mathrm{E}\left[X_1^2L_2\left(|X_1|\vee e^e\right)\right]<\infty. \tag{5.1}$$

Let $S_n:=X_1+\cdots+X_n$ $(n\ge1)$ denote the corresponding random walk.

5.1 An Application to Random Walks

According to Theorem 2 of Einmahl (1987), there exists a probability space on which one can construct $\{S_n\}_{n=1}^{\infty}$ together with a Brownian motion B such that $|S_n - B(n)|^2 = o(n/L_2 n)$ a.s. On the other hand, by the reflection principle and the Borel–Cantelli lemma, $|B(t) - B(n)|^2 = o(n/L_2 n)$ uniformly for all $t \in [n, n+1]$ a.s., and with room to spare. These remarks, and a few more lines of elementary computations, together yield the following.

Theorem 5.1. *If* (5.1) *holds and* $x \in \mathbf{R}$*, then as* $n \to \infty$*,*

$$\mathrm{P}\left\{2L_2 n\left(\sup_{k\geq n}\frac{S_k}{\sqrt{2kL_2k}}-1\right)-\frac{3}{2}L_3 n+L_4 n+L\left(\frac{3}{\sqrt{2}}\right)\leq x\right\}\to\Lambda(x), \tag{5.2}$$

$$\mathrm{P}\left\{2L_2 n\left(\sup_{k\geq n}\frac{|S_k|}{\sqrt{2kL_2k}}-1\right)-\frac{3}{2}L_3 n+L_4 n+L\left(\frac{3}{2\sqrt{2}}\right)\leq x\right\}\to\Lambda(x). \tag{5.3}$$

It would be interesting to know if the preceding remains valid if only $\mathrm{E}[X_1] = 0$ and $\mathrm{E}[X_1^2] = 1$. The answer may well be, "No"; see Einmahl (1989) for a related result. We owe this observation to an anonymous referee.

5.2 On the Darling–Erdős Theorem

An anonymous referee has suggested that our Theorem 1.1 is similar to the Darling–Erdős theorem. Here, we show this is true at the technical level. We do so by presenting a simplified version of our proof of Theorem 1.1 which yields also the following "Darling–Erdős (1956) theorem." Similar ideas appeared earlier in Bertoin (1998).

Theorem 5.2. *For all* $x \in \mathbf{R}$*, as* $n \to \infty$*,*

$$\mathrm{P}\left\{2L_2 n\left(\frac{\max_{1\leq k\leq n}(S_k/\sqrt{k})}{\sqrt{2L_2 n}}-1\right)-\frac{L_3 n}{2}+\frac{L(4\pi)}{2}\leq x\right\}\to\Lambda(x). \tag{5.4}$$

There is also a related result about $|S_k|$ that is proved by similar means. We will restrict attention to the statement of Theorem 5.2 only.

In their original paper, Darling and Erdős (1956) proved this under the more restrictive condition that $\mathrm{E}\{|X_1|^3\} < \infty$. They proceed by first working on the Gaussian case and then using a "weak invariance principle." The present formulation requires only that $\mathrm{E}\{X_1^2 L_2(|X_1| \vee e^e)\} < \infty$; it can be shown to follow directly from the Darling–Erdős theorem, in the Gaussian case, and the strong invariance principle of Einmahl (1987) in place of the said weak invariance principle. Einmahl (1989, p. 242) attributes this observation to David M. Mason; see also Einmahl and Mason (1989). Furthermore, Einmahl proves that the P-integrability of $X_1^2 L_2(|X_2| \vee e^e)$ is optimal. For other related results see, for example, the works of Bertoin (1998), Oodaira (1976), and Shorack (1979).

Proof. We will use the notation of the proof of Theorem 1.1 throughout.

By (2.22) and independence,

$$\lim_{n\to\infty}\mathrm{P}\left\{\max_{1\leq j\leq n}M_j\leq\sqrt{2Ln+L_2n-2L\gamma}\right\}=e^{-\gamma/\sqrt{2}}, \tag{5.5}$$

valid for all $\gamma > 0$. But $\max_{j\le n} M_j = X^*(\tau(n))$, where $X^*(t) := \sup_{s\le t} X(s)$. This, (2.4), and monotonicity together imply that for all $x \in \mathbf{R}$,

$$\lim_{n\to\infty} \mathrm{P}\left\{ Ln\left(\frac{X^*\left(n\sqrt{2\pi}\right)}{\sqrt{2Ln}} - 1\right) - \frac{L_2 n}{4} \le x \right\} = \exp\left(-\frac{e^{-2x}}{\sqrt{2}}\right). \tag{5.6}$$

Yet another appeal to monotonicity yields that for all $x \in \mathbf{R}$,

$$\lim_{t\to\infty} \mathrm{P}\left\{ Lt\left(\frac{X^*(t)}{\sqrt{2L\left(t/\sqrt{2\pi}\right)}} - 1\right) - \frac{L_2 t}{4} \le x \right\} = \exp\left(-\frac{e^{-2x}}{\sqrt{2}}\right). \tag{5.7}$$

We have used also the well-known fact that $X^*(t) \sim \sqrt{2Lt}$ a.s. But

$$Lt\left(\frac{X^*(t)}{\sqrt{2L\left(t/\sqrt{2\pi}\right)}} - 1\right) = o(1) + Lt\left(\frac{X^*(t)}{\sqrt{2Lt}} - 1\right) + \frac{\ln(2\pi)}{4} \quad \text{a.s.} \tag{5.8}$$

Hence, for all $x \in \mathbf{R}$,

$$\lim_{t\to\infty} \mathrm{P}\left\{ Lt\left(\frac{X^*(t)}{\sqrt{2Lt}} - 1\right) - \frac{L_2 t}{4} \le x \right\} = \exp\left(-\frac{e^{-2x}}{2\sqrt{\pi}}\right). \tag{5.9}$$

The preceding is the "Darling–Erdős theorem for Brownian motion." The remainder of the theorem follows from strong approximations. □

Remark 5.3. Several times in the previous proof we used the classical fact that $X^*(t) \sim \sqrt{2Lt}$ a.s. This too follows from the methods of the paper. We include a proof in order to illustrate the power of these techniques. Firstly, we note that $X^*(t) \le (1 + o(1))\sqrt{2Lt}$ by the LIL. Secondly, in accord with (2.22), $\mathrm{P}\{\max_{j\le n} M_j \le (1-\epsilon)\sqrt{2Ln}\} \le \exp(-cn^{\epsilon})$ for some $c > 0$ which does not depend on (n, ϵ). This and the Borel–Cantelli lemma together prove that $\max_{j\le n} M_j \ge (1 + o(1))\sqrt{2Ln}$. Equation (2.4) and monotonicity together prove that $X^*(t) \ge (1 + o(1))\sqrt{2Lt}$, which has the desired effect.

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