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BERRY-ESSEEN BOUNDS FOR PROJECTIONS OF COORDINATE SYMMETRIC RANDOM VECTORS

LARRY GOLDSTEIN
Department of Mathematics, University of Southern California, Los Angeles, CA 90089-2532, USA
email: `larry@math.usc.edu`

QI-MAN SHAO[1]
Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China
email: `maqmshao@ust.hk`

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Abstract
For a coordinate symmetric random vector $(Y_1,\ldots,Y_n)=\mathbf{Y}\in\mathbb{R}^n$, that is, one satisfying $(Y_1,\ldots,Y_n)=_d(e_1Y_1,\ldots,e_nY_n)$ for all $(e_1,\ldots,e_n)\in\{-1,1\}^n$, for which $P(Y_i=0)=0$ for all $i=1,2,\ldots,n$, the following Berry Esseen bound to the cumulative standard normal Φ for the standardized projection $W_\theta=Y_\theta/v_\theta$ of $\mathbf{Y}$ holds:

$$\sup_{x\in\mathbb{R}}|P(W_\theta\le x)-\Phi(x)|\le 2\sum_{i=1}^n|\theta_i|^3E|X_i|^3+8.4E(V_\theta^2-1)^2,$$

where $Y_\theta=\theta\cdot\mathbf{Y}$ is the projection of $\mathbf{Y}$ in direction $\theta\in\mathbb{R}^n$ with $||\theta||=1$, $v_\theta=\sqrt{\mathrm{Var}(Y_\theta)}$, $X_i=|Y_i|/v_\theta$ and $V_\theta=\sum_{i=1}^n\theta_i^2X_i^2$. As such coordinate symmetry arises in the study of projections of vectors chosen uniformly from the surface of convex bodies which have symmetries with respect to the coordinate planes, the main result is applied to a class of coordinate symmetric vectors which includes cone measure $\mathscr{C}_p^n$ on the ℓ_p^n sphere as a special case, resulting in a bound of order $\sum_{i=1}^n|\theta_i|^3$.

1 Introduction and main result

Properties of the distributions of vectors uniformly distributed over the surface, or interior, of compact, convex bodies, such as the unit sphere in $\mathbb{R}^n$, have been well studied. When the convex body has symmetry with respect to all n coordinate planes, a vector $\mathbf{Y}$ chosen uniformly from its surface satisfies

$$(Y_1,\ldots,Y_n)=_d(e_1Y_1,\ldots,e_nY_n)\quad\text{for all }(e_1,\ldots,e_n)\in\{-1,1\}^n$$

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and is said to be coordinate symmetric. Projections

$$Y_\theta = \theta \cdot \mathbf{Y} = \sum_{i=1}^{n} \theta_i Y_i \tag{1.1}$$

of $\mathbf{Y}$ along $\theta \in \mathbb{R}^n$ with $||\theta|| = 1$ have generated special interest, and in many cases normal approximations, and error bounds, can be derived for W_θ, the projection Y_θ standardized to have mean zero and variance 1. In this note we show that when a random vector is coordinate symmetric, even though its components may be dependent, results for independent random variables may be applied to derive error bounds to the normal for its standardized projection. Bounds in the Kolmogorov and total variation metric for projections of vectors with symmetries are given also in [8], but the bounds are not optimal; the bounds provided here, in particular those in Theorem 2.1 for the normalized projections of the generalization $\mathscr{C}_{p,F}^n$ of cone measure, are of order $\sum_{i=1}^n |\theta_i|^3$. In related work, many authors study the measure of the set of directions on the unit sphere along which projections are approximately normally distributed, but in most cases bounds are not provided; see in particular [12], [1] and [2]. One exception is [6] where the surprising order $\sum_{i=1}^n |\theta_i|^4$ is obtained under the additional assumption that a joint density function of $\mathbf{Y}$ exists, and is log-concave.

When the components $Y_1, \ldots, Y_n$ of a coordinate symmetric vector $\mathbf{Y}$ have finite variances $v_1^2, \ldots, v_n^2$, respectively, it follows easily from $Y_i =_d -Y_i$ and $(Y_i, Y_j) =_d (-Y_i, Y_j)$ for $i \neq j \in \{1, \ldots, n\}$ that

$$EY_i = 0, \quad \text{and} \quad EY_iY_j = v_i^2 \delta_{ij},$$

and hence, that

$$EY_\theta = 0 \quad \text{and} \quad \mathrm{Var}(Y_\theta) = v_\theta^2 \quad \text{where} \quad v_\theta^2 = \sum_{i=1}^n \theta_i^2 v_i^2.$$

Standardizing to variance 1, write

$$W_\theta = Y_\theta / v_\theta \quad \text{and} \quad X_i = |Y_i|/v_\theta. \tag{1.2}$$

When $v_i^2 = v^2$ is constant in i then $v_\theta^2 = v^2$, the common variance of the components, for all θ with $||\theta|| = 1$.

One conclusion of Theorem 1.1 gives a Kolmogorov distance bound between the standardized projection W_θ and the normal in terms of expectations of functions of $V_\theta = \sum_{i=1}^n \theta_i^2 X_i^2$ and $\sum_{i=1}^n |\theta_i|^3 |X_i|^3$. We apply Theorem 1.1 to standardized projections of a family of coordinate symmetric random vectors, generalizing cone measure $\mathscr{C}_p^n$ on the sphere $S(\ell_p^n)$, defined as follows. With $p > 0$, let

$$S(\ell_p^n) = \{x \in \mathbb{R}^n : \sum_{i=1}^n |x_i|^p = 1\} \quad \text{and} \quad B(\ell_p^n) = \{x \in \mathbb{R}^n : \sum_{i=1}^n |x_i|^p \le 1\}.$$

With μ^n Lebesgue measure on $\mathbb{R}^n$, the cone measure of $A \subset S(\ell_p^n)$ is given by

$$\mathscr{C}_p^n(A) = \frac{\mu^n([0,1]A)}{\mu^n(B(\ell_p^n))} \quad \text{where} \quad [0,1]A = \{ta : a \in A, t \in [0,1]\}. \tag{1.3}$$

The cases $p = 1$ and $p = 2$ are of special interest, corresponding to the uniform distribution over the unit simplex and unit sphere, respectively. In particular, the authors of [4] compute bounds for the total variation distance between the normal and the components of $\mathbf{Y}$ in the case $p = 2$.

In [5] an L^1 bound between the standardized variable W_θ in (1.2) and the normal is obtained when $\mathbf{Y}$ has the cone measure distribution. Here an application of Theorem 1.1 yields Theorem 2.1, which gives Kolmogorov distance bounds of the order $\sum_{i=1}^n |\theta_i|^3$ for a class of distributions $\mathscr{C}_{p,F}^n$ which include cone measure as a special case.
We note that if $\theta \in \mathbb{R}^n$ satisfies $||\theta|| = 1$, so Hölder's inequality with $1/s + 1/t = 1$ yields

$$1 = \left(\sum_{i=1}^n \theta_i^2\right) \le \left(\sum_{i=1}^n |\theta_i|^{2s}\right)^{1/s} n^{1/t} \quad \text{hence} \quad n^{-s/t} \le \sum_{i=1}^n |\theta_i|^{2s}. \tag{1.4}$$

In particular, with $s = 3/2, t = 3$ we have $n^{-1/2} \le \sum_{i=1}^n |\theta_i|^3$, and therefore, for any sequence of norm one vectors θ in $\mathbb{R}^n$ for $n = 1, 2, \ldots$ we have $n^{-\beta} = o(\sum_{i=1}^n |\theta_i|^3)$ for all $\beta > 1/2$. We note that equality is achieved in (1.4) when $\theta = n^{-1/2}(1, 1, \ldots, 1)$, the case recovering the standardized sum of the coordinates of $\mathbf{Y}$.
We have the following simple yet crucial result, shown in Section 3.

Lemma **1.1.** *Let $\mathbf{Y}$ be a coordinate symmetric random variable in $\mathbb{R}^n$ such that $P(Y_i = 0) = 0$ for all $i = 1, 2\ldots, n$, and let $\varepsilon_i = sign(Y_i)$, the sign of Y_i. Then the signs $\varepsilon_1, \ldots, \varepsilon_n$ of the coordinates $Y_1, \ldots, Y_n$ are i.i.d. variables taking values uniformly in $\{-1, 1\}$, and*

$$(\varepsilon_1, \ldots, \varepsilon_n) \quad \textit{and} \quad (|Y_1|, \ldots, |Y_n|) \quad \textit{are independent.}$$

The independence property provided by Lemma 1.1 is the key ingredient in the following theorem.

Theorem **1.1.** *Let $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be a coordinate symmetric random vector in $\mathbb{R}^n$ whose components satisfy $P(Y_i = 0) = 0$ and have variances v_i^2 for $i = 1, \ldots, n$. For $\theta \in \mathbb{R}^n$ with $||\theta|| = 1$ let*

$$Y_\theta = \theta \cdot \mathbf{Y}, \quad v_\theta^2 = \sum_{i=1}^n \theta_i^2 v_i^2, \quad \textit{and} \quad V_\theta^2 = \sum_{i=1}^n \theta_i^2 X_i^2, \tag{1.5}$$

where $X_i = |Y_i|/v_\theta$. Then, with $\Phi(x)$ the cumulative distribution function of the standard normal, the normalized projection $W_\theta = Y_\theta / v_\theta$ satisfies

$$\begin{aligned} &\sup_{x\in\mathbb{R}} |P(W_\theta \le x) - \Phi(x)| \\ &\quad \le \quad 4.2E|V_\theta^2 - 1| I\{|V_\theta^2 - 1| > 1/2\} + 0.4E(V_\theta^2 - 1)^2 I\{|V_\theta^2 - 1| \le 1/2\} \\ &\qquad\quad + 2\sum_{i=1}^n |\theta_i|^3 E|X_i|^3. \end{aligned} \tag{1.6}$$

In particular,

$$\sup_{x\in\mathbb{R}} |P(W_\theta \le x) - \Phi(x)| \le 8.4E(V_\theta^2 - 1)^2 + 2\sum_{i=1}^n |\theta_i|^3 E|X_i|^3. \tag{1.7}$$

We remark that in related work, Theorem 4 in [3] gives an exponential non-uniform Berry-Esseen bound for the Studentized sums

$$\sum_{i=1}^n \theta_i Y_i / (\sum_{i=1}^n \theta_i^2 Y_i^2)^{1/2}.$$

A simplification of the bounds in Theorem 1.1 result when $\mathbf{Y}$ has the 'square negative correlation property,' see [9], that is, when

$$\mathrm{Cov}(Y_i^2, Y_j^2) \le 0 \;\text{ for }\; i \ne j, \tag{1.8}$$

as then

$$E(V_\theta^2 - 1)^2 \le \sum_{i=1}^n \theta_i^4 \mathrm{Var}(X_i^2),$$

and hence the first term on the right hand side of (1.7) can be replaced by $8.4 \sum_{i=1}^n \theta_i^4 \mathrm{Var}(X_i^2)$. Proposition 3 of [9] shows that cone measure $\mathscr{C}_p^n$ satisfies a correlation condition much stronger than (1.8); see also [1] regarding negative correlation in the interior of $B(\ell_p^n)$.

2 Application

One application of Theorem 1.1 concerns the following generalization of cone measure $\mathscr{C}_p^n$. Let $n \ge 2$ and $G_1, \dots, G_n$ be i.i.d. nontrivial, positive random variables with distribution function F, and set

$$G_{1,n} = \sum_{i=1}^n G_i.$$

In addition, let $\varepsilon_1, \dots, \varepsilon_n$ be i.i.d. random variables, independent of $G_1, \dots, G_n$, taking values uniformly in $\{-1, 1\}$. Let $\mathscr{C}_{p,F}^n$ be the distribution of the vector

$$\mathbf{Y} = \left(\varepsilon_1 (\frac{G_1}{G_{1,n}})^{1/p}, \dots, \varepsilon_n (\frac{G_n}{G_{1,n}})^{1/p} \right). \tag{2.1}$$

By results in [10], for instance, cone measure $\mathscr{C}_p^n$ as given in (1.3) is the special case when F is the Gamma distribution $\Gamma(1/p, 1)$.

THEOREM **2.1.** *Let* $\mathbf{Y}$ *have distribution* $\mathscr{C}_{p,F}^n$ *given by (2.1) with* $p > 0$ *and* F *for which* $EG_1^{2+4/p} < \infty$ *when* G_1 *is distributed according to* F*. Then there exists a constant* $c_{p,F}$ *depending on* p *and* F *such that for all* $\theta \in \mathbb{R}^n$ *for which* $||\theta|| = 1$ *we have*

$$\sup_{x \in \mathbb{R}} |P(W_\theta \le x) - \Phi(x)| \le c_{p,F} \sum_{i=1}^n |\theta_i|^3, \tag{2.2}$$

where

$$W_\theta = Y_\theta / v_\theta \quad \text{with} \quad Y_\theta = \theta \cdot \mathbf{Y} \quad \text{and} \quad v_\theta^2 = E\Big(\frac{G_1}{G_{1,n}}\Big)^{2/p}.$$

As the Gamma distribution $\Gamma(1/p, 1)$ has moments of all orders, the conclusion of Theorem 2.1 holds, in particular, for cone measure $\mathscr{C}_p^n$.

3 Proofs

Proof of Lemma 1.1 Let $A_1, \dots, A_n$ be measurable subsets of $(0, \infty)$ and $(e_1, \dots, e_n) \in \{-1, 1\}^n$.

Then, using the coordinate symmetry property to obtain the fourth equality, we have

$$\begin{aligned}
&P(\varepsilon_1 = e_1, \ldots, \varepsilon_n = e_n, |Y_1| \in A_1, \ldots, |Y_n| \in A_n) \\
&= P(\varepsilon_1 = e_1, \ldots, \varepsilon_n = e_n, \varepsilon_1 Y_1 \in A_1, \ldots, \varepsilon_n Y_n \in A_n) \\
&= P(e_1 Y_1 \in A_1, \ldots, e_n Y_n \in A_n) \\
&= P(Y_1 \in e_1 A_1, \ldots, Y_n \in e_n A_n) \\
&= \frac{1}{2^n} \sum_{(\gamma_1,\ldots,\gamma_n)\in\{-1,1\}^n} P(Y_1 \in \gamma_1 A_1, \ldots, Y_n \in \gamma_n A_n) \\
&= \left(\prod_{i=1}^n P(\varepsilon_i = e_i)\right) P(|Y_1| \in A_1, \ldots, |Y_n| \in A_n). \quad \square
\end{aligned}$$

Before proving Theorem 1.1, we invoke the following well-known Berry-Esseen bound for independent random variables (see [11]): *if $\xi_1, \cdots, \xi_n$ are independent random variables satisfying $E\xi_i = 0$, $E|\xi_i|^3 < \infty$ for $1 \le i \le n$ and $\sum_{i=1}^n E\xi_i^2 = 1$,*

$$\sup_{x\in\mathbb{R}} |P(\sum_{i=1}^n \xi_i \le x) - \Phi(x)| \le \min(1, 0.7056 \sum_{i=1}^n E|\xi_i|^3).$$

In particular, if $\varepsilon_1, \ldots, \varepsilon_n$ are independent random variables taking the values $-1, +1$ with equal probability, and $b_1, \ldots, b_n$ are any nonzero constants, then $W = \sum_{i=1}^n b_i \varepsilon_i$ satisfies

$$\sup_{x\in\mathbb{R}} |P(W \le x) - \Phi(x/V)| \le \min(1, 0.7056 \sum_{i=1}^n |b_i|^3/V^3), \tag{3.1}$$

where $V^2 = \sum_{i=1}^n b_i^2$.

Proof of Theorem 1.1. By Lemma 1.1, recalling $X_i = |Y_i|/\nu_\theta$, we may write

$$W_\theta = \sum_{i=1}^n \varepsilon_i \theta_i X_i$$

where $\{\varepsilon_i, 1 \le i \le n\}$ is a collection of i.i.d. random variables with $P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = 1/2$, independent of $X_1, \ldots, X_n$. Note that, by construction, $\sum_{i=1}^n \theta_i^2 E X_i^2 = 1$.

Now,

$$\begin{aligned}
&P(W_\theta \le x) - P(Z \le x) \\
&= E\Big(P(W_\theta \le x | \{X_i\}_{1\le i\le n}) - \Phi(x/V_\theta)\Big) + E\Big\{\Phi(x/V_\theta) - \Phi(x)\Big\} \\
&:= R_1 + R_2.
\end{aligned} \tag{3.2}$$

By (3.1),

$$\begin{aligned}
|R_1| &\le E\Big\{\min(1, \frac{0.7056 \sum_{i=1}^n |\theta_i|^3 |X_i|^3}{V_\theta^3})\Big\} \\
&\le P(V_\theta^2 < 1/2) + 0.7056(2^{3/2}) \sum_{i=1}^n |\theta_i|^3 E|X_i|^3 \\
&\le 2E|V_\theta^2 - 1| I\{|V_\theta^2 - 1| > 1/2\} + 2 \sum_{i=1}^n |\theta_i|^3 E|X_i|^3.
\end{aligned} \tag{3.3}$$

As to R_2, letting $Z \sim N(0,1)$ be independent of V_θ we have

$$
\begin{aligned}
|R_2| &= |P(Z \le x/V_\theta) - P(Z \le x)| \\
&\le |P(Z \le x/V_\theta, |V_\theta^2 - 1| \le 1/2) - P(Z \le x, |V_\theta^2 - 1| \le 1/2)| \\
&\quad + |P(Z \le x/V_\theta, |V_\theta^2 - 1| > 1/2) - P(Z \le x, |V_\theta^2 - 1| > 1/2)| \\
&\le |P(Z \le x/V_\theta, |V_\theta^2 - 1| \le 1/2) - P(Z \le x, |V_\theta^2 - 1| \le 1/2)| \\
&\quad + P(|V_\theta^2 - 1| > 1/2) \\
&\le |P(Z \le x/V_\theta, |V_\theta^2 - 1| \le 1/2) - P(Z \le x, |V_\theta^2 - 1| \le 1/2)| \\
&\quad + 2E|V_\theta^2 - 1| I\{|V_\theta^2 - 1| > 1/2\} \\
&:= R_3 + 2E|V_\theta^2 - 1| I\{|V_\theta^2 - 1| > 1/2\},
\end{aligned}
$$

where

$$R_3 = |E\Big((\Phi(x/V_\theta) - \Phi(x)) I\{|V_\theta^2 - 1| \le 1/2\}\Big)|.$$

By monotonicity, it is easy to see that

$$\frac{|(1+x)^{-1/2} - 1 + x/2|}{x^2} \le c_0 := 4\sqrt{2} - 5 \tag{3.4}$$

for $|x| \le 1/2$. Hence, assuming $|V_\theta^2 - 1| \le 1/2$

$$1/V_\theta = (1 + V_\theta^2 - 1)^{-1/2} = 1 - (1/2)(V_\theta^2 - 1) + \gamma_1 (V_\theta^2 - 1)^2$$

with $|\gamma_1| \le c_0$. A Taylor expansion of Φ yields

$$
\begin{aligned}
&\Phi(x/V_\theta) - \Phi(x) \\
&\quad = x\phi(x)(1/V_\theta - 1) + (1/2)x^2 (1/V_\theta - 1)^2 \phi'(x\gamma_2) \\
&\quad = x\phi(x)\Big\{ -(1/2)(V_\theta^2 - 1) + \gamma_1 (V_\theta^2 - 1)^2 \Big\} \\
&\qquad + (1/2)x^2 \phi'(x\gamma_2) \frac{(V_\theta^2 - 1)^2}{(V_\theta (V_\theta + 1))^2}
\end{aligned}
$$

where $(2/3)^{1/2} \le \gamma_2 \le \sqrt{2}$ whenever $|V_\theta^2 - 1| \le 1/2$. Let

$$c_1 = \sup_{x \in \mathbb{R}} |x\phi(x)| = \frac{1}{\sqrt{2\pi}} e^{-1/2} \le 0.24198$$

and

$$
\begin{aligned}
&\sup_x \sup_{(2/3)^{1/2} \le \gamma_2 \le 2^{1/2}} |x^2 \phi'(x\gamma_2)| \\
&= \sup_x \sup_{(2/3)^{1/2} \le \gamma_2 \le 2^{1/2}} |x^3 \gamma_2 \phi(x\gamma_2)| \\
&= \sup_x \sup_{(2/3)^{1/2} \le \gamma_2 \le 2^{1/2}} \gamma_2^{-2} |x\gamma_2|^3 \phi(x\gamma_2) \\
&\le \frac{3}{2} \sup_x |x|^3 \phi(x) = \frac{3(3/e)^{3/2}}{2\sqrt{2\pi}} = c_2 \le 0.6939.
\end{aligned}
$$

Since $E(V_\theta^2-1)=0$, we have

$$\begin{aligned}
R_3 &= |E\Big\{\Big(x\phi(x)\Big\{-(1/2)(V_\theta^2-1)+\gamma_1(V_\theta^2-1)^2\Big\}\\
&\quad +(1/2)x^2\phi'(x\gamma_2)\frac{(V_\theta^2-1)^2}{(V_\theta(V_\theta+1))^2}\Big)I\{|V_\theta^2-1|\le 1/2\}\Big\}|\\
&= |(1/2)x\phi(x)E(V_\theta^2-1)I\{|V_\theta^2-1|>1/2\}\\
&\quad +x\phi(x)\gamma_1E(V_\theta^2-1)^2I\{|V_\theta^2-1|\le 1/2\}\\
&\quad +(1/2)x^2\phi'(x\gamma_2)E\Big\{\frac{(V_\theta^2-1)^2}{(V_\theta(V_\theta+1))^2}I\{|V_\theta^2-1|\le 1/2\}\Big\}|\\
&\le (1/2)c_1E|V_\theta^2-1|I\{|V_\theta^2-1|>1/2\}\\
&\quad +(c_0c_1+(1/2)c_2c_3)E(V_\theta^2-1)^2I\{|V_\theta^2-1|\le 1/2\}\\
&= (1/2)c_1E|V_\theta^2-1|I\{|V_\theta^2-1|>1/2\}\\
&\quad +c_4E(V_\theta^2-1)^2I\{|V_\theta^2-1|\le 1/2\}
\end{aligned}$$

where

$$c_3=\Big(\frac{1}{2^{-1/2}(1+2^{-1/2})}\Big)^2 \quad \text{and} \quad c_4=c_0c_1+\frac{1}{2}c_2c_3\le 0.4.$$

Collecting the bounds above yields

$$\begin{aligned}
&|P(W\le x)-P(Z\le x)|\\
&\le (4+c_1/2)E|V_\theta^2-1|I\{|V_\theta^2-1|>1/2\}\\
&\quad +c_4E(V_\theta^2-1)^2I\{|V_\theta^2-1|\le 1/2\}+2\sum_{i=1}^n|\theta_i|^3\,E|X_i|^3\\
&\le 4.2E|V_\theta^2-1|I\{|V_\theta^2-1|>1/2\}\\
&\quad +0.4E(V_\theta^2-1)^2I\{|V_\theta^2-1|\le 1/2\}+2\sum_{i=1}^n|\theta_i|^3\,E|X_i|^3
\end{aligned}$$

as desired.

Lastly, (1.7) follows from (1.6) and the fact that

$$\begin{aligned}
&4.2E|V_\theta^2-1|I\{|V_\theta^2-1|>1/2\}+0.4E(V_\theta^2-1)^2I\{|V_\theta^2-1|\le 1/2\}\\
\le\ &8.4E(V_\theta^2-1)^2I\{|V_\theta^2-1|>1/2\}+0.4E(V_\theta^2-1)^2I\{|V_\theta^2-1|\le 1/2\}\\
\le\ &8.4E(V_\theta^2-1)^2.
\end{aligned}$$

□

Proof of Theorem 2.1. Let $\mathbf{Y}$ be distributed as $\mathscr{C}_{p,F}^n$. With $r=1/p$ for convenience, first we claim that

$$n^{-2r}=O(v_n^2) \quad \text{where} \quad v_n^2=\mathrm{Var}(Y_1), \tag{3.5}$$

where the implicit constant in the order here, and below, may depend on p and F. For $r\ge 1/2$ Jensen's inequality yields

$$v_n^2=\mathrm{Var}(\varepsilon_1(\frac{G_1}{G_{1,n}})^r)=E\left(\frac{G_1}{G_{1,n}}\right)^{2r}\ge\left(E\left(\frac{G_1}{G_{1,n}}\right)\right)^{2r}=n^{-2r}. \tag{3.6}$$

For $0 < r < 1/2$, we apply the following exponential inequality for non-negative independent random variables (see, for example, Theorem 2.19 in [7]): *For $\xi_i, 1 \le i \le n$, independent non-negative random variables with $a := \sum_{i=1}^n E\xi_i$ and $b^2 := \sum_{i=1}^n E\xi_i^2 < \infty$, and any $0 < x < a$,*

$$P(\sum_{i=1}^n \xi_i \le x) \le \exp\Big(-\frac{(a-x)^2}{2b^2}\Big). \tag{3.7}$$

Let $c = E(G_1)/(2(EG_1 + 2\sqrt{\mathrm{Var}(G_1)}))$. Observe that

$$\begin{aligned} E\left(\frac{G_1}{G_{1,n}}\right)^{2r} &\ge E\left(\frac{G_1}{G_{1,n}} I\{G_1/G_{1,n} \ge c/n\}\right)^{2r} \\ &\ge (c/n)^{2r} P(G_1/G_{1,n} \ge c/n) \end{aligned}$$

and

$$\begin{aligned} &P(G_1/G_{1,n} \ge c/n) \\ &\quad\ge P((n-1)G_1 \ge c(G_{1,n} - G_1)) \\ &\quad\ge P(G_1 \ge E(G_1)/2, G_{1,n} - G_1 \le (n-1)(EG_1 + 2\sqrt{\mathrm{Var}(G_1)})) \\ &\quad= \Big(1 - P(G_1 < E(G_1)/2)\Big)\Big(1 - P(G_{1,n} - G_1 > (n-1)(EG_1 + 2\sqrt{\mathrm{Var}(G_1)}))\Big) \\ &\quad\ge \Big(1 - \exp(-(EG_1)^2/(8EG_1^2))\Big)(1 - \frac{1}{4(n-1)}), \end{aligned}$$

obtaining the final inequality by applying (3.7) with $n = 1$ to the first factor and Chebyshev's inequality to the second. This proves (3.5).

As $\mathscr{C}_{p,F}^n$ is coordinate symmetric with exchangeable coordinates, we apply Theorem 1.1 with $v_\theta = v_n$ as in (3.6), and claim that it suffices to show

$$E(G_1/G_{1,n})^{3r} = O(n^{-3r}) \tag{3.8}$$

and

$$E(V_\theta^2 - 1)^2 = O(\sum_{i=1}^n \theta_i^4). \tag{3.9}$$

In particular, regarding the second term in (1.7), we have by (3.5) and (3.8)

$$\sum_{i=1}^n |\theta_i|^3 E|X_i|^3 = v_n^{-3} \sum_{i=1}^n |\theta_i|^3 E(\frac{G_i}{G_{1,n}})^{3r} = O(\sum_{i=1}^n |\theta_i|^3),$$

which dominates (3.9), the order of the first term in (1.7), thus yielding the theorem.

Letting $\mu = EG_1$, the main idea is to use (i) that $G_{1,n}/n$ is close to μ with probability one by the law of large numbers; and (ii) the Taylor expansions

$$(1+x)^{-2r} = 1 - 2rx + \gamma_1 x^2 \quad \text{for } x > -1/2 \tag{3.10}$$

and

$$(1+x)^{-2r} = 1 + \gamma_2 x \quad \text{for } x > -1/2 \tag{3.11}$$

where $|\gamma_1| \le r(2r+1)2^{2r+2}$ and $|\gamma_2| \le r2^{2r+2}$.

We first show that

$$(n\mu)^{2r}E(G_1/G_{1,n})^{2r} = EG_1^{2r} + O(n^{-1}). \tag{3.12}$$

Let $\Delta_n = (G_{1,n} - n\mu)/(n\mu)$ and write

$$G_{1,n} = n\mu(1+\Delta_n).$$

Then

$$\begin{aligned}(n\mu)^{2r}E(G_1/G_{1,n})^{2r} &= (n\mu)^{2r}E(G_1/G_{1,n})^{2r}I\{G_{1,n} \le n\mu/2\}\\ &\quad +(n\mu)^{2r}E(G_1/G_{1,n})^{2r}I\{G_{1,n} > n\mu/2\}\\ &= (n\mu)^{2r}E(G_1/G_{1,n})^{2r}I\{G_{1,n} \le n\mu/2\}\\ &\quad +E(G_1^{2r}(1+\Delta_n)^{-2r}I\{\Delta_n > -1/2\}\\ &:= R_4 + R_5. \end{aligned} \tag{3.13}$$

By (3.7), we have

$$P(G_{1,n} \le n\mu/2) \le \exp\Big(-\frac{(n\mu/2)^2}{2nEG_1^2}\Big) = \exp\Big(-\frac{n\mu^2}{8EG_1^2}\Big) \tag{3.14}$$

and hence

$$R_4 \le (n\mu)^{2r}P(G_{1,n} \le n\mu/2) = O(n^{-2}). \tag{3.15}$$

By (3.10),

$$\begin{aligned} R_5 &= E(G_1^{2r}(1-2r\Delta_n+\gamma_1\Delta_n^2)I\{\Delta_n > -1/2\}\\ &= E(G_1^{2r}(1-2r\Delta_n) - E(G_1^{2r}(1-2r\Delta_n)I\{\Delta_n \le -1/2\}\\ &\quad +EG_1^{2r}\gamma_1\Delta_n^2 I\{\Delta_n > -1/2\}\\ &= EG_1^{2r} - 2rEG_1^{2r}(G_{1,n}-n\mu)/(n\mu) - E(G_1^{2r}(1-2r\Delta_n)I\{\Delta_n \le -1/2\}\\ &\quad +EG_1^{2r}\gamma_1\Delta_n^2 I\{\Delta_n > -1/2\}\\ &= EG_1^{2r} - 2rEG_1^{2r}(G_1-\mu)/(n\mu) + R_{5,1}, \end{aligned} \tag{3.16}$$

where

$$R_{5,1} = -E(G_1^{2r}(1-2r\Delta_n)I\{\Delta_n \le -1/2\} + EG_1^{2r}\gamma_1\Delta_n^2 I\{\Delta_n > -1/2\}.$$

Applying Hölder's inequality to the first term in $R_{5,1}$, and that $\Delta_n \ge -1$, yields

$$|R_{5,1}| \le E(G_1^{2r}(1+2r)I\{\Delta_n \le -1/2\} + O(1)EG_1^{2r}\Delta_n^2 \tag{3.17}$$

$$\begin{aligned} &\le (1+2r)\Big(EG_1^{2r(2r+2)/(2r+1)}\Big)^{(2r+1)/(2+2r)}P^{1/(2+2r)}(\Delta_n \le -1/2)\\ &\quad +O(1)EG_1^{2r}\Delta_n^2\\ &= O(1)P^{1/(2+2r)}(\Delta_n \le -1/2)\\ &\quad +O(1)(n\mu)^{-2}\Big(EG_1^{2r}(G_1-\mu)^2 + EG_1^{2r}E(\sum_{i=2}^{n}(G_i-\mu)^2\Big)\\ &= O(n^{-1}), \end{aligned} \tag{3.18}$$

proving (3.12) by (3.13) and (3.14) - (3.18).

As to (3.8), again applying (3.14), we have

$$\begin{aligned} n^{3r}E(G_1/G_{1,n})^{3r} &= n^{3r}E(G_1/G_{1,n})^{3r}I\{G_{1,n} \le n\mu/2\} \\ &\quad + n^{3r}E(G_1/G_{1,n})^{3r}I\{G_{1,n} > n\mu/2\} \\ &\le n^{3r}P(G_{1,n} \le n\mu/2) + EG_1^{3r}/(\mu/2)^{3r} \\ &= O(1). \end{aligned}$$

Now, to prove (3.9), write

$$V_\theta^2 - 1 = (V_\theta^2 - 1)I\{G_{1,n} \le n\mu/2\} + (V_\theta^2 - 1)I\{G_{1,n} > n\mu/2\}. \tag{3.19}$$

Note that $V_\theta^2 = O(n^{2r})$ by (3.5). Similarly to (3.15), by (3.14) again

$$E(V_\theta^2 - 1)^2 I\{G_{1,n} \le n\mu/2\} = O(n^{4r})P(G_{1,n} \le n\mu/2) = O(n^{-1}). \tag{3.20}$$

For the next term in (3.19) observe that

$$\begin{aligned} &(V_\theta^2 - 1)I\{G_{1,n} > n\mu/2\} \qquad (3.21) \\ &= \frac{I\{\Delta_n > -1/2\}}{v_\theta^2}\Big\{G_{1,n}^{-2r}\sum_{i=1}^n \theta_i^2 G_i^{2r} - v_\theta^2\Big\} \\ &= \frac{I\{\Delta_n > -1/2\}(n\mu)^{-2r}}{v_\theta^2}\Big\{(1+\Delta_n)^{-2r}\sum_{i=1}^n \theta_i^2 G_i^{2r} - (n\mu)^{2r}E(G_1/G_{1,n})^{2r}\Big\} \\ &= \frac{I\{\Delta_n > -1/2\}(n\mu)^{-2r}}{v_\theta^2}\Big\{(1+\gamma_2\Delta_n)\sum_{i=1}^n \theta_i^2 G_i^{2r} - EG_1^{2r} + O(n^{-1})\Big\} \\ &\qquad\qquad [\text{by (3.11) and (3.12)}] \\ &= \frac{I\{\Delta_n > -1/2\}(n\mu)^{-2r}}{v_\theta^2}\Big\{\sum_{i=1}^n \theta_i^2 (G_i^{2r} - EG_i^{2r}) \\ &\quad + \gamma_2\Delta_n \sum_{i=1}^n \theta_i^2 G_i^{2r} + O(n^{-1})\Big\} \\ &:= R_6 + R_7 + R_8, \qquad (3.22) \end{aligned}$$

where

$$\begin{aligned} R_6 &= \frac{I\{\Delta_n > -1/2\}(n\mu)^{-2r}}{v_\theta^2}\sum_{i=1}^n \theta_i^2 (G_i^{2r} - EG_i^{2r}), \\ R_7 &= \gamma_2 \frac{I\{\Delta_n > -1/2\}(n\mu)^{-2r}}{v_\theta^2}\Delta_n \sum_{i=1}^n \theta_i^2 G_i^{2r}, \\ R_8 &= \frac{I\{\Delta_n > -1/2\}(n\mu)^{-2r}}{v_\theta^2} O(n^{-1}). \end{aligned}$$

From (3.12) it follows that

$$\frac{(n\mu)^{-2r}}{v_\theta^2} = O(1). \tag{3.23}$$

Hence

$$\begin{aligned} ER_6^2 &= O(1)E\Big\{\sum_{i=1}^n \theta_i^2(G_i^{2r} - EG_i^{2r})\Big\}^2 \\ &= O(1)E(G_1^{2r} - EG_1^{2r})^2\sum_{i=1}^n \theta_i^4 \\ &= O(1)\sum_{i=1}^n \theta_i^4. \end{aligned} \tag{3.24}$$

As to R_7, here using the assumption that $EG_1^{2+4r} < \infty$, using the Cauchy-Schwarz inequality for the second step, we have

$$\begin{aligned} ER_7^2 &= O(1)E\Delta_n^2(\sum_{i=1}^n \theta_i^2 G_i^{2r})^2 \\ &= O(1)E\Delta_n^2\sum_{i=1}^n \theta_i^2 G_i^{4r}\sum_{i=1}^n \theta_i^2 \\ &= O(1)E\Delta_n^2 G_1^{4r} = O(n^{-2})EG_1^{4r}(\sum_{i=1}^n (G_i-\mu))^2 \\ &= O(n^{-2})E\Big(G_1^{4r}\Big(\sum_{i=1}^n (G_i-\mu)^2 + 2\sum_{1\le i<j\le n}(G_i-\mu)(G_j-\mu)\Big)\Big) \\ &= O(n^{-2})E\Big(G_1^{4r}\Big(\sum_{i=1}^n (G_i-\mu)^2\Big) \\ &= O(n^{-1}). \end{aligned} \tag{3.25}$$

Lastly, for R_8, by (3.23) we have

$$ER_8^2 = O(n^{-2}). \tag{3.26}$$

Noting that (1.4) with $s = t = 2$ yields $n^{-1} \le \sum_{i=1}^n \theta_i^4$, (3.9) now follows by (3.19), (3.20), (3.22), and (3.24)-(3.26).

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