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THE VOLUME FRACTION OF A NON–OVERLAPPING GERM–GRAIN MODEL

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Abstract
We discuss the volume fraction of a model of non–overlapping convex grains. It is obtained from thinning a Poisson process where each point has a weight and is the centre of a grain, by removing any grain that is overlapped by one of larger or equal weight. In the limit as the intensity of the Poisson process tends to infinity, the model can be identified with the intact grains in the dead leaves model if the weights are independent of the grain sizes. In this case we can show that the volume fraction is at most $1/2^d$ for $d = 1$ or 2 if the shape is fixed, but the size and the orientation are random. The upper bound is achieved for centrally symmetric sets of the same size and orientation. For general d we can show the upper bound, $1/2^d$, for spherical grains with two–point radius distribution. If dependence between weight and size is allowed, it is possible to achieve a volume fraction arbitrarily close to one.

1 Introduction

The model considered in this paper is a non–overlapping germ–grain model, which is a generalisation of one of Matérns hard–core models in [6]. It was proposed by Månsson and Rudemo in [5]. The model is constructed by generating a Poisson process in $\mathbb{R}^d$ and letting each point be the centre of a grain. The sizes and orientations of the grains are random and each grain is given a weight which may depend on its size. The process is thinned by rejecting any grain that intersects with another grain that has equal or higher weight. In [5] the intensity and size

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distribution of the grains after thinning for this model were given. Furthermore, the asymptotic value of the volume fraction as the intensity before thinning tends to infinity was derived in the case of fixed-sized grains. One result is that centrally symmetric sets of equal size give the volume fraction $1/2^d$.

The aim of the present paper is to study the asymptotic volume fraction, namely if fixed-sized grains give the highest volume fraction in the case where the weights are independent of the grain size and if it is possible to choose weights so that the volume fraction can become arbitrarily close to 1. We believe that $1/2^d$ is an upper limit for the volume fraction in $\mathbb{R}^d$ for any d if the weights are independent of the grain sizes. However we can only show it in general for $d = 1$ or 2 and for spherical grains with two–point distribution for any d. Furthermore, we show that it is possible to achieve a volume fraction arbitrarily close to one by a particular choice of radius distribution and weights depending on the radii.

If the weight distribution is continuous and the intensity tends to infinity, the grains kept in our model are the same as the intact grains in Matheron's dead leaves model, [7]. It can be defined as follows. Consider a stationary Poisson process $\{(x_i, t_i)\}$ with unit intensity in $\mathbb{R}^d \times (-\infty, 0]$. Interpret t_i as the arrival time of the point $x_i \in \mathbb{R}^d$. Let d–dimensional, possibly random, compact grains be implanted at the points x_i sequentially in time, so that a new grain deletes portions of the "older" ones. At time $t = 0$ the space $\mathbb{R}^d$ is completely occupied, and the grains which are not completely deleted constitute a tessellation of $\mathbb{R}^d$.

The grains which are intact, that is not intersected by any later grains, constitute a model of non-intersecting grains. The intact grains can also be considered as the limit of the generalisation of Matérn's hard-core model under study here. Let the weights be continuously distributed on $(-\infty, 0]$, independent of each other and of the radii. Then the weights can be identified with the time coordinate in the description of the dead leaves model given above. The connection between Matérn's hard-core model and the dead leaves model in the case of fixed-sized spheres was noted by Stoyan and Schlater [10]. The dead leaves model and generalisations of it, for instance the colour dead leaves, are studied in a number of papers by Jeulin, see e.g. [4]. Results on the intensity and size distribution of the intact grains can be found in [3].

When the intensity of the Poisson process tends to infinity and the grains are spherical an alternative formulation of our model, which is related to the description of the dead leaves model above, can be found in [2]. Consider a $(d+1)$–dimensional space $\mathbb{R}^d \times \mathbb{R}^+$ where $\mathbb{R}^+$ is a time dimension. Each point in a Poisson process in this space is the centre of a sphere in $\mathbb{R}^d$ which is tried to be added to the model and the final coordinate represents the time of the trial. A sphere has radius $R(t)$ at time t. A sphere is not added if it overlaps with any sphere with smaller value of t regardless of whether this sphere was rejected or not. The only difference from the formulation in [5] is that the sizes of the spheres are not random. Large times corresponds to small weights in our model and the function $R(t)$ is similar to weights depending deterministically on the radius.

Obviously a volume fraction of one is impossible to achieve. However, Gilbert, [2], proves that the volume fraction can be made arbitrarily close to one by choosing the function $R(t)$ carefully. One choice is

$$R(t) = \left(1 + \frac{a(d+a)t}{A}\right)^{1/(d+a)}, \tag{1.1}$$

where a and $A > 0$ are constants and $|a| < 1$. Volume fractions close to one are achievable if A and $|a|$ are small. If a is negative, in addition $A/|a|$ needs to be large. Here we will give an alternative proof of the achievability of volume fractions close to one, based on a "separation of size" argument somewhat reminiscent of the construction of Meester, Roy and Sarkar, [8], to demonstrate the nonuniversality of critical volume fractions in the so–called Boolean model of continuum percolation.

The paper is outlined as follows. In Section 2 we give a detailed description of the model with spherical grains and show that it is stochastically increasing in the intensity of the Poisson process if the weight distribution is independent of the radius. In Section 3 we discuss the volume fraction when the intensity of the Poisson process tends to infinity and the weight distribution is independent of the radius. Our alternative proof that the volume fraction can be made arbitratily close to one if the weight distribution is dependent of the radius is given in Section 4. The use of more general convex sets in place of spheres is considered in Section 5.

2 Model

For simplicity we give the description of the model for spherical grains, but the generalisation to convex grains is obvious. In Section 5 we give the counterpart to (2.2) for convex grains. The model is constructed by thinning a marked Poisson process, also known as a Boolean model, with proposal intensity λ_{pr} in $\mathbb{R}^d$. Each point in the Poisson process is given two marks. One of the marks is the radius of a sphere centred at the point and the other mark is a weight that is allowed to depend on the radius. Points are assigned radii independently and according to a proposal radius distribution F_{pr}. The radii are independent of the point process. Weights are also assigned independently of the point process but to stress the possible dependence on radius, the weight distribution is denoted $F_{W|r}$. A point is kept in the thinning only if its sphere does not intersect any other sphere with equal or higher weight. Note that the radii of the spheres are no longer independent after thinning. One way of quantifying the dependence is by the mark–correlation function, see [1]. Some further notation is needed. Let κ_d be the volume of the unit sphere in $\mathbb{R}^d$ and define $\bar{F}(x) = \mathbb{P}(X \geq x)$ for a random variable X with distribution function F.

In Sections 3 and 4 we will need some properties of the model, primarily the volume fraction ρ. For a stationary model with intensity λ and non-overlapping grains of random size it can be written as the intensity times the mean volume of a typical grain $\bar{v}$, that is

$$\rho = \lambda \bar{v}. \tag{2.1}$$

One useful property is the probability that a randomly chosen point with radius r is kept when thinning, henceforth called the retention probability, which from [5] is

$$g(r) = \int_0^\infty \exp\left\{-\lambda_{pr}\kappa_d \int_0^\infty \bar{F}_{W|y}(w)(r+y)^d F_{pr}(\mathrm{d}y)\right\} F_{W|r}(\mathrm{d}w). \tag{2.2}$$

The probability of retaining a point with weight w and sphere of radius r is the same as the probability that there are no spheres that intersect it that have higher or equal weight. The centers of those spheres constitute an inhomogeneous Poisson process with expectation given in the exponent of (2.2). Also from [5] the intensity after thinning is

$$\lambda = \lambda_{pr} \int_0^\infty g(r) F_{pr}(\mathrm{d}r), \tag{2.3}$$

i.e. the intensity times the probability of retaining a random point. The distribution function of the radius of a randomly chosen sphere after thinning is

$$F(r) = 1 - \frac{\lambda_{pr}}{\lambda} \int_r^\infty g(s) F_{pr}(\mathrm{d}s). \tag{2.4}$$

Observe that λ/λ_{pr} is the probability of retaining a random point.
In the following we will mostly be concerned with the case when the intensity of the Poisson process tends to infinity. When the weight distribution is independent of radius, the intensity and the volume fraction after thinning are strictly increasing as functions of the intensity before thinning. In fact the process is increasing in the intensity before thinning as can be seen in the following theorem.

Theorem 2.1 *Consider the model with continuous weight distribution independent of the radii and let $\lambda_1 < \lambda_2$. Let X be the union of the resulting spheres for $\lambda_{pr} = \lambda_1$ and let Y be the union of the resulting spheres for $\lambda_{pr} = \lambda_2$. Then X is stochastically dominated by Y.*

Proof. We prove the theorem by a coupling argument. Take a Poisson process in $\mathbb{R}^d$ with intensity λ_2 and give each point independently a radius with distribution F_{pr}. Furthermore give each point a weight that is uniform $(0, \lambda_2)$. Let $\tilde{Y}$ consist of the spheres that are left when the thinning is performed. This process has the same distribution as Y. In the Poisson process, consider only those spheres that have weights in the interval $(\lambda_2 - \lambda_1, \lambda_2)$. The intensity of this process is λ_1 and the radius distribution is still F_{pr} because the weights are independent of the radii. Carry out the thinning and call the resulting process of spheres $\tilde{X}$. It has the same distribution as X. A sphere before thinning with weight greater than or equal to $\lambda_2 - \lambda_1$ will belong to $\tilde{Y}$ if and only if it belongs to $\tilde{X}$. A sphere with weight $\lambda_2 - \lambda_1$ will only be contained in $\tilde{Y}$. We have shown

$$\tilde{X} \subseteq \tilde{Y}$$

and hence X is stochastically dominated by Y. ■

The condition that the weight distribution is continuous is necessary in the argument above.

Example 2.1. Let the spheres have equal radii, r, and let the weights be constant. Then all spheres will be removed except those that do not intersect with any other sphere. The intensity after thinning is by using (2.2) and (2.3)

$$\lambda_{pr} \exp\{-\lambda_{pr} \kappa_d 2^d r^d\}.$$

The intensity after thinning is at most $1/(\kappa_d 2^d r^d e)$ for $\lambda_{pr} = 1/(\kappa_d 2^d r^d)$ and it tends to zero as λ_{pr} tends to infinity. □

If the weights are continuous but depend on the radii, the process is not necessarily increasing.

Example 2.2. Let the radii take value 1 or a with probabilities p and $q = 1 - p$ respectively. Let the weight distribution be uniform in $(0, 1)$ given radius 1 and let it be uniform in $(1, 2)$ given radius a. Then the intensity, by (2.2) and (2.3), is

$$\frac{1}{\kappa_d 2^d} \left\{ \exp\big(-\lambda_{pr} \kappa_d (1 + a)^d q\big)\big(1 - \exp(-\lambda_{pr} \kappa_d 2^d p)\big) + \frac{1 - \exp\big(-\lambda_{pr} \kappa_d 2^d a^d q\big)}{a^d} \right\}.$$

When λ_{pr} tends to infinity λ tends to $1/(\kappa_d 2^d a^d)$. Let $d = 2$, $a = 2$ and $p = q = 1/2$, then numerical inspection shows that the intensity has maximum approximately 0.027 for $\lambda_{pr} \approx 0.088$. The value of λ as λ_{pr} tends to infinity is $1/(16\pi) \approx 0.020$. □

Theorem 2.1 implies that the process exists in the limit as λ_{pr} tends to infinity. If the weights are allowed to depend on the radii, the limit process does not necessarily exist.

Example 2.3. Suppose we have a model with two different radii of the spheres, 1 and 2, with respective probabilities $N/(N+1)$ and $1/(N+1)$. Let N be large, $N = 100$ say, and let the weight of a sphere of radius 1 be uniform in

$$\bigcup_{i=0}^{\infty} \left(\frac{N^{2i}-1}{N^{2i}}, \frac{N^{2i+1}-1}{N^{2i+1}} \right)$$

and let the weight of a sphere of radius 2 be uniform in

$$\bigcup_{i=0}^{\infty} \left(\frac{N^{2i+1}-1}{N^{2i+1}}, \frac{N^{2i+2}-1}{N^{2i+2}} \right).$$

The limit process is not well defined since as $\lambda_{pr} \to \infty$ the process will fluctuate between consisting mostly of spheres of radius 1 and consisting mostly of spheres of radius 2.
Alternatively, the example could be formulated in the following manner. Suppose we have a model where the weight distribution is uniform in $[0, 1]$, and which has two possible radii, 1 and 2, depending on the weight w as follows. Fix N large and set the radius $r = 1$ if

$$w \in \bigcup_{i=0}^{\infty} \left(\frac{N^{2i}-1}{N^{2i}}, \frac{N^{2i+1}-1}{N^{2i+1}} \right)$$

and $r = 2$ otherwise. □

3 Volume fraction for the spherical case if the weight distribution is independent of the radius

In this section we will consider the case where the weight distribution is continuous and independent of the radii and $\lambda_{pr} \to \infty$. As noted earlier the model then coincides with the intact grains of the dead leaves model. We will show that the largest volume fraction achievable is that of the process with all radii being equal. In that case the volume fraction, as shown in [5], is 2^{-d}.

Theorem 3.1 *If the weight distribution is continuous and independent of the radii and $\lambda_{pr} \to \infty$, then, for $\mathbb{R}^d$ with $d = 1$ or 2, the volume fraction is at most*

$$\frac{1}{2^d},$$

with equality if and only if the spheres have equal radii.

Proof. First we need to find an expression for the volume fraction. From (2.2) the retention probability for fixed r, when λ_{pr} is the intensity of the Poisson process, is

$$g(r) = \frac{1 - \exp\{-\lambda_{pr}\kappa_d \mathbb{E}[(r+Y)^d]\}}{\lambda_{pr}\kappa_d \mathbb{E}[(r+Y)^d]},$$

where Y has distribution F_{pr}. By (2.4), the expectation of R^d is

$$\mathbb{E}[R^d] = \frac{\lambda_{pr}}{\lambda} \int_0^\infty r^d g(r) F_{pr}(\mathrm{d}r)$$

and hence the volume fraction is by (2.1),

$$\rho = \int_0^\infty r^d \frac{1 - \exp\{-\lambda_{pr}\kappa_d \mathbb{E}[(r+Y)^d]\}}{\mathbb{E}[(r+Y)^d]} F_{pr}(\mathrm{d}r).$$

Letting the intensity tend to infinity gives

$$\lim_{\lambda_{pr}\to\infty} \rho = \int_0^\infty \frac{r^d}{\mathbb{E}[(r+Y)^d]} F_{pr}(\mathrm{d}r). \tag{3.1}$$

If $d = 1$ the function

$$\frac{r}{r + \mathbb{E}Y}$$

is concave and we can use Jensen's inequality to deduce

$$\int_0^\infty \frac{r}{r + \mathbb{E}Y} F_{pr}(\mathrm{d}r) \leq \frac{\mathbb{E}Y}{\mathbb{E}Y + \mathbb{E}Y} = \frac{1}{2}.$$

We have equality above only if the radius is constant, since otherwise the function is strictly concave.
If $d = 2$ the function

$$f(r) = \frac{r^2}{r^2 + 2r\mathbb{E}Y + \mathbb{E}Y^2}$$

is not concave but it can be shown to lie below a tangent passing through the origin. Let $\mu = \mathbb{E}Y$ and $\gamma = \mathbb{E}Y^2$ and the equation for the tangent is

$$t(r) = \frac{r}{2(\mu + \sqrt{\gamma})}.$$

The difference between the tangent and the curve is

$$t(r) - f(r) = \frac{r(r - \sqrt{\gamma})^2}{2(\mu + \sqrt{\gamma})(r^2 + 2r\mu + \gamma)}.$$

Hence $t(r) - f(r) \geq 0$ and

$$\int_0^\infty \frac{r^2}{r^2 + 2\mu r + \gamma} F_{pr}(\mathrm{d}r) \leq \int_0^\infty \frac{r}{2(\mu + \sqrt{\gamma})} F_{pr}(\mathrm{d}r) = \frac{\mu}{2(\mu + \sqrt{\gamma})} \leq \frac{1}{4},$$

where in the last inequality we used $\gamma \geq \mu^2$. Since equality holds only for fixed radius, the volume fraction is 1/4 only if that is the case. ■

We cannot prove that the upper bound of the volume fraction is $1/2^d$ for general d. In fact the method used in the proof above gives an upper bound for the volume fraction in $d = 3$ as 4/27. This can be seen by considering the function

$$f(r) = \frac{r^3}{\mathbb{E}[(r+Y)^3]}.$$

Since $\mathbb{E}Y^3 \geq (\mathbb{E}Y)^3$ for $Y \geq 0$ we have

$$f(r) \leq \frac{r^3}{(r+\mathbb{E}Y)^3}.$$

As before this function lies below a tangent that passes through the origin. The equation of the tangent is

$$\frac{4r}{27\mu}.$$

Proposition 3.2 *For a two point radius distribution and continuous weight distribution independent of the radius in $\mathbb{R}^d$ and $\lambda_{pr} \to \infty$, the volume fraction is at most $1/2^d$. The upper bound is achieved only if the radius is fixed.*

Proof. Let the radius take value 1 with probability p and value a with probability $q = 1 - p$. From (3.1) the volume fraction as the intensity of the Poisson process tends to infinity is

$$\rho = \frac{p}{2^d p + (1+a)^d q} + \frac{a^d q}{(1+a)^d p + 2^d a^d q}.$$

Rewriting with a common divisor gives,

$$\rho = \frac{(1+a)^d p^2 + 2^{d+1} a^d pq + (1+a)^d a^d q^2}{(2^d p + (1+a)^d q)((1+a)^d p + 2^d a^d q)}.$$

By subtracting the volume fraction from $1/2^d$ we have

$$\frac{1}{2^d} - \rho = \frac{\big((1+a)^{2d} - 2^{2d} a^d\big) pq}{2^d (2^d p + (1+a)^d q)((1+a)^d p + 2^d a^d q)}.$$

It is easy to see that $a = 1$ is a root to $(1+a)^{2d} - 2^{2d} a^d = 0$. It is actually a double root and the expression is clearly 0 only for $a = 1$ and positive otherwise. ∎

Proposition 3.2 gives an indication that Theorem 3.1 holds for any d. Hence we state the following conjecture.

Conjecture 3.3 *If the weight distribution is continuous and independent of the radii and $\lambda_{pr} \to \infty$, then in $\mathbb{R}^d$ for any d, the volume fraction is at most*

$$\frac{1}{2^d},$$

attained by spheres of equal radius.

4 Volume fraction if the weight distribution depends on the radius

As can be seen in the Introduction, Gilbert [2], showed that the volume fraction can be made arbitrarily close to one by choosing the right function $R(t)$. This is similar in our view to let the weight distribution depend deterministically on the radius. We will give an alternative proof of this fact. The idea is the same in our setting as in Gilberts, namely letting the function $R(t)$ decrease in such a way that not much space is wasted. In Gilbert's case $R(t)$, see (1.1), is continuous while we have discrete radii.

Theorem 4.1 *If the weight distribution depends on the radius, it is possible to achieve a volume fraction arbitrarily close to 1 in* $\mathbb{R}^d$ *for any* d.

Proof. The theorem will be proved by considering a model with spheres having discrete radius distribution with k possible values. The weight will be proportional to the radius of the sphere. The idea is to let each size of spheres have sufficiently low intensity so that they do not overlap spheres of the same size and to let smaller spheres be so much smaller that not much space is wasted if they overlap partially with a larger sphere.
Fix small $\alpha > 0$ and $\delta > 0$. Below we will show that we can achieve a volume fraction of at least

$$1 - \alpha\kappa_d(3^d - 1) - 2\delta. \tag{4.1}$$

The volume fraction can be made arbitrarily close to one by picking α and δ small. Let the radius of a sphere before thinning take value $r_i = \epsilon^{i-1}$ with probability $p_i = \lambda_i/\lambda_{pr}$, $i = 1, \dots, k$, where λ_{pr} is the intensity of the Poisson process. Think of $\epsilon > 0$ as being small and k large. Let the weight of a sphere with radius r_i be uniform $((r_{i-1}+r_i)/2, (r_i+r_{i+1})/2)$. The intensity of spheres of radius r_i is λ_i before thinning.
The volume fraction after thinning is the same as the probability that the origin is covered after thinning and can be written

$$\begin{aligned}\rho =& 1 - \mathbb{P}(\text{The origin is not covered after thinning})\\ =& 1 - \mathbb{P}(\text{The origin is not covered before thinning})\\ & - \mathbb{P}(\text{All spheres covering the origin are deleted}).\end{aligned} \tag{4.2}$$

The number of spheres with radius r_i that covers the origin before thinning is Poisson distributed with expectation $\lambda_i\kappa_d r_i^d$ and hence

$$\mathbb{E}[\#\text{ spheres covering the origin before thinning}] = \sum_i^k \kappa_d r_i^d \lambda_i.$$

Letting $\lambda_i = \alpha/r_i^d$ the expectation becomes $k\kappa_d\alpha$. Pick k large enough so that

$$\mathbb{P}(\text{The origin is not covered before thinning}) = \exp(-k\kappa_d\alpha) \leq \delta. \tag{4.3}$$

To obtain the probability that all spheres covering the origin are deleted we assume that at least one sphere covers the origin before thinning. Let the largest of all such spheres be denoted B. In case several spheres having the same radius cover the origin we let B be the one with highest weight. If B has radius r_i, a centre of a sphere with higher weight than B, having radius $r_j \geq r_i$, that intersects B must be separated by at least a distance of r_j from the origin, otherwise we get a contradiction of the definition of B. On the other hand, the centre of B is at most a distance r_i from the origin and hence the centre of a sphere with radius r_j overlapping B cannot be further away from the origin than $2r_i + r_j$. Now we can get an upper bound for the probability that all spheres covering the origin are deleted by

$$\begin{aligned}&\mathbb{P}(\text{All spheres covering the origin are deleted})\\ &\leq \mathbb{P}(\text{A sphere with radius larger than or equal to } r_i \text{ overlaps } B)\\ &\leq \mathbb{E}[\#\text{ spheres with radius larger than or equal to } r_i \text{ overlapping } B]\\ &\leq \sum_{j=1}^{i} \mathbb{E}\begin{bmatrix}\#\text{ spheres with radius } r_j \text{ and center at}\\ \text{distance between } r_j \text{ and } 2r_i + r_j \text{ from the origin}\end{bmatrix}.\end{aligned}$$

The number of spheres with radius r_j is Poisson distributed and

$$\begin{aligned}
&\mathbb{P}(\text{All spheres covering the origin are deleted}) \\
&\le \sum_{j=1}^{i} \lambda_j \kappa_d \big((2r_i + r_j)^d - r_j^d\big) = \sum_{j=1}^{i} \alpha\kappa_d \big((2\epsilon^{i-j}+1)^d - 1\big) \\
&= \alpha\kappa_d(3^d - 1) + \alpha\kappa_d \sum_{j=1}^{i-1} \big((1+2\epsilon^{i-j})^d - 1\big).
\end{aligned} \tag{4.4}$$

We can choose a small ϵ such that, for all i simultaneously,

$$\alpha\kappa_d \sum_{j=1}^{i-1} \big((1+2\epsilon^{i-j})^d - 1\big) < \delta.$$

Insert this estimation of (4.4) together with (4.3) in (4.2) and we have shown (4.1). ■

5 Convex grains

In our model we may replace the spheres with convex sets of different sizes. We introduce a minimum of notation to prove a counterpart to Theorem 3.1 and refer to [5] for a more detailed description.
We begin with some definitions. First, $D(A)$, denotes the diameter of a set A, that is

$$D(A) = \sup_{x,y\in A} \|x - y\|.$$

We let half the diameter be called the size. Let $\mathcal{C}^d$ be the set of all convex, compact sets C in $\mathbb{R}^d$ such that the origin belongs to C and $D(C)/2 = 1$. Moreover let $C(x,r)$ be the set C translated by x and with half its diameter equal to r and let $\check{C} = \{-x : x \in C\}$ be the reflection of C in the origin. Finally we denote the Lebesgue measure in d dimensions by l_d.
In the following we will only consider $\mathbb{R}^2$ and $C \in \mathcal{C}^2$. Replacing $\kappa_2(r+y)^2$ in (2.2) with $l_2(\{x : C(o,r) \cap C(x,y) \neq \emptyset\})$ gives the retention probability for convex sets with the same shape and orientation as C. Let $\nu(C,\check{C})$ be the mixed volume of C and $\check{C}$, then

$$l_2(\{x : C(o,r) \cap C(x,y) \neq \emptyset\}) = (r^2 + y^2) l_2(C) + 2ry\nu(C,\check{C}).$$

If the sets are uniformly rotated about the origin, then $\kappa_2(r+y)^2$ should be replaced by $\mathbb{E}[l_2(\{x : C(o,r) \cap mC(x,y) \neq \emptyset\})]$, where m is a rotation matrix, i.e. orthogonal with determinant 1, and the expectation is taken with respect to an angle of rotation that is uniform $(0, 2\pi)$. Let $S_1(C)$ be the perimeter of C, then by the generalised Steiner formula

$$\mathbb{E}[l_2(\{x : C(o,r) \cap mC(x,y) \neq \emptyset\})] = (r^2 + y^2) l_2(C) + \frac{ryS_1(C)^2}{2\pi}.$$

Just as in the spherical case the maximal volume fraction, at least in $\mathbb{R}^2$, is given by grains of equal size.

Proposition 5.1 *Let the grains be convex of the same shape as $C \in \mathcal{C}^2$ and let the weight distribution be continuous and independent of the size. For grains of the same orientation and when $\lambda_{pr} \to \infty$, the volume fraction is at most*

$$\frac{l_2(C)}{2(l_2(C) + \nu(C, \check{C}))}.$$

For grains of random orientation and when $\lambda_{pr} \to \infty$, the volume fraction is at most

$$\frac{l_2(C)}{2l_2(C) + S_1(C)^2/(2\pi)}.$$

In both cases the upper bound is attained if and only if all the grains have the same size.

Proof. The volume fraction as $\lambda_{pr} \to \infty$ is deduced similar to (3.1). For convex sets of the same orientation we have volume fraction

$$\rho = \int_0^\infty \frac{r^2 l_2(C)}{\int_0^\infty \Big(r^2 + y^2)l_2(C) + 2ry\nu(C, \check{C})\Big) F_{pr}(\mathrm{d}y)} F_{pr}(\mathrm{d}r),$$

and for uniformly rotated convex sets we have volume fraction

$$\rho_{rot} = \int_0^\infty \frac{r^2 l_2(C)}{\int_0^\infty \Big((r^2 + y^2)l_2(C) + \frac{ryS_1(C)^2}{2\pi}\Big) F_{pr}(\mathrm{d}y)} F_{pr}(\mathrm{d}r).$$

In both cases we take the expectation of a function that can be written as

$$\frac{r^2}{r^2 + ar + b},$$

for some positive constants a and b. The result is shown exactly as for the $d = 2$ case in the proof of Theorem 3.1. ■

In two dimensions it is well-known that for convex C

$$l_2(C) \leq \nu(C, \check{C}) \leq 2l_2(C)$$

with equality to the left if and only if C is centrally symmetric and to the right if and only if C is a triangle. No convex set has a larger perimeter relative to its area than a circle, more precisely $S_1(C)^2 \geq l_2(C)4\pi$. By these bounds and Proposition 5.1 it follows that among all dead leaves models with convex grains of equal shape, fixed or uniformly distributed orientations, and independent random radii, the highest volume fraction results for fixed-sized centrally symmetric sets of equal orientation. In this case the volume fraction is 1/4 if $d = 2$ and we believe that the bound $1/2^d$ holds in any dimension. Finally, we generalise Conjecture 3.3 to hold among convex grains of fixed or random orientation and the upper bound is achieved for centrally symmetric sets of fixed size.

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