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THE BEST BOUNDS IN A THEOREM OF RUSSELL LYONS

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Abstract
We sharpen a bound in a theorem of Russell Lyons [4] for percolation on a tree and associated random walk.

1 Introduction

Consider a Bernoulli percolation on a tree which assigns to every edge σ a survival probability p_σ. Russell Lyons [1] associates to this percolation an electric network defined as follows: if σ is an edge, denote by A_σ the set of edges connecting σ to the root, including σ. Then put on σ a resistance R_σ such that

$$1+\sum_{\rho\in A_\sigma} R_\rho = \prod_{\rho\in A_\sigma} p_\rho^{-1}$$

Note that if we construct an augmented tree by adding to the original tree a vertex E that is only connected to the root via a unit resistance, the total resistance between E and σ is exactly $\prod_{\rho\in A_\sigma} p_\rho^{-1}$.

It is well-known that one can define a random walk associated to the electric network on the augmented tree (see [3] for instance). If we let this random walk start from the root, the probability that it reaches the boundary of the tree before visiting E is $C/(1+C)$, where C denotes the total conductance of the network between the root and the boundary (see [4]). If we compare with the probability that a ray survives the percolation, we have the following result:

Theorem 1 (Lyons)

$$\frac{C}{1+C} \le P \text{ (a ray survives the percolation)} \le 2\frac{C}{1+C}$$

For a proof, see [1] or [4]. The lower bound is sharp (it corresponds to the case of a tree where each vertex has at most one child). Our aim is to prove the following refinement for the upper bound:

Theorem 2 *We have:*

$$P\ (\text{a ray survives the percolation}) \leq 1 - e^{-2C}$$

and this inequality is optimal.

As in [4], the inequalities are established by induction for finite trees. The key difference with [4] is that the induction is based on the number of vertices rather than the height of the tree. The remainder of this note is organized as follows: after recalling some basic definition, we present a short proof of the lower bound in Theorem 1, then we establish the upper bound in Theorem 2 and finally we show the optimality.

2 Proof of the Theorem

2.1 Preliminaries

We start by recalling some standard definitions related to the subject.
A tree is a connected, acyclic graph with a distinguished vertex called the root. A ray is a maximal self-avoiding sequence of adjacent vertices starting from the root, and the set of rays is called the boundary of the tree. If the tree is finite, the boundary can be identified as the set of leaves, i.e. the set of vertices having no child. We say that a random walk on a tree reaches the boundary if it visits all vertices of some ray.
A Bernoulli percolation on the tree is obtained by keeping or removing each edge σ independently with probability p_σ and $1 - p_\sigma$ respectively.
For the link between random walks and electric networks, we refer to Doyle and Snell [3].

2.2 The lower bound

We prove the lower bound of Theorem 1 using a simple method. As pointed out by Lyons, it suffices to prove the inequality for finite trees and the general result follows by monotone convergence. So we proceed by induction on the number of vertices of the tree. We have to prove:

$$C \leq \frac{P}{1-P} \tag{1}$$

where C denotes the total conductance and P the percolation probability. When the tree has two vertices, the inequality (1) is in fact an equality. Assume now that the result is established for $1, 2, \ldots, n$ vertices. To show it for $n+1$ vertices, we shall distinguish two cases.
First, suppose that the root has at least two children. Then we can split the tree into two subtrees whose sole common vertex is the root, and (1) holds for both subtrees by induction. Let C_1 (resp. C_2, C) denote the total conductance of the first subtree (resp. the second subtree, the tree) and P_1 (resp P_2, P) its percolation probability. Then:

$$C = C_1 + C_2 \leq \frac{P_1}{1-P_1} + \frac{P_2}{1-P_2} \leq \frac{1-(1-P_1)(1-P_2)}{(1-P_1)(1-P_2)} = \frac{P}{1-P}$$

We now treat the case when the root has only one child. Consider the tree obtained by removing the root and keeping the same survival probability for each edge. Then we get the electric network associated to this new tree by multiplying all the resistances of the first tree by p, where p is the survival probability of the edge adjacent to the root. Let C_1 (resp. C) denote the total conductance of the little tree (resp. the total tree) and P_1 (resp. P) denote its percolation probability. The induction formula (1) yields:

$$C_1 \leq \frac{P_1}{1-P_1}$$

On the other hand,

$$\frac{1}{C} = \frac{1}{pC_1} + \frac{1-p}{p}$$

We deduce:

$$C \leq \frac{1}{1/p - 1 + (1-P_1)/pP_1} = \frac{pP_1}{1-pP_1} = \frac{P}{1-P}$$

2.3 The upper bound

Next, we show the majoration of Theorem 2. As in the preceding proof, we consider only finite trees and proceed by induction on the number of vertices. We also distinguish the two cases. First suppose that the root has at least two children. Then with the same notations as above, we get immediately:

$$P = 1 - (1-P_1)(1-P_2) \leq 1 - e^{-2(C_1+C_2)} = 1 - e^{-2C}$$

If the root has only one child, the induction formula yields:

$$P \leq p(1 - e^{-2C_1}) \tag{2}$$

But we also have:

$$C_1 = \frac{C}{p(1+C) - C}$$

Hence (2) becomes:

$$P \leq p\left(1 - \exp\left(\frac{-2C}{p(1+C)-C}\right)\right)$$

It suffices now to establish the following inequality for every $C \geq 0$ and every $p \in [C/(1+C), 1]$:

$$p\left(1 - \exp\left(\frac{-2C}{p(1+C)-C}\right)\right) \leq 1 - e^{-2C} \tag{3}$$

Fix $C \geq 0$ and define the function f by:

$$f(p) = 1 - \exp(-2C) - p\left(1 - \exp\left(\frac{-2C}{p(1+C)-C}\right)\right)$$

Then $f(1) = 0$,

$$f'(1) = -1 + \exp(-2C) + 2C(1+C)\exp(-2C) \leq 0$$

because $e^{2C} \geq 1 + 2C + 2C^2$, and finally the second derivative equals

$$f''(p) = \frac{4C^3(1+C)\exp(-2C/(p(1+C)-C))}{(p(1+C)-C)^4} \geq 0$$

so that the function is convex. Thus $f(p) \geq 0$ for $p \in [C/(1+C), 1]$, which proves (3), and the upper bound of Theorem 2 is established.

2.4 Optimality

It remains to prove that the upper bound is sharp. To this end, fix a conductance C and consider the critical branching process Z with binomial offspring distribution $B(2, 1/2)$ and initial population 1. Its variance σ^2 equals $1/2$. Put $t = 1/C\sigma^2$. It is well-known (see for instance Athreya-Ney [1], p. 19) that:

$$P_n = P(Z_{[nt]} > 0) \sim \frac{2}{[nt]\sigma^2} \sim \frac{2}{nt\sigma^2} = \frac{2C}{n}$$

Consider now a branching process $Y^{(n)}$ with the same offspring distribution and initial population n. Then:

$$Q_n = P(Y^{(n)}_{[nt]} > 0) = 1 - (1 - P_n)^n \to 1 - e^{-2C}$$

This branching process is associated to a Bernoulli percolation on a tree of height $[nt] + 1$ where the root has n children, any other vertex has 2 children, the survival probability is 1 for the edges at height 1 and $1/2$ for the other edges. The percolation probability of this tree is Q_n. As the resistance of an edge is $2^{m-1} - 2^{m-2}$ at level $m \geq 2$ and zero at level 1, straightforward calculations show that the total conductance is $(nt/[nt])C$. Since $Q_n \to 1 - e^{-2C}$ and $(nt/[nt])C \to C$ as n goes to infinity, we see that our majoration is sharp.

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