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KENDALL'S IDENTITY FOR THE FIRST CROSSING TIME REVISITED

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Abstract
We give a new relatively compact proof of the famous identity for the distribution of the first hitting time of a linear boundary by a skip-free process with stationary independent increments. The proof uses martingale identities and change of measure.

Let $\{X_t\}_{t\geq 0}$, $X_0 = 0$, be a Lévy process which is skip-free in the positive direction. That is, X_t has no positive jumps and its increments are stationary independent. For an $x > 0$ set

$$\tau(x) := \inf\{t > 0 : X_t \geq x\}$$

with $\tau(x) = \infty$ on the event $\{\sup_{t\geq 0} X_t < x\}$.
The following result is well known.

For any $y, s > 0$,

$$\int_y^\infty \mathbf{P}\left(\tau(x) \leq s\right)\frac{dx}{x} = \int_0^s \mathbf{P}\left(X_t > y\right)\frac{dt}{t}. \tag{1}$$

If X_t has a density $p_X(t,x)$ at x, then $\tau(x)$ also has a density $p_\tau(t,x)$ at t and

$$\frac{1}{x}p_\tau(t,x) = \frac{1}{t}p_X(t,x). \tag{2}$$

Relation (2) was first observed in a special case in [7]. Later the theorem was proved in [6], [9], [11] (under additional conditions) and [2]. Moreover, it was shown in [8] that (1) is a necessary condition for the process $\{X_t\}$ to be skip-free.

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A discrete time (and, of course, space) analog of (2) is closely related to the classical ballot problem and was first given (in a special case) as early as in 1711 by A. de Moivre. For historical references and comments (and some generalizations) see e.g. [1] and Section 21 in [10]. Some interesting extensions to the multidimensional case were given in [4].
Known proofs of the identities are either analytical (exploiting Laplace transforms in time and their multiplicative structure) or are based on a limiting procedure and a combinatorial argument (given, in particular in [5] and [10]) or factorization identities [3]. We present a new, rather short and elegant (from our point of view) proof of the above result which uses (a) martingale techniques to find the Laplace transform of the crossing time and then (b) change of measure to find the desired representation for the distribution of the time in terms of the distribution of X_t. To avoid a trivial situation, we assume that $\mathbf{P}\,(X_t > 0) > 0$ for $t > 0$.

Proof. (i) Since there are no positive jumps in the process $\{X_t\}$, the m.g.f. $\varphi(\lambda) := \mathbf{E}\, e^{\lambda X_1} < \infty$ for any $\lambda \geq 0$ and is analytic on the positive half-line, and by independence of increments,

$$M_t := e^{\lambda X_t}/\varphi^t(\lambda), \quad t \geq 0,$$

is a martingale. Denote by $\mathbf{P}_\lambda$ the probability corresponding to the respective Cramér transform of the distribution of $\{X_t\}$, so that

$$\mathbf{P}_\lambda(X_t \in dy) = \frac{e^{\lambda y}}{\varphi^t(\lambda)}\mathbf{P}\,(X_t \in dy) \tag{3}$$

and the process $\{X_t\}$ still remains a Lévy process under $\mathbf{P}_\lambda$.
Next set $\lambda_0 := \sup\{\lambda \geq 0 : \varphi(\lambda) = 1\}$. Clearly, $0 \leq \lambda_0 < \infty$ always, and under $\mathbf{P}_\lambda$ the drift in $\{X_t\}$ is linear with the coefficient

$$m_\lambda := \mathbf{E}_\lambda X_1 = \frac{\varphi'(\lambda)}{\varphi(\lambda)} = (\ln \varphi(\lambda))' > 0 \quad \text{for} \quad \lambda > \lambda_0 \tag{4}$$

(just recall the elementary fact that $\ln \varphi(\lambda)$ is convex and, when $\lambda_0 > 0$, equal to 0 for both $\lambda = 0$ and $\lambda = \lambda_0$). Note also that, for $\lambda > \lambda_0$, one has $\varphi(\lambda) > 1$.

(ii) The last inequality, together with the obvious fact that $M_t < e^{\lambda x}/\varphi^t(\lambda)$ on the event $\{\tau(x) > t\}$, implies that, as $t \to \infty$, $M_t \to 0$ on $\{\tau(x) = \infty\}$ (so that we can formally set $M_{\tau(x)} = 0$ on this event) and also $\mathbf{E}\,(M_t; \tau(x) > t) \to 0$. These relations ensure that the optional stopping theorem holds:

$$1 = M_0 = \mathbf{E}\, M_{\tau(x)} = e^{\lambda x}\mathbf{E}\, e^{-\mu\tau(x)} \quad \text{with} \quad \mu := \ln \varphi(\lambda) \tag{5}$$

(this can be shown e.g. by applying the theorem to the bounded stopping time $\tau(x) \wedge t$ and then letting $t \to \infty$). It is also clear that $\mu = \mu(\lambda)$ is an increasing function mapping (λ_0, ∞) onto $(0, \infty)$ and hence has an inverse function $\lambda = \lambda(\mu)$, $\mu \in (0, \infty)$, with

$$\frac{d\lambda}{d\mu} = \frac{1}{d\mu/d\lambda} = \frac{\varphi(\lambda)}{\varphi'(\lambda)} = \frac{1}{m_\lambda} \tag{6}$$

from (4).
Now (5) is equivalent to $\mathbf{E}\, e^{-\mu\tau(x)} = e^{-\lambda x}$, and differentiating this relation w.r.t. μ yields

$$\int_0^\infty e^{-\mu t} t\, \mathbf{P}\,(\tau(x) \in dt) = x e^{-\lambda x}\frac{d\lambda}{d\mu} = \frac{x}{m_\lambda e^{\lambda x}}. \tag{7}$$

(iii) Next denote by $T_A = \int_0^\infty \mathbf{1}_A(X_t)\,dt$ the time spent by our process in the set A (i.e. the sojourn time of A). We will make use of the following fact: for any $0 < a < \infty$ and $\lambda > \lambda_0$,

$$\mathbf{E}_\lambda T_{(0,a]} = \frac{a}{m_\lambda}. \tag{8}$$

To prove it, note that due to (4) and Wald's identity one has for any $a > 0$

$$a = \mathbf{E}_\lambda X_{\tau(a)} = m_\lambda \mathbf{E}_\lambda \tau(a). \tag{9}$$

Further, one can easily see that

$$\tau(a) - T_{(-\infty,0]} \le T_{(0,a]} \le \tau(a) + T^{(a)}_{(-\infty,0]}, \tag{10}$$

where $T_A^{(a)}$ denotes the time spent in A by the process $\{X_{t+\tau(a)} - a\}_{t\ge0}$, which clearly has the same distribution as $\{X_t\}_{t\ge0}$, and hence $T_{(-\infty,0]}$ and $T^{(a)}_{(-\infty,0]}$ are identically distributed, too. Moreover, for $\lambda > \lambda_0$, they have a finite expectation under $\mathbf{P}_\lambda$:

$$\begin{aligned} I_\lambda &:= \mathbf{E}_\lambda T_{(-\infty,0]} = \mathbf{E}_\lambda \int_0^\infty \mathbf{1}_{(-\infty,0]}(X_t)\,dt \le \mathbf{E}_\lambda \int_0^\infty e^{-\lambda X_t}dt \\ &= \int_0^\infty \mathbf{E}_\lambda e^{-\lambda X_t}dt = \int_0^\infty dt \int e^{-\lambda y}\frac{e^{\lambda y}}{\varphi^t(\lambda)}\mathbf{P}\,(X_t \in dy) = \int_0^\infty \frac{dt}{\varphi^t(\lambda)} < \infty \end{aligned}$$

since $\varphi(\lambda) > 1$.

Now this and the RHS of (10) imply that $\mathbf{E}_\lambda T_{(a,b]} < \infty$ for any finite interval $(a,b]$ [the last fact actually following from the well-known known recurrence-transience dichotomy for Lévy processes as well]. Together with the obvious observation that, for any $0 < a < b < c < \infty$,

$$T_{(a,b]} + T_{(b,c]} = T_{(a,c]}$$

this means that a corresponding equality holds for the (finite) $\mathbf{E}_\lambda$-expectations of these variables, and hence we have

$$\mathbf{E}_\lambda T_{(a,b]} = \gamma(b-a)$$

for some constant $\gamma < \infty$. But from (9) and (10) one gets

$$|\mathbf{E}_\lambda T_{(0,a]} - a/m_\lambda| \le I_\lambda,$$

and letting here $a \to \infty$ we immediately see that this constant γ must be equal to $1/m_\lambda$ which completes the proof of (8).

(iv) Note that equality (8) can be re-written as

$$\begin{aligned} \frac{a}{m_\lambda} &= \mathbf{E}_\lambda \int_0^\infty \mathbf{1}_{(0,a]}(X_t)\,dt = \int_0^\infty \mathbf{P}_\lambda(X_t \in (0,a])\,dt \\ &= \int_0^\infty dt \int_{(0,a]} \frac{e^{\lambda x}}{\varphi^t(\lambda)}\mathbf{P}\,(X_t \in dx) = \int_{(0,a]} \left[e^{\lambda x}\int_0^\infty e^{-\mu t}\mathbf{P}\,(X_t \in dx)\,dt \right]. \end{aligned}$$

Since the above holds for any $a > 0$, the expression in the brackets is (for any $\lambda > \lambda_0$) nothing else but a multiple of the Lebesgue measure: $[\cdots] = m_\lambda^{-1}dx$ for $x \in (0,\infty)$, so that

$$\frac{dx}{m_\lambda e^{\lambda x}} = \int_0^\infty e^{-\mu t}\mathbf{P}\,(X_t \in dx)\,dt.$$

Now dividing the LHS and RHS of (7) by x and integrating them w.r.t. dx over (y,∞) we get, using the above formula, that

$$\begin{aligned}\int_y^\infty \frac{dx}{x}\int_0^\infty e^{-\mu t} t\,\mathbf{P}(\tau(x)\in dt) &= \int_0^\infty e^{-\mu t} t \int_y^\infty \frac{dx}{x}\mathbf{P}(\tau(x)\in dt)\\ =\int_y^\infty \frac{dx}{m_\lambda e^{\lambda x}} &= \int_0^\infty e^{-\mu t}\mathbf{P}(X_t>y)dt.\end{aligned}$$

for any $\mu>0$. The equality of the Laplace transforms (in μ) implies that the measure

$$tL(dt) := t\int_y^\infty \frac{dx}{x}\mathbf{P}(\tau(x)\in dt)$$

has the density $\mathbf{P}(X_t>y)$, or, equivalently, that $L(dt)$ has the density $\mathbf{P}(X_t>y)/t$. And this completes the proof since the desired identity (1) just represents that fact in terms of the respective distribution functions.

References

[1] Barton, D. E., and Mallows, C. L. (1965) Some aspects of the random sequence. *Ann. Math. Statist.* **36**, 236–260.

[2] Borovkov, A. A. (1965) On the first passage time for a class of processes with independent increments. *Theor. Probab. Appl.* **10**, 360–364. [In Russian.]

[3] Borovkov, A. A. (1976) *Stochastic processes in queueing theory.* Springer: New York.

[4] Borovkov, K. A. (1995) On crossing times for multidimensional walks with skip-free components. *J. Appl. Prob.* **32**, 991–1006.

[5] Feller, W. (1971) *An introduction to probability theory and its applications.* Vol. 2. 2nd edn. Wiley: New York.

[6] Keilson, J. (1963) The first passage time density for homogeneous skip-free walks on the continuum. *Ann. Math. Statist.* **34**, 1003–1011.

[7] Kendall, D. G. (1957) Some problems in the theory of dams. *J. Royal Stat. Soc. B*, **19**, 207–212.

[8] Rogozin, B. A. (1966) Distribution of certain functionals related to boundary value problems for processes with independent increments. *Theor. Probab. Appl.* **11**, 161–169. [In Russian.]

[9] Skorohod, A. V. (1964) *Random processes with independent increments.* Nauka: Moscow [In Russian. English translation of the second Russian edition was published in 1991 (Kluwer: Dordrecht).]

[10] Takács, L. (1967) *Combinatorial methods in the theory of stochastic processes.* Wiley: New York.

[11] Zolotarev, V. M. (1964) The first passage time of a level and the behaviour at infinity of a class of processes with independent increments. *Theor. Probab. Appl.* **9**, 724–733. [In Russian.]