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SOLUTION TO A SYSTEM OF EQUATIONS MODELLING COMPRESSIBLE FLUID FLOW WITH CAPILLARY STRESS EFFECTS

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Abstract. We study the initial-value problem for a system of nonlinear equations that models the flow of a compressible fluid with capillary stress effects. The system includes hyperbolic equations for the density and for the velocity, and an algebraic equation (the equation of state) for the pressure. We prove the existence of a unique classical solution to an initial-value problem for this system of equations under periodic boundary conditions. The key to the proof is an a priori estimate for the density and velocity in a high Sobolev norm.

1. Introduction

We begin by considering a system of equations which arises from a model of the multi-dimensional flow of a compressible fluid with capillary stresses. When viscosity is neglected, the model consists of the following equations:

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\mathbf{v}$$
$$\frac{D\mathbf{v}}{Dt} = -\rho^{-1}\nabla p + c\nabla\Delta\rho$$

where ρ is the density, p is the pressure, and $\mathbf{v}$ is the velocity. Here c is a coefficient of capillarity which is a small, positive constant. The material derivative $D/Dt = \partial/\partial t + \mathbf{v}\cdot\nabla$. The term $c\nabla\Delta\rho$ is due to capillary stresses, from the theory of Korteweg-type materials described by Dunn and Serrin [5]. The fluid's thermodynamic state is determined by the density ρ, and the pressure p is then determined from the density by an equation of state $p = \hat{p}(\rho)$. A derivation of the model's equations appears in [4]. Anderson, McFadden and Wheeler [1] have reviewed related theories, as well as applications to diffuse-interface modelling. Other researchers have proven the existence of solutions to other versions of this model which include viscosity and an evolution equation for temperature (see, e.g., [2, 8, 9, 10]). To our knowledge, this system of equations for inviscid fluid flow with capillary stresses has not been previously studied.

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With the change of variables

$$\mathbf{u} = \rho\mathbf{v},$$

the system of equations becomes

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot\mathbf{u} \tag{1.1}$$

$$\begin{aligned}\frac{\partial\mathbf{u}}{\partial t} &= -\rho^{-1}\mathbf{u}\cdot\nabla\mathbf{u} + \rho^{-2}(\mathbf{u}\cdot\nabla\rho)\mathbf{u}\\ &\quad - \rho^{-1}(\nabla\cdot\mathbf{u})\mathbf{u} - \nabla p + c\rho\nabla\Delta\rho\end{aligned} \tag{1.2}$$

Let $\bar{\rho} = \rho - |\Omega|^{-1}\int_\Omega \rho d\mathbf{x}$. We assume that $\bar{\rho}$ is small. Since the capillary coefficient c is very small, we assume that $c\bar{\rho}$ is neglibly small, and we will approximate the capillary stress term as follows:

$$c\rho\nabla\Delta\rho = c\Big(\bar{\rho} + |\Omega|^{-1}\int_\Omega \rho d\mathbf{x}\Big)\nabla\Delta\rho \approx c\Big(|\Omega|^{-1}\int_\Omega \rho d\mathbf{x}\Big)\nabla\Delta\rho$$

Then using the equation of state for the pressure, we make the following approximation to equation (1.2):

$$\begin{aligned}\frac{\partial\mathbf{u}}{\partial t} &= -\rho^{-1}\mathbf{u}\cdot\nabla\mathbf{u} + \rho^{-2}(\mathbf{u}\cdot\nabla\rho)\mathbf{u} - \rho^{-1}(\nabla\cdot\mathbf{u})\mathbf{u} - p'(\rho)\nabla\rho\\ &\quad + c\left(|\Omega|^{-1}\textstyle\int_\Omega \rho d\mathbf{x}\right)\nabla\Delta\rho\end{aligned} \tag{1.3}$$

The purpose of this paper is to prove the existence of a unique classical solution $\mathbf{u}$, ρ to the initial-value problem for equations (1.1), (1.3), for $0 \le t \le T$, using periodic boundary conditions. Hence, we choose for our domain the N-dimensional torus $\mathbb{T}^N$, where $N = 2$ or $N = 3$. We will show that a unique solution exists, provided that $T\|D\mathbf{u}_0\|_s$ and $T\|\nabla\rho_0\|_{s+1}$ are sufficiently small, where $\mathbf{u}_0$, ρ_0 is the given initial data.

2. Existence theorem

In this section, we prove the existence of a unique classical solution to the initial-value problem for equations (1.1), (1.3) with periodic boundary conditions.

We will be using the Sobolev space $H^s(\Omega)$ (where $s \ge 0$ is an integer) of real-valued functions in $L^2(\Omega)$ whose distribution derivatives up to order s are in $L^2(\Omega)$, with norm given by $\|f\|_s^2 = \sum_{|\alpha|\le s}\int_\Omega |D^\alpha f|^2 d\mathbf{x}$. We use the standard multi-index notation. We will be using the standard function spaces $L^\infty([0,T], H^s(\Omega))$ and $C([0,T], H^s(\Omega))$. $L^\infty([0,T], H^s(\Omega))$ is the space of bounded measurable functions from $[0,T]$ into $H^s(\Omega)$, with the norm $\|f\|_{s,T}^2 = \operatorname{ess\,sup}_{0\le t\le T}\|f(t)\|_s^2$.

The set $C([0,T], H^s(\Omega))$ is the space of continuous functions from $[0,T]$ into $H^s(\Omega)$. We will also be using the notation $|f|_{L^\infty,T} = \operatorname{ess\,sup}_{0\le t\le T}|f(t)|_{L^\infty(\Omega)}$.

Theorem 2.1. *Let $\rho_0(\mathbf{x}) = \rho(\mathbf{x},0) \in H^{s+2}(\Omega)$, $\mathbf{u}_0(\mathbf{x}) = \mathbf{u}(\mathbf{x},0) \in H^{s+1}(\Omega)$ be the given initial data, with $s > \frac{N}{2}+1$, and $\Omega = \mathbb{T}^N$, with $N = 2$ or $N = 3$. Let max $\{|\rho_0|_{L^\infty}, |\mathbf{u}_0|_{L^\infty}\} \le L_0$, for some positive constant L_0. Let $p = \hat{p}(\rho)$ be a given equation of state for the pressure p as a function of ρ. We assume that p is a sufficiently smooth function of ρ for any $\rho \in G$, where $G \subset \mathbf{R}$ is an open set. We assume that in G, ρ is positive and $p'(\rho)$ is positive. We fix convex, bounded open sets G_0 and G_1 such that $\bar{G}_0 \subset G_1$ and $\bar{G}_1 \subset G$, and we require that the initial data satisfy $\rho_0(\mathbf{x}) \in G_0$, for all $\mathbf{x} \in \Omega$. Then the initial-value problem for* (1.1), (1.3) *with* $\Omega = \mathbb{T}^N$ *has a unique, classical solution* ρ, $\mathbf{u}$ *for* $0 \le t \le T$, *where* $\rho \in \bar{G}_1$, *and*

$$\rho \in C([0,T], C^3(\Omega)) \cap L^\infty([0,T], H^{s+2}(\Omega))$$
$$\mathbf{u} \in C([0,T], C^2(\Omega)) \cap L^\infty([0,T], H^{s+1}(\Omega))$$

provided $T\|D\mathbf{u}_0\|_s$ *and* $T\|\nabla\rho_0\|_{s+1}$ *are sufficiently small.*

Proof. The proof of the theorem is based on the method of successive approximations, in which an iteration scheme, based on solving a linearized version of the equations, is designed and convergence of the sequence of approximating solutions is sought. Convergence of the sequence is proven in two steps: first, we prove the uniform boundedness of the approximating sequence $\{\rho^k\}$, $\{\mathbf{u}^k\}$, in a high Sobolev norm, and then we prove contraction of the sequence in a low Sobolev norm. Standard compactness arguments complete the proof. □

We will construct the solution of the initial-value problem for (1.1), (1.3) with $\Omega = \mathbb{T}^N$ through the following iteration scheme. Set $\rho^0(\mathbf{x}, t) = \rho_0(\mathbf{x})$, and $\mathbf{u}^0(\mathbf{x}, t) = \mathbf{u}_0(\mathbf{x})$. For $k = 0, 1, 2, \dots.$ construct ρ^{k+1}, $\mathbf{u}^{k+1}$ from the previous iterates ρ^k, $\mathbf{u}^k$ by solving

$$\frac{\partial \rho^{k+1}}{\partial t} = -\nabla \cdot \mathbf{u}^{k+1} \tag{2.1}$$

$$\begin{aligned}\frac{\partial \mathbf{u}^{k+1}}{\partial t} &= -(\rho^k)^{-1}\mathbf{u}^k \cdot \nabla\mathbf{u}^{k+1} + (\rho^k)^{-2}\mathbf{u}^k \cdot \nabla\rho^{k+1}\mathbf{u}^k - (\rho^k)^{-1}(\nabla \cdot \mathbf{u}^{k+1})\mathbf{u}^k \\ &\quad - p'(\rho^k)\nabla\rho^{k+1} + c\Big(\frac{1}{|\Omega|}\int_\Omega \rho^k d\mathbf{x}\Big)\nabla\Delta\rho^{k+1}\end{aligned} \tag{2.2}$$

with initial data $\rho^{k+1}(\mathbf{x},0) = \rho_0(\mathbf{x})$, $\mathbf{u}^{k+1}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x})$.

Existence of a solution to equations (2.1), (2.2) for fixed k is proven in Appendix A. The a priori estimates used in the proof are proven in Appendix B. We proceed now to prove convergence of the iterates as $k \to \infty$ to a unique, classical solution of (1.1), (1.3).

Since $\rho^k(\mathbf{x}, 0) = \rho_0 \in G_0$, where $\bar{G}_0 \subset G_1$ and $\bar{G}_1 \subset G$, we fix $\delta = \hat{\delta}(G_1)$ so that if $|\rho - \rho_0|_{L^\infty,T} \le \delta$, then $\rho \in \bar{G}_1$. And we fix $c_1 = \hat{c}_1(G_1) > 0$ and $c_2 = \hat{c}_2(G_1) > 0$, where $c_1 < 1$, so that $c_1 < \rho < c_2$ and $c_1 < p'(\rho) < c_2$ for $\rho \in \bar{G}_1$.

Next, we proceed with the proof of uniform boundedness of the approximating sequence in a high Sobolev norm.

Proposition 2.2. *Assume that the hypotheses of Theorem 2.1 hold. Let* δ, R *be given positive constants. Then there are constants* L_1, L_2, *such that for* $k = 0, 1, 2, 3 \dots$ *the following estimates hold*

(a) $\|\nabla\rho^k\|_{s,T} \le L_1$, $\|\Delta\rho^k\|_{s,T} \le L_1$, $\|D\mathbf{u}^k\|_{s,T} \le L_1$;
(b) $|\rho^k - \rho_0|_{L^\infty,T} \le \delta$, $|\mathbf{u}^k - \mathbf{u}_0|_{L^\infty,T} \le R$;
(c) $\|\rho^k\|_{0,T} \le L_1$, $\|\mathbf{u}^k\|_{0,T} \le L_1$;
(d) $\|\frac{\partial\rho^k}{\partial t}\|_{s,T} \le L_2$, $\|\frac{\partial\mathbf{u}^k}{\partial t}\|_{s-1,T} \le L_2$

provided $T\|D\mathbf{u}_0\|_s$ *and* $T\|\nabla\rho_0\|_{s+1}$ *are sufficiently small.*

Proof. The proof is by induction on k, of which we show only the inductive step. We will derive estimates for ρ^{k+1} and $\mathbf{u}^{k+1}$, and then use these estimates to prescribe L_1 and L_2 a priori, independent of k, so that if ρ^k and $\mathbf{u}^k$ satisfy (a)–(d), then ρ^{k+1} and $\mathbf{u}^{k+1}$ also satisfy (a)–(d).

From the statement of the theorem, we have $\max\{|\rho_0|_{L^\infty}, |\mathbf{u}_0|_{L^\infty}\} \le L_0$, for some positive constant L_0. By the induction hypothesis, we have $|\mathbf{u}^k - \mathbf{u}_0|_{L^\infty,T} \le R$. It follows that $|\mathbf{u}^k|_{L^\infty,T} \le |\mathbf{u}_0|_{L^\infty} + R \le L_0 + R < c_3$, for some constant $c_3 > 1$ which depends on L_0 and R. Then applying Lemma B.2 from Appendix B to equations (2.1)–(2.2), where we let $\mathbf{F} = 0$ and $Q_k\mathbf{g} = 0$ in equation (B.2) of Lemma B.2, yields the estimate

$$\begin{aligned} &\|D\mathbf{u}^{k+1}\|_s^2 + \|\nabla\rho^{k+1}\|_s^2 + \|\Delta\rho^{k+1}\|_s^2 \\ &\le C_4(1 + C_4K_4Te^{C_4K_4T})(\|D\mathbf{u}_0\|_s^2 + \|\nabla\rho_0\|_{s+1}^2) \end{aligned} \tag{2.3}$$

where $C_4 = \hat{C}_4(s, c, c_1, c_2, c_3)$, where $s > \frac{N}{2} + 1$ with $N = 2$ or $N = 3$, so that $s \ge 3$, and where from Lemma B.2

$$\begin{aligned} K_4 = \max\Big\{&1,\ \|(\rho^k)^{-1}\|_{s+1,T}^2\|\mathbf{u}^k\|_{s+1,T}^2,\ \|p'(\rho^k)\|_{s+1,T}^2,\ \|(\rho^k)^{-2}\|_{s+1,T}^2\|\mathbf{u}^k\|_{s+1,T}^4, \\ &\|(\rho^k)_t^{-1}\|_{2,T}^2\|\mathbf{u}^k\|_{2,T}^2,\ \|(\rho^k)^{-1}\|_{2,T}^2\|(\mathbf{u}^k)_t\|_{2,T}^2,\ \|(\rho^k)_t\|_{2,T},\ \|(p'(\rho^k))_t\|_{2,T}\Big\} \end{aligned}$$

We estimate $K_4 \le C_6$, where the constant $C_6 = \hat{C}_6(c_1, L_1, L_2)$, by the induction hypothesis. Then after using this estimate for K_4 in equation (2.3), we obtain

$$\begin{aligned} &\|D\mathbf{u}^{k+1}\|_s^2 + \|\nabla\rho^{k+1}\|_s^2 + \|\Delta\rho^{k+1}\|_s^2 \\ &\le C_4(1 + C_4C_6Te^{C_4C_6T})(\|D\mathbf{u}_0\|_s^2 + \|\nabla\rho_0\|_{s+1}^2) \\ &= (C_4 + C_7Te^{C_7T})(\|D\mathbf{u}_0\|_s^2 + \|\nabla\rho_0\|_{s+1}^2) \end{aligned} \tag{2.4}$$

where $C_7 = \hat{C}_7(s, c, c_1, c_2, c_3, L_1, L_2)$. Recall that C_4 does not depend on L_1 or L_2. Therefore, it follows that

$$\|D\mathbf{u}^{k+1}\|_{s,T}^2 + \|\nabla\rho^{k+1}\|_{s,T}^2 + \|\Delta\rho^{k+1}\|_{s,T}^2 \le L_1^2$$

provided that we choose L_1 large enough so that

$$\frac{L_1^2}{2} \ge C_4(\|D\mathbf{u}_0\|_s^2 + \|\nabla\rho_0\|_{s+1}^2), \tag{2.5}$$

and provided that $T\|D\mathbf{u}_0\|_s$ and $T\|\nabla\rho_0\|_{s+1}$ are sufficiently small so that

$$C_7Te^{C_7T}(\|D\mathbf{u}_0\|_s^2 + \|\nabla\rho_0\|_{s+1}^2) \le \frac{L_1^2}{2}. \tag{2.6}$$

Thus, either the time interval $0 \le t \le T$ is chosen to be sufficiently small, or the norms of the initial gradients, $\|D\mathbf{u}_0\|_s$ and $\|\nabla\rho_0\|_{s+1}$, are sufficiently small, or both are small. This completes the proof of part (a).

Next, from (2.1) for ρ^{k+1}, we have

$$\begin{aligned} |\rho^{k+1} - \rho_0| &\le \int_0^t |\rho_t^{k+1}|_{L^\infty}d\tau \le C\int_0^T \|\nabla\cdot\mathbf{u}^{k+1}\|_s dt \\ &\le CT\|D\mathbf{u}^{k+1}\|_{s,T} \\ &\le CT\Big((C_4 + C_7Te^{C_7T})(\|D\mathbf{u}_0\|_s^2 + \|\nabla\rho_0\|_{s+1}^2)\Big)^{1/2} \end{aligned} \tag{2.7}$$

Similarly, from equation (2.2), we obtain

$$\begin{aligned} |\mathbf{u}^{k+1}-\mathbf{u}_0| &\le \int_0^t |\mathbf{u}_t^{k+1}|_{L^\infty} d\tau \\ &\le C\int_0^T \|(\rho^k)^{-1}\|_{s-1}\|\mathbf{u}^k\|_{s-1}\|D\mathbf{u}^{k+1}\|_{s-1} d\tau \\ &\quad + C\int_0^T \|(\rho^k)^{-2}\|_{s-1}\|\mathbf{u}^k\|_{s-1}^2\|\nabla\rho^{k+1}\|_{s-1} d\tau \\ &\quad + C\int_0^T \|(\rho^k)^{-1}\|_{s-1}\|\mathbf{u}^k\|_{s-1}\|\nabla\cdot\mathbf{u}^{k+1}\|_{s-1} d\tau \\ &\quad + C\int_0^T \|p'(\rho^k)\|_{s-1}\|\nabla\rho^{k+1}\|_{s-1} d\tau \\ &\quad + C\int_0^T \|c\Big(\frac{1}{|\Omega|}\int_\Omega \rho^k d\mathbf{x}\Big)\nabla\Delta\rho^{k+1}\|_{s-1} d\tau \\ &\le C_8T(\|D\mathbf{u}^{k+1}\|_{s,T}+\|\nabla\rho^{k+1}\|_{s,T}+\|\Delta\rho^{k+1}\|_{s,T}) \\ &\le 3C_8T\Big((C_4+C_7Te^{C_7T})(\|D\mathbf{u}_0\|_s^2+\|\nabla\rho_0\|_{s+1}^2)\Big)^{1/2} \end{aligned} \tag{2.8}$$

where $C_8=\hat{C}_8(s,c,c_1,c_2,L_1)$. It follows from (2.7), (2.8) that

$$|\rho^{k+1}-\rho_0|_{L^\infty,T}\le\delta,$$
$$|\mathbf{u}^{k+1}-\mathbf{u}_0|_{L^\infty,T}\le R$$

provided that $T\|D\mathbf{u}_0\|_s$ and $T\|\nabla\rho_0\|_{s+1}$ are small enough to satisfy

$$CT\Big((C_4+C_7Te^{C_7T})(\|D\mathbf{u}_0\|_s^2+\|\nabla\rho_0\|_{s+1}^2)\Big)^{1/2}\le\delta \tag{2.9}$$

and provided that $T\|D\mathbf{u}_0\|_s$ and $T\|\nabla\rho_0\|_{s+1}$ are small enough to satisfy

$$3C_8T\Big((C_4+C_7Te^{C_7T})(\|D\mathbf{u}_0\|_s^2+\|\nabla\rho_0\|_{s+1}^2)\Big)^{1/2}\le R \tag{2.10}$$

This completes the proof of part (b).

Using the fact that max $\{|\rho_0|_{L^\infty},|\mathbf{u}_0|_{L^\infty}\}\le L_0$, and the result just obtained for part (b), it follows that $|\rho^{k+1}|_{L^\infty,T}\le|\rho_0|_{L^\infty}+\delta\le L_0+\delta$ and $|\mathbf{u}^{k+1}|_{L^\infty,T}\le|\mathbf{u}_0|_{L^\infty}+R\le L_0+R$. Therefore, we have

$$\|\rho^{k+1}\|_{0,T}\le|\Omega|^{1/2}|\rho^{k+1}|_{L^\infty,T}\le|\Omega|^{1/2}(L_0+\delta)\le L_1$$

and

$$\|\mathbf{u}^{k+1}\|_{0,T}\le|\Omega|^{1/2}|\mathbf{u}^{k+1}|_{L^\infty,T}\le|\Omega|^{1/2}(L_0+R)\le L_1$$

provided that we choose L_1 large enough so that

$$L_1\ge|\Omega|^{1/2}(L_0+\delta) \tag{2.11}$$

and we choose L_1 large enough so that

$$L_1\ge|\Omega|^{1/2}(L_0+R) \tag{2.12}$$

This completes the proof of part (c). Since $\|\nabla\rho^k\|_{s+1,T}^2\le C\|\Delta\rho^k\|_{s,T}^2$ when $\Omega=\mathbb{T}^N$ (a proof appears in [3]), it follows from parts (a) and (c) that $\rho^{k+1}\in L^\infty([0,T],H^{s+2})$.

Finally, using equations (2.1), (2.2), and using the results just obtained in parts (a) and (c), we can directly estimate

$$\|\rho_t^{k+1}\|_{s,T} \le C_9, \quad \|\mathbf{u}_t^{k+1}\|_{s-1,T} \le C_{10}$$

where $C_9 = \hat{C}_9(s, L_1)$ and $C_{10} = \hat{C}_{10}(s, c, c_1, L_1)$. Therefore, $\|\rho_t^{k+1}\|_{s,T} \le L_2$ and $\|\mathbf{u}_t^{k+1}\|_{s-1,T} \le L_2$ provided we choose L_2 large enough so that

$$L_2 \ge C_9, \quad L_2 \ge C_{10} \tag{2.13}$$

This completes the proof of part (d).

Summarizing, if we fix L_1, L_2, a priori and independent of k, so that (2.5), (2.6), (2.9), (2.10), (2.11), (2.12), (2.13) are satisfied, then ρ^k and $\mathbf{u}^k$ satisfy (a)–(d) for all $k \ge 0$. This completes the proof. □

Next, we give the proof of contraction in low norm.

Proposition 2.3. *Assume that the hypotheses of Theorem 2.1 hold. Then it follows that*

$$\sum_{k=1}^{\infty} \big(\|\rho^{k+1} - \rho^k\|_{3,T}^2 + \|\mathbf{u}^{k+1} - \mathbf{u}^k\|_{2,T}^2\big) < \infty$$

Proof. Subtracting (2.1), (2.2) for ρ^k, $\mathbf{u}^k$ from (2.1), (2.2) for ρ^{k+1}, $\mathbf{u}^{k+1}$ yields

$$\frac{\partial(\rho^{k+1} - \rho^k)}{\partial t} = -\nabla \cdot (\mathbf{u}^{k+1} - \mathbf{u}^k), \tag{2.14}$$

$$\begin{aligned} \frac{\partial(\mathbf{u}^{k+1} - \mathbf{u}^k)}{\partial t} &= -(\rho^k)^{-1}\mathbf{u}^k \cdot \nabla(\mathbf{u}^{k+1} - \mathbf{u}^k) + (\rho^k)^{-2}\mathbf{u}^k \cdot \nabla(\rho^{k+1} - \rho^k)\mathbf{u}^k \\ &\quad - (\rho^k)^{-1}(\nabla \cdot (\mathbf{u}^{k+1} - \mathbf{u}^k))\mathbf{u}^k - p'(\rho^k)\nabla(\rho^{k+1} - \rho^k) \\ &\quad + c\Big(|\Omega|^{-1} \int_\Omega \rho^k d\mathbf{x}\Big)\nabla\Delta(\rho^{k+1} - \rho^k) + \mathbf{F} \end{aligned} \tag{2.15}$$

where $(\rho^{k+1} - \rho^k)(\mathbf{x}, 0) = 0$, and $(\mathbf{u}^{k+1} - \mathbf{u}^k)(\mathbf{x}, 0) = 0$, and where

$$\begin{aligned} \mathbf{F} &= -((\rho^k)^{-1}\mathbf{u}^k - (\rho^{k-1})^{-1}\mathbf{u}^{k-1}) \cdot \nabla\mathbf{u}^k \\ &\quad + (((\rho^k)^{-2}\mathbf{u}^k - (\rho^{k-1})^{-2}\mathbf{u}^{k-1}) \cdot \nabla\rho^k)\mathbf{u}^k + (\rho^{k-1})^{-2}(\mathbf{u}^{k-1} \cdot \nabla\rho^k)(\mathbf{u}^k - \mathbf{u}^{k-1}) \\ &\quad - (\nabla \cdot \mathbf{u}^k)((\rho^k)^{-1}\mathbf{u}^k - (\rho^{k-1})^{-1}\mathbf{u}^{k-1}) - (p'(\rho^k) - p'(\rho^{k-1}))\nabla\rho^k \\ &\quad + c\Big(|\Omega|^{-1} \int_\Omega (\rho^k - \rho^{k-1}) d\mathbf{x}\Big)\nabla\Delta\rho^k \end{aligned}$$

From Lemma B.2 in Appendix B, using $r = 1$, where we let $Q_k\mathbf{g} = 0$ in equation (B.2) of Lemma B.2, we obtain the following inequality

$$\|D(\mathbf{u}^{k+1} - \mathbf{u}^k)\|_1^2 + \|\nabla(\rho^{k+1} - \rho^k)\|_1^2 + \|\Delta(\rho^{k+1} - \rho^k)\|_1^2 \le C_{11} \int_0^t \|\mathbf{F}\|_2^2 d\tau \tag{2.16}$$

where $C_{11} = \hat{C}_{11}(c, c_1, c_2, c_3, L_1, L_2, T)$, and where we have used the results from Proposition 2.2.

From Lemma B.2 in Appendix B, where we let $Q_k\mathbf{g} = 0$ in equation (B.2) of Lemma B.2, and using the results from Proposition 2.2, we obtain the L^2 estimate

$$\begin{aligned} &\|\mathbf{u}^{k+1} - \mathbf{u}^k\|_0^2 + \|\rho^{k+1} - \rho^k\|_0^2 + \|\nabla(\rho^{k+1} - \rho^k)\|_0^2 \\ &\le C_{12} \int_0^t (\|D(\mathbf{u}^{k+1} - \mathbf{u}^k)\|_0^2 + \|\mathbf{F}\|_0^2) d\tau \end{aligned} \tag{2.17}$$

where $C_{12} = \hat{C}_{12}(c, c_1, c_2, c_3, L_1, L_2, T)$. After adding (2.16), (2.17), and putting additional terms on the right-hand side, we obtain

$$
\begin{aligned}
&\|\mathbf{u}^{k+1} - \mathbf{u}^k\|_0^2 + \|\rho^{k+1} - \rho^k\|_0^2 + \|\nabla(\rho^{k+1} - \rho^k)\|_0^2 \\
&+ \|D(\mathbf{u}^{k+1} - \mathbf{u}^k)\|_1^2 + \|\nabla(\rho^{k+1} - \rho^k)\|_1^2 + \|\Delta(\rho^{k+1} - \rho^k)\|_1^2 \\
&\le C_{13} \int_0^t (\|\mathbf{u}^{k+1} - \mathbf{u}^k\|_0^2 + \|\rho^{k+1} - \rho^k\|_0^2 + \|\nabla(\rho^{k+1} - \rho^k)\|_0^2) d\tau \\
&\quad + C_{13} \int_0^t (\|D(\mathbf{u}^{k+1} - \mathbf{u}^k)\|_1^2 + \|\nabla(\rho^{k+1} - \rho^k)\|_1^2) d\tau \\
&\quad + C_{13} \int_0^t (\|\Delta(\rho^{k+1} - \rho^k)\|_1^2 + \|\mathbf{F}\|_2^2) d\tau
\end{aligned} \tag{2.18}
$$

where $C_{13} = \hat{C}_{13}(c, c_1, c_2, c_3, L_1, L_2, T)$. From the definition of $\mathbf{F}$, and using Proposition 2.2, we obtain the estimate

$$
\|\mathbf{F}\|_2^2 \le C_{14}(\|\mathbf{u}^k - \mathbf{u}^{k-1}\|_2^2 + \|\rho^k - \rho^{k-1}\|_2^2) \tag{2.19}
$$

where $C_{14} = \hat{C}_{14}(c, c_1, L_1)$. Here, we used the fact that $s > \frac{N}{2} + 1$, so that $s \ge 3$, and we used the Sobolev inequality $|f|_{L^\infty} \le C\|f\|_{s_0}$ (see, e.g., [3], [6]), where $s_0 = [\frac{N}{2}] + 1 = 2$, when we estimated the term

$$
\|c\big(|\Omega|^{-1} \int_\Omega (\rho^k - \rho^{k-1}) d\mathbf{x}\big) \nabla\Delta\rho^k\|_2^2 \le c|\rho^k - \rho^{k-1}|_{L^\infty}^2 \|\nabla\Delta\rho^k\|_2^2 \le CL_1^2 \|\rho^k - \rho^{k-1}\|_2^2
$$

in the definition of $\mathbf{F}$. Applying Gronwall's inequality to (2.18), and using (2.19), yields

$$
\begin{aligned}
&\|\mathbf{u}^{k+1} - \mathbf{u}^k\|_0^2 + \|\rho^{k+1} - \rho^k\|_0^2 + \|\nabla(\rho^{k+1} - \rho^k)\|_0^2 \\
&+ \|D(\mathbf{u}^{k+1} - \mathbf{u}^k)\|_1^2 + \|\nabla(\rho^{k+1} - \rho^k)\|_1^2 + \|\Delta(\rho^{k+1} - \rho^k)\|_1^2 \\
&\le C_{15} \int_0^t \|\mathbf{F}\|_2^2 d\tau \\
&\le C_{16} \int_0^t (\|\rho^k - \rho^{k-1}\|_2^2 + \|\mathbf{u}^k - \mathbf{u}^{k-1}\|_2^2) d\tau
\end{aligned} \tag{2.20}
$$

where $C_{15} = \hat{C}_{15}(c, c_1, c_2, c_3, L_1, L_2, T)$, $C_{16} = \hat{C}_{16}(c, c_1, c_2, c_3, L_1, L_2, T)$. It follows that

$$
\|\rho^{k+1} - \rho^k\|_3^2 + \|\mathbf{u}^{k+1} - \mathbf{u}^k\|_2^2 \le C_{17} \int_0^t (\|\rho^k - \rho^{k-1}\|_3^2 + \|\mathbf{u}^k - \mathbf{u}^{k-1}\|_2^2) d\tau \tag{2.21}
$$

where $C_{17} = \hat{C}_{17}(c, c_1, c_2, c_3, L_1, L_2, T)$. Here we used the fact that $\|\nabla(\rho^{k+1} - \rho^k)\|_2^2 \le C\|\Delta(\rho^{k+1} - \rho^k)\|_1^2$ when $|\Omega| = \mathbb{T}^N$ (a proof appears in [3]).

Repeatedly applying (2.21) yields

$$
\|\rho^{k+1} - \rho^k\|_{3,T}^2 + \|\mathbf{u}^{k+1} - \mathbf{u}^k\|_{2,T}^2 \le \frac{(C_{17}T)^k}{k!} (\|\rho^1 - \rho^0\|_{3,T}^2 + \|\mathbf{u}^1 - \mathbf{u}^0\|_{2,T}^2)
$$

It follows that

$$
\sum_{k=1}^\infty \big(\|\rho^{k+1} - \rho^k\|_{3,T}^2 + \|\mathbf{u}^{k+1} - \mathbf{u}^k\|_{2,T}^2\big) < \infty
$$

This completes the proof. □

Using Propositions 2.2 and 2.3, we now complete the proof of Theorem 2.1 by using a standard argument (see, for example, [6], [11]). From Proposition 2.3, we conclude that there exist $\rho \in C([0,T], H^3(\Omega))$, and $\mathbf{u} \in C([0,T], H^2(\Omega))$ so that $\|\rho^k - \rho\|_{3,T} \to 0$, and $\|\mathbf{u}^k - \mathbf{u}\|_{2,T} \to 0$ as $k \to \infty$. Using the standard interpolation inequalities (see, e.g., [6])

$$\|\rho^{k+1} - \rho^k\|_{s'+2} \leq C\|\rho^{k+1} - \rho^k\|_3^{\beta}\|\rho^{k+1} - \rho^k\|_{s+2}^{1-\beta}$$
$$\|\mathbf{u}^{k+1} - \mathbf{u}^k\|_{s'+1} \leq C\|\mathbf{u}^{k+1} - \mathbf{u}^k\|_2^{\beta}\|\mathbf{u}^{k+1} - \mathbf{u}^k\|_{s+1}^{1-\beta}$$

with $\beta = \frac{s-s'}{s-1}$, and Propositions 2.2 and 2.3, we can conclude that $\|\rho^k - \rho\|_{s'+2,T} \to 0$, and $\|\mathbf{u}^k - \mathbf{u}\|_{s'+1,T} \to 0$ as $k \to \infty$ for any $s' < s$. For $s' > \frac{N}{2} + 1$, Sobolev's lemma implies that $\rho^k \to \rho$ in $C([0,T], C^3(\Omega))$, and $\mathbf{u}^k \to \mathbf{u}$ in $C([0,T], C^2(\Omega))$. From the linear system of equations (2.1), (2.2) it follows that $\|\rho_t^k - \rho_t\|_{s',T} \to 0$, and $\|\mathbf{u}_t^k - \mathbf{u}_t\|_{s'-1,T} \to 0$ as $k \to \infty$, so that $\rho_t^k \to \rho_t \in C([0,T], C^1(\Omega))$, and $\mathbf{u}_t^k \to \mathbf{u}_t$ in $C([0,T], C(\Omega))$, and ρ, $\mathbf{u}$ is a classical solution of the system of equations (1.1), (1.3).

The additional facts that $\rho \in L^\infty([0,T], H^{s+2}(\Omega))$, $\mathbf{u} \in L^\infty([0,T], H^{s+1}(\Omega))$, can be deduced from the uniform boundedness of $\{\rho^k\}$ in $L^\infty([0,T], H^{s+2}(\Omega))$ and of $\{\mathbf{u}^k\}$ in $L^\infty([0,T], H^{s+1}(\Omega))$ from Proposition 2.2, and from the weak-* compactness of bounded sets in $L^\infty([0,T], H^r(\Omega))$, i.e., by Alaoglu's theorem (see, for example, [6], [11]). The uniqueness of the solution follows by a standard proof, using estimates similar to the proof of Proposition 2.3.

Appendix A. Existence for the linear problem

We now present a proof of the existence of a classical solution ρ, $\mathbf{u}$ to the linear equations (2.1), (2.2):

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{u} \tag{A.1}$$

$$\begin{aligned}\frac{\partial \mathbf{u}}{\partial t} &= -a_1^{-1}\mathbf{v} \cdot \nabla \mathbf{u} - a_1^{-1}(\nabla \cdot \mathbf{u})\mathbf{v} + a_1^{-2}(\mathbf{v} \cdot \nabla \rho)\mathbf{v} - a_2 \nabla \rho \\ &\quad + c\Big(|\Omega|^{-1} \int_\Omega a_1 d\mathbf{x}\Big)\nabla \Delta \rho\end{aligned} \tag{A.2}$$

Lemma A.1. *Given*

$$\mathbf{v} \in C([0,T], H^0(\Omega)) \cap L^\infty([0,T], H^{s+1}(\Omega)),$$
$$a_1 \in C([0,T], H^0(\Omega)) \cap L^\infty([0,T], H^{s+2}(\Omega)),$$
$$a_2 \in C([0,T], H^0(\Omega)) \cap L^\infty([0,T], H^{s+2}(\Omega)),$$
$$\mathbf{v}_t \in L^\infty([0,T], H^{s-1}(\Omega)),$$
$$(a_1)_t,\ (a_2)_t \in L^\infty([0,T], H^s(\Omega)),$$

where $s > \frac{N}{2} + 1$, $\Omega = \mathbb{T}^N$, *with* $N = 2$ *or* $N = 3$, *and where* $0 < c_1 < a_1(\mathbf{x},t) < c_2$, $0 < c_1 < a_2(\mathbf{x},t) < c_2$, *and* $|\mathbf{v}(\mathbf{x}, t)| < c_3$ *for some constants* c_1, c_2, c_3, *with* $c_1 < 1$, $c_3 > 1$ *and* $0 \leq t \leq T$, *there is a classical solution* ρ, $\mathbf{u}$ *of the initial value problem for* (A.1), (A.2), *with initial data* $\rho(\mathbf{x}, 0) = \rho_0(\mathbf{x}) \in H^{s+2}(\Omega)$, $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \in H^{s+1}(\Omega)$, *and*

$$\rho \in C([0,T], C^3(\Omega)) \cap L^\infty([0,T], H^{s+2}(\Omega)),$$
$$\mathbf{u} \in C([0,T], C^2(\Omega)) \cap L^\infty([0,T], H^{s+1}(\Omega)).$$

Proof. Since we are solving the initial-value problem under periodic boundary conditions, we will use Galerkin's method, with the standard orthonormal basis in L^2 of trigonometric functions $\{w_i\}_{i=1}^{\infty}$, to construct the solution. Here w_i has the form $cos(2\pi\mathbf{n}_i \cdot \mathbf{x})$ or $sin(2\pi\mathbf{n}_i \cdot \mathbf{x})$ with $\mathbf{n}_i \in \mathbb{Z}_+^N$. The proof by Galerkin's method is a standard one, and is included here for the sake of completeness.

We will write the system of equations (A.1), (A.2) equivalently as follows:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{u}, \tag{A.3}$$

$$\begin{aligned}\frac{\partial u_i}{\partial t} &= -a_1^{-1}\mathbf{v} \cdot \nabla u_i - a_1^{-1}(\nabla \cdot \mathbf{u})v_i + a_1^{-2}(\mathbf{v} \cdot \nabla \rho)v_i \\ &\quad - a_2\frac{\partial \rho}{\partial x_i} + c\Big(|\Omega|^{-1}\int_\Omega a_1 d\mathbf{x}\Big)\frac{\partial}{\partial x_i}(\Delta \rho),\end{aligned} \tag{A.4}$$

where $i = 1, \dots, N$. Here u_i is the ith component of the vector $\mathbf{u}$ and v_i is the ith component of the vector $\mathbf{v}$.

Let P_k denote the orthogonal projection of L^2 onto the finite dimensional subspace $V_k = \text{span}\{w_1, \dots, w_k\}$. The finite-dimensional approximation $\rho^k \in V_k$ and $u_i^k \in V_k$, where u_i^k is the ith component of $\mathbf{u}^k$, is the solution of the equations

$$\frac{\partial \rho^k}{\partial t} = -\nabla \cdot \mathbf{u}^k, \tag{A.5}$$

$$\begin{aligned}\frac{\partial u_i^k}{\partial t} &= -P_k(a_1^{-1}\mathbf{v} \cdot \nabla u_i^k) - P_k(a_1^{-1}(\nabla \cdot \mathbf{u}^k)v_i) + P_k(a_1^{-2}(\mathbf{v} \cdot \nabla \rho^k)v_i) \\ &\quad - P_k\Big(a_2\frac{\partial \rho^k}{\partial x_i}\Big) + P_k\Big(c\Big(|\Omega|^{-1}\int_\Omega a_1 d\mathbf{x}\Big)\frac{\partial}{\partial x_i}(\Delta \rho^k)\Big),\end{aligned} \tag{A.6}$$

with $\rho^k(\mathbf{x},0) = P_k\rho(\mathbf{x}, 0)$, and $u_i^k(\mathbf{x},0) = P_k u_i(\mathbf{x}, 0)$, for $i = 1, \dots, N$.

Because $\rho^k \in V_k$ and $u_i^k \in V_k$, we can write

$$\rho^k = \sum_{j=1}^{k} \alpha_j(t) w_j, \tag{A.7}$$

$$u_i^k = \sum_{j=1}^{k} \gamma_{i,j}(t) w_j. \tag{A.8}$$

After substituting (A.7), (A.8) into (A.5) and (A.6) we take the L^2 inner product of (A.5) and (A.6) with w_l for $l = 1, \dots, k$, which transforms (A.5) and (A.6) into the following equivalent linear system of ordinary differential equations for the coefficients $\alpha_l(t)$ and $\gamma_{i,l}(t)$, where $i = 1, \dots, N$, and $l = 1, \dots, k$:

$$\frac{d\alpha_l}{dt} = -\sum_{j=1}^{k}(\sum_{m=1}^{N} \gamma_{m,j}(t)\frac{\partial w_j}{\partial x_m}, w_l),$$

$$\begin{aligned}\frac{d\gamma_{i,l}}{dt} &= -\sum_{j=1}^{k}\Big((a_1^{-1}\mathbf{v} \cdot \nabla w_j, w_l)\gamma_{i,j}(t) - (a_1^{-1}(\sum_{m=1}^{N}\gamma_{m,j}(t)\frac{\partial w_j}{\partial x_m})v_i, w_l)\Big) \\ &\quad + \sum_{j=1}^{k}\Big((a_1^{-2}(\mathbf{v} \cdot \nabla w_j)v_i, w_l)\alpha_j(t) - (a_2\frac{\partial w_j}{\partial x_i}, w_l)\alpha_j(t)\Big)\end{aligned}$$

$$+\sum_{j=1}^{k}\Big(\Big(c\Big(|\Omega|^{-1}\int_\Omega a_1 d\mathbf{x}\Big)\frac{\partial}{\partial x_i}(\Delta w_j), w_l\Big)\Big)\alpha_j(t)\Big).$$

Also $\alpha_l(0) = (\rho(\mathbf{x}, 0), w_l)$, and $\gamma_{i,l}(0) = (u_i(\mathbf{x}, 0), w_l)$.

The coefficients in this system of equations are continuous, and it has a unique solution $\{\alpha_l(t)\}_{l=1}^k \in C^1([0,T])$ and $\{\gamma_{i,l}(t)\}_{l=1}^k \in C^1([0,T])$, for $i = 1,\dots,N$. It follows that $\rho^k \in C^1([0,T], H^r(\Omega))$ and $u_i^k \in C^1([0,T], H^r(\Omega))$ for any $r \geq 0$.

Next, we obtain estimates for ρ^k, $\mathbf{u}^k$ in high Sobolev norm. Let $Q_k = I - P_k$, where I is the identity operator. Then we write (A.5), (A.6) equivalently as follows:

$$\frac{\partial \rho^k}{\partial t} = -\nabla\cdot\mathbf{u}^k \tag{A.9}$$

$$\begin{aligned}\frac{\partial \mathbf{u}^k}{\partial t} &= -a_1^{-1}\mathbf{v}\cdot\nabla\mathbf{u}^k - a_1^{-1}(\nabla\cdot\mathbf{u}^k)\mathbf{v} + a_1^{-2}(\mathbf{v}\cdot\nabla\rho^k)\mathbf{v} - a_2\nabla\rho^k\\ &\quad + c\Big(|\Omega|^{-1}\int_\Omega a_1 d\mathbf{x}\Big)\nabla\Delta\rho^k - Q_k\mathbf{g}\end{aligned} \tag{A.10}$$

where

$$Q_k\mathbf{g} = -Q_k(a_1^{-1}\mathbf{v}\cdot\nabla\mathbf{u^k}) - Q_k(a_1^{-1}(\nabla\cdot\mathbf{u}^k)\mathbf{v}) + Q_k(a_1^{-2}(\mathbf{v}\cdot\nabla\rho^k)\mathbf{v}) - Q_k(a_2\nabla\rho^k)$$

Note that by the orthogonality of the projections P_k and Q_k, we have $(Q_k\mathbf{g}, \mathbf{u}^k) = 0$, $(\nabla\cdot(Q_k\mathbf{g})_\alpha, \nabla\cdot\mathbf{u}^k_\alpha) = 0$, and $(\nabla\times(Q_k\mathbf{g})_\alpha, \nabla\times\mathbf{u}^k_\alpha) = 0$ for $|\alpha| \geq 0$. Also, note that $Q_k(c\left(|\Omega|^{-1}\int_\Omega a_1 d\mathbf{x}\right)\nabla\Delta\rho^k) = 0$. Then applying Lemma B.2 in Appendix B to equations (A.9), (A.10) yields the following estimates

$$\|D\mathbf{u}^k\|_s^2 + \|\nabla\rho^k\|_s^2 + \|\Delta\rho^k\|_s^2 \leq C_4(1 + C_4K_4Te^{C_4K_4T})(\|D\mathbf{u}_0\|_s^2 + \|\nabla\rho_0\|_{s+1}^2) \tag{A.11}$$

and

$$\begin{aligned}&\|\mathbf{u}^k\|_0^2 + \|\rho^k\|_0^2 + \|\nabla\rho^k\|_0^2\\ &\leq C_5(1 + C_5K_4Te^{C_5K_4T})(\|\mathbf{u}_0\|_0^2 + \|\rho_0\|_0^2 + \|\nabla\rho_0\|_0^2)\\ &\quad + C_5(1 + C_5K_4Te^{C_5K_4T})\int_0^t \|D\mathbf{u}^k\|_0^2 d\tau\\ &\leq C_5(1 + C_5K_4Te^{C_5K_4T})(\|\mathbf{u}_0\|_0^2 + \|\rho_0\|_0^2 + \|\nabla\rho_0\|_0^2)\\ &\quad + C_5(1 + C_5K_4Te^{C_5K_4T})TC_4(1 + C_4K_4Te^{C_4K_4T})(\|D\mathbf{u}_0\|_s^2 + \|\nabla\rho_0\|_{s+1}^2)\end{aligned} \tag{A.12}$$

where the constants C_4, C_5, K_4 are defined in Lemma B.2. Here, we used the fact that $\|P_k\rho_0\|_r \leq \|\rho_0\|_r$ and $\|P_k\mathbf{u}_0\|_r \leq \|\mathbf{u}_0\|_r$. And we used estimate (A.11) in the right-hand side of estimate (A.12).

From (A.11), (A.12) it follows that $\{\rho^k\}$ is bounded in $L^\infty([0,T], H^{s+2}(\Omega))$ and $\{\mathbf{u}^k\}$ is bounded in $L^\infty([0,T], H^{s+1}(\Omega))$. Here we used the fact that $\|\nabla\rho^k\|_{s+1,T}^2 \leq C\|\Delta\rho^k\|_{s,T}^2$ when $\Omega = \mathbb{T}^N$ (a proof appears in [3]). From equations (A.9), (A.10), it follows that $\|\rho_t^k\|_0$ and $\|\mathbf{u}_t^k\|_0$ are bounded for all $k \geq 1$. Here we used the fact that $\|Q_k\mathbf{g}\|_0 \leq \|\mathbf{g}\|_0$. It follows that $\{\rho^k\}$ and $\{\mathbf{u}^k\}$ are bounded and equicontinuous in $C([0,T], H^0(\Omega))$. Using the Arzela-Ascoli theorem together with the weak-* compactness of bounded sets in $L^\infty([0,T], H^r(\Omega))$, it follows that there exist subsequences ρ^{k_j} of ρ^k and $\mathbf{u}^{k_j}$ of $\mathbf{u}^k$, and there exist functions $\rho \in C([0,T], H^0(\Omega)) \cap L^\infty([0,T], H^{s+2}(\Omega))$, $\mathbf{u} \in C([0,T], H^0(\Omega)) \cap L^\infty([0,T], H^{s+1}(\Omega))$, such that as $j \to \infty$,

$$\rho^{k_j} \to \rho \quad \text{strongly in } C([0,T], H^0(\Omega)),$$

$$\rho^{k_j} \to \rho \quad \text{weak-* in } L^\infty([0,T], H^{s+2}(\Omega)),$$
$$\mathbf{u}^{k_j} \to \mathbf{u} \quad \text{strongly in } C([0,T], H^0(\Omega)),$$
$$\mathbf{u}^{k_j} \to \mathbf{u} \quad \text{weak-* in } L^\infty([0,T], H^{s+1}(\Omega))$$

Using the standard interpolation inequalities (see, e.g., [6]),

$$\|\mathbf{u}^{k_{j+1}} - \mathbf{u}^{k_j}\|_{s'+1} \le C\|\mathbf{u}^{k_{j+1}} - \mathbf{u}^{k_j}\|_0^{\theta_1}\|\mathbf{u}^{k_{j+1}} - \mathbf{u}^{k_j}\|_{s+1}^{1-\theta_1}$$
$$\|\rho^{k_{j+1}} - \rho^{k_j}\|_{s'+2} \le C\|\rho^{k_{j+1}} - \rho^{k_j}\|_0^{\theta_2}\|\rho^{k_{j+1}} - \rho^{k_j}\|_{s+2}^{1-\theta_2}$$

with $\theta_1 = \frac{s-s'}{s+1}$, $\theta_2 = \frac{s-s'}{s+2}$, it follows that $\rho^{k_j} \to \rho$ in $C([0,T], H^{s'+2}(\Omega))$ and $\mathbf{u}^{k_j} \to \mathbf{u}$ in $C([0,T], H^{s'+1}(\Omega))$ for any $s' < s$.

From applying the Lebesgue dominated convergence theorem to equations (A.9), (A.10) and using a standard argument (see, for example, Embid [6] and Majda [11]), it follows that ρ, $\mathbf{u}$ is a classical solution of (A.1), (A.2). □

Appendix B. A priori estimates

To obtain a priori estimates, we will be using the Sobolev space $H^s(\Omega)$ (where $s \ge 0$ is an integer) of real-valued functions in $L^2(\Omega)$ whose distribution derivatives up to order s are in $L^2(\Omega)$, with norm given by $\|f\|_s^2 = \sum_{|\alpha|\le s}\int_\Omega |D^\alpha f|^2 d\mathbf{x}$. We use the standard multi-index notation. For convenience, we will be denoting derivatives by $f_\alpha = D^\alpha f$. And we will be letting Df denote the gradient of f. In addition, we will be denoting the L^2 inner product by $(f,g) = \int_\Omega f\cdot g \, d\mathbf{x}$. We will also be using the notation $|f|_{L^\infty,T} = \operatorname{ess\,sup}_{0\le t\le T}|f(t)|_{L^\infty(\Omega)}$. The following lemmas will yield the a priori estimates needed for the proof of Theorem 2.1.

Lemma B.1 (Low-Norm Commutator Estimate)**.** *If $Df \in H^{r_1}(\Omega)$, $g \in H^{r-1}(\Omega)$, where $r_1 = \max\{r-1, s_0\}$, $s_0 = [\frac{N}{2}] + 1$, then for any $r \ge 1$, f, g satisfy the estimate $\|D^\alpha(fg) - fD^\alpha g\|_0 \le C\|Df\|_{r_1}\|g\|_{r-1}$, where $r = |\alpha|$, and the constant C depends on r, Ω.*

The proof of the above lemma is based on standard Sobolev calculus inequalities and appears in [3]. The next lemma provides the key a priori estimate for the existence proof.

Lemma B.2. *Let a_1, a_2, $\mathbf{v}$, $\mathbf{F}$ be sufficiently smooth given functions in the system of equations*

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot\mathbf{u} \tag{B.1}$$

$$\begin{aligned}\frac{\partial\mathbf{u}}{\partial t} &= -a_1^{-1}\mathbf{v}\cdot\nabla\mathbf{u} - a_1^{-1}(\nabla\cdot\mathbf{u})\mathbf{v} + a_1^{-2}(\mathbf{v}\cdot\nabla\rho)\mathbf{v} - a_2\nabla\rho \\ &\quad + c\Big(|\Omega|^{-1}\int_\Omega a_1 d\mathbf{x}\Big)\nabla\Delta\rho + \mathbf{F} - Q_k\mathbf{g}\end{aligned} \tag{B.2}$$

where Q_k is the orthogonal projection operator from Lemma A.1 in Appendix A and

$$Q_k\mathbf{g} = -Q_k(a_1^{-1}\mathbf{v}\cdot\nabla\mathbf{u}) - Q_k(a_1^{-1}(\nabla\cdot\mathbf{u})\mathbf{v}) + Q_k(a_1^{-2}(\mathbf{v}\cdot\nabla\rho)\mathbf{v}) - Q_k(a_2\nabla\rho) \tag{B.3}$$

and where $(Q_k\mathbf{g},\mathbf{u}) = 0$, $(\nabla\cdot(Q_k\mathbf{g})_\alpha, \nabla\cdot\mathbf{u}_\alpha) = 0$, $(\nabla\times(Q_k\mathbf{g})_\alpha, \nabla\times\mathbf{u}_\alpha) = 0$ for $|\alpha| \ge 0$. And $0 < c_1 < a_1(\mathbf{x},t) < c_2$, $0 < c_1 < a_2(\mathbf{x},t) < c_2$, and $|\mathbf{v}(\mathbf{x},t)| < c_3$ for some constants c_1, c_2, c_3, where $c_1 < 1$, $c_3 > 1$. Here, $0 \le t \le T$, and the domain

$\Omega = \mathbb{T}^N$. Let $\rho_0(\mathbf{x}) = \rho(\mathbf{x},0)$, $\mathbf{u}_0(\mathbf{x}) = \mathbf{u}(\mathbf{x},0)$ be the given initial data, which is assumed to be sufficiently smooth.

Then ρ, $\mathbf{u}$ satisfy the following two inequalities

$$\begin{aligned}\|D\mathbf{u}\|_r^2 + \|\nabla\rho\|_r^2 + \|\Delta\rho\|_r^2 &\le C_4(1 + C_4K_4Te^{C_4K_4T})(\|D\mathbf{u}_0\|_r^2 + \|\nabla\rho_0\|_{r+1}^2)\\ &\quad + C_4(1 + C_4K_4Te^{C_4K_4T})\int_0^t \|\mathbf{F}\|_{r+1}^2 d\tau\end{aligned}$$

and

$$\begin{aligned}\|\mathbf{u}\|_0^2 + \|\rho\|_0^2 + \|\nabla\rho\|_0^2 &\le C_5(1 + C_5K_4Te^{C_5K_4T})(\|\mathbf{u}_0\|_0^2 + \|\rho_0\|_0^2 + \|\nabla\rho_0\|_0^2)\\ &\quad + C_5(1 + C_5K_4Te^{C_5K_4T})\int_0^t (\|D\mathbf{u}\|_0^2 + \|\mathbf{F}\|_0^2) d\tau\,,\end{aligned}$$

where $C_4 = \hat{C}_4(r, c, c_1, c_2, c_3)$, $C_5 = \hat{C}_5(c, c_1, c_2)$, and $r \ge 1$, and where

$$\begin{aligned}K_4 = \max\Big\{&1,\ \|a_1^{-1}\|_{q+1,T}^2\|\mathbf{v}\|_{q+1,T}^2,\ \|a_2\|_{q+1,T}^2,\quad \|a_1^{-2}\|_{q+1,T}^2\|\mathbf{v}\|_{q+1,T}^4,\\ &\|(a_1^{-1})_t\|_{2,T}^2\|\mathbf{v}\|_{2,T}^2,\ \|a_1^{-1}\|_{2,T}^2\|\mathbf{v}_t\|_{2,T}^2,\ \|(a_1)_t\|_{2,T},\ \|(a_2)_t\|_{2,T}\Big\}\end{aligned}$$

where $q = \max\{r, s_0\}$, where $r \ge 1$, and where $s_0 = [\frac{N}{2}] + 1 = 2$ for $N = 2$ or $N = 3$.

Proof. First, we will obtain an L^2 estimate. Then we will obtain estimates for $\nabla\cdot\mathbf{u}$ and for $\nabla\times\mathbf{u}$, which will be combined to obtain an estimate for $D\mathbf{u}$.

Using the fact that $(Q_k\mathbf{g}, \mathbf{u}) = 0$, we obtain an L^2 estimate as follows:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_0^2 &= (\mathbf{u}_t, \mathbf{u})\\ &= -(a_1^{-1}\mathbf{v}\cdot\nabla\mathbf{u}, \mathbf{u}) - (a_1^{-1}(\nabla\cdot\mathbf{u})\mathbf{v}, \mathbf{u}) + (a_1^{-2}(\mathbf{v}\cdot\nabla\rho)\mathbf{v}, \mathbf{u})\\ &\quad - (a_2\nabla\rho, \mathbf{u}) + c((|\Omega|^{-1}\int_\Omega a_1 d\mathbf{x})\nabla\Delta\rho, \mathbf{u}) + (\mathbf{F}, \mathbf{u}) - (Q_k\mathbf{g}, \mathbf{u})\\ &= \frac{1}{2}(\mathbf{u}\nabla\cdot(a_1^{-1}\mathbf{v}), \mathbf{u}) - (a_1^{-1}(\nabla\cdot\mathbf{u})\mathbf{v}, \mathbf{u}) + (a_1^{-2}(\mathbf{v}\cdot\nabla\rho)\mathbf{v}, \mathbf{u})\\ &\quad + (\rho\nabla a_2, \mathbf{u}) + (a_2\rho, \nabla\cdot\mathbf{u}) - c((|\Omega|^{-1}\int_\Omega a_1 d\mathbf{x})\Delta\rho, \nabla\cdot\mathbf{u}) + (\mathbf{F}, \mathbf{u})\\ &\le C(|a_1^{-1}|_{L^\infty}|\nabla\cdot\mathbf{v}|_{L^\infty} + |D(a_1^{-1})|_{L^\infty}|\mathbf{v}|_{L^\infty})\|\mathbf{u}\|_0^2\\ &\quad + C|a_1^{-1}|_{L^\infty}|\mathbf{v}|_{L^\infty}\|\nabla\cdot\mathbf{u}\|_0\|\mathbf{u}\|_0 + C|a_1^{-2}|_{L^\infty}|\mathbf{v}|_{L^\infty}^2\|\nabla\rho\|_0\|\mathbf{u}\|_0\\ &\quad + C|Da_2|_{L^\infty}\|\rho\|_0\|\mathbf{u}\|_0 - (a_2\rho, \rho_t) + c((|\Omega|^{-1}\int_\Omega a_1 d\mathbf{x})\Delta\rho, \rho_t)\\ &\quad + C\|\mathbf{F}\|_0\|\mathbf{u}\|_0\\ &\le C(1 + |a_1^{-1}|_{L^\infty}|D\mathbf{v}|_{L^\infty} + |D(a_1^{-1})|_{L^\infty}|\mathbf{v}|_{L^\infty})\|\mathbf{u}\|_0^2\\ &\quad + C(|a_1^{-1}|_{L^\infty}^2|\mathbf{v}|_{L^\infty}^2 + |a_1^{-2}|_{L^\infty}^2|\mathbf{v}|_{L^\infty}^4 + |Da_2|_{L^\infty}^2)\|\mathbf{u}\|_0^2 + C\|\nabla\cdot\mathbf{u}\|_0^2\\ &\quad - \frac{1}{2}\frac{d}{dt}(a_2\rho, \rho) + \frac{1}{2}((a_2)_t\rho, \rho) + C\|\rho\|_0^2 + C\|\nabla\rho\|_0^2 + C\|\mathbf{F}\|_0^2\\ &\quad - \frac{c}{2}\frac{d}{dt}((|\Omega|^{-1}\int_\Omega a_1 d\mathbf{x})\nabla\rho, \nabla\rho) + \frac{c}{2}((|\Omega|^{-1}\int_\Omega (a_1)_t d\mathbf{x})\nabla\rho, \nabla\rho)\end{aligned} \tag{B.4}$$

where C is a generic constant, and where we used equation (B.1) to substitute for $\nabla \cdot \mathbf{u}$. Here, we have used Holder's inequality $(f, g) \leq \|f\|_0 \|g\|_0$. Also, we used Cauchy's inequality $fg \leq \frac{1}{2}(f^2 + g^2)$.

Integrating (B.4) with respect to time, and using the fact that $0 < c_1 < a_1(\mathbf{x}, t) < c_2$ and $0 < c_1 < a_2(\mathbf{x},t) < c_2$ yields

$$\begin{aligned} &\|\mathbf{u}\|_0^2 + \|\rho\|_0^2 + \|\nabla\rho\|_0^2 \\ &\leq C_1\Big(\|\mathbf{u}_0\|_0^2 + \|\rho_0\|_0^2 + \|\nabla\rho_0\|_0^2\Big) \\ &\quad + C_1K_1 \int_0^t (\|\mathbf{u}\|_0^2 + \|\rho\|_0^2 + \|\nabla\rho\|_0^2) d\tau + C_1 \int_0^t (\|D\mathbf{u}\|_0^2 + \|\mathbf{F}\|_0^2) d\tau \end{aligned} \tag{B.5}$$

where $C_1 = \hat{C}_1(c, c_1, c_2)$, and where we define K_1, which is an upper bound for the coefficients in (B.4), as follows:

$$\begin{aligned} K_1 = \max\Big\{&1,\ |a_1^{-1}|^2_{L^\infty,T}|D\mathbf{v}|^2_{L^\infty,T},\ |D(a_1^{-1})|^2_{L^\infty,T}|\mathbf{v}|^2_{L^\infty,T},\ |a_1^{-1}|^2_{L^\infty,T}|\mathbf{v}|^2_{L^\infty,T}, \\ &|a_1^{-2}|^2_{L^\infty,T}|\mathbf{v}|^4_{L^\infty,T},\ |Da_2|^2_{L^\infty,T},\ |(a_2)_t|_{L^\infty,T},\ |(a_1)_t|_{L^\infty,T}\} \end{aligned} \tag{B.6}$$

where in K_1 we have used Cauchy's inequality $fg \leq \frac{1}{2}(f^2 + g^2)$, with $g = 1$ for some of the terms. Applying Gronwall's inequality to (B.5) yields

$$\begin{aligned} \|\mathbf{u}\|_0^2 + \|\rho\|_0^2 + \|\nabla\rho\|_0^2 &\leq C_1(1 + C_1K_1Te^{C_1K_1T})(\|\mathbf{u}_0\|_0^2 + \|\rho_0\|_0^2 + \|\nabla\rho_0\|_0^2) \\ &\quad + C_1(1 + C_1K_1Te^{C_1K_1T}) \int_0^t (\|D\mathbf{u}\|_0^2 + \|\mathbf{F}\|_0^2) d\tau \end{aligned} \tag{B.7}$$

Next, we will obtain estimates for $\nabla \cdot \mathbf{u}$ and for $\nabla \times \mathbf{u}$. Recall that we use the notation $f_\alpha = D^\alpha f$. We will let C denote a generic constant which may change from one instance to the next, but which will depend only on r, where $|\alpha| \leq r$.

After applying the operator D^α to (B.1), (B.2), we obtain

$$\frac{\partial \rho_\alpha}{\partial t} = -\nabla \cdot \mathbf{u}_\alpha \tag{B.8}$$

$$\begin{aligned} \frac{\partial \mathbf{u}_\alpha}{\partial t} &= -a_1^{-1}\mathbf{v} \cdot \nabla\mathbf{u}_\alpha - a_1^{-1}(\nabla \cdot \mathbf{u}_\alpha)\mathbf{v} + a_1^{-2}(\mathbf{v} \cdot \nabla\rho_\alpha)\mathbf{v} \\ &\quad - a_2\nabla\rho_\alpha + c\Big(\frac{1}{|\Omega|}\int_\Omega a_1 d\mathbf{x}\Big)\nabla\Delta\rho_\alpha - (Q_k\mathbf{g})_\alpha + \mathbf{G}_\alpha \end{aligned} \tag{B.9}$$

where we define $\mathbf{G}_\alpha$ as follows:

$$\begin{aligned} \mathbf{G}_\alpha = \mathbf{F}_\alpha &- [(a_1^{-1}\mathbf{v} \cdot \nabla\mathbf{u})_\alpha - a_1^{-1}\mathbf{v} \cdot \nabla\mathbf{u}_\alpha] - [(a_1^{-1}(\nabla \cdot \mathbf{u})\mathbf{v})_\alpha - a_1^{-1}(\nabla \cdot \mathbf{u}_\alpha)\mathbf{v}] \\ &+ [(a_1^{-2}(\mathbf{v} \cdot \nabla\rho)\mathbf{v})_\alpha - a_1^{-2}(\mathbf{v} \cdot \nabla\rho_\alpha)\mathbf{v}] - [(a_2\nabla\rho)_\alpha - a_2\nabla\rho_\alpha] \end{aligned} \tag{B.10}$$

Next, we will obtain an estimate for $\nabla\cdot\mathbf{u}$. We apply the divergence operator to equation (B.9), and obtain

$$
\begin{aligned}
\frac{\partial\nabla\cdot\mathbf{u}_\alpha}{\partial t} &= -2a_1^{-1}\mathbf{v}\cdot\nabla(\nabla\cdot\mathbf{u}_\alpha) - \nabla(a_1^{-1})\cdot(\mathbf{v}\cdot\nabla\mathbf{u}_\alpha) - a_1^{-1}(\nabla\mathbf{v}^T:\nabla\mathbf{u}_\alpha)\\
&\quad - (\nabla\cdot\mathbf{u}_\alpha)\mathbf{v}\cdot\nabla(a_1^{-1}) - a_1^{-1}(\nabla\cdot\mathbf{u}_\alpha)\nabla\cdot\mathbf{v} + (\mathbf{v}\cdot\nabla\rho_\alpha)\mathbf{v}\cdot\nabla(a_1^{-2})\\
&\quad + a_1^{-2}\nabla(\mathbf{v}\cdot\nabla\rho_\alpha)\cdot\mathbf{v} + a_1^{-2}(\mathbf{v}\cdot\nabla\rho_\alpha)\nabla\cdot\mathbf{v} - \nabla\cdot(a_2\nabla\rho_\alpha)\\
&\quad + c\left(\frac{1}{|\Omega|}\int_\Omega a_1 d\mathbf{x}\right)\Delta^2\rho_\alpha - \nabla\cdot(Q_k\mathbf{g})_\alpha + \nabla\cdot\mathbf{G}_\alpha
\end{aligned}
\tag{B.11}
$$

From equation (B.11), and using the fact that $(\nabla\cdot(Q_k\mathbf{g})_\alpha, \nabla\cdot\mathbf{u}_\alpha) = 0$ we obtain the estimate

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\cdot\mathbf{u}_\alpha\|_0^2\\
&= (\frac{\partial\nabla\cdot\mathbf{u}_\alpha}{\partial t}, \nabla\cdot\mathbf{u}_\alpha)\\
&= -2(a_1^{-1}\mathbf{v}\cdot\nabla(\nabla\cdot\mathbf{u}_\alpha), \nabla\cdot\mathbf{u}_\alpha) - (\nabla(a_1^{-1})\cdot(\mathbf{v}\cdot\nabla\mathbf{u}_\alpha), \nabla\cdot\mathbf{u}_\alpha)\\
&\quad - (a_1^{-1}(\nabla\mathbf{v}^T:\nabla\mathbf{u}_\alpha), \nabla\cdot\mathbf{u}_\alpha) - ((\nabla\cdot\mathbf{u}_\alpha)\mathbf{v}\cdot\nabla(a_1^{-1}), \nabla\cdot\mathbf{u}_\alpha)\\
&\quad - (a_1^{-1}(\nabla\cdot\mathbf{u}_\alpha)\nabla\cdot\mathbf{v}, \nabla\cdot\mathbf{u}_\alpha) + ((\mathbf{v}\cdot\nabla\rho_\alpha)\mathbf{v}\cdot\nabla(a_1^{-2}), \nabla\cdot\mathbf{u}_\alpha)\\
&\quad + (a_1^{-2}\nabla(\mathbf{v}\cdot\nabla\rho_\alpha)\cdot\mathbf{v}, \nabla\cdot\mathbf{u}_\alpha) + (a_1^{-2}(\mathbf{v}\cdot\nabla\rho_\alpha)\nabla\cdot\mathbf{v}, \nabla\cdot\mathbf{u}_\alpha)\\
&\quad - (\nabla\cdot(a_2\nabla\rho_\alpha), \nabla\cdot\mathbf{u}_\alpha) + (c(|\Omega|^{-1}\int_\Omega a_1 d\mathbf{x})\Delta^2\rho_\alpha, \nabla\cdot\mathbf{u}_\alpha)\\
&\quad - (\nabla\cdot(Q_k\mathbf{g})_\alpha, \nabla\cdot\mathbf{u}_\alpha) + (\nabla\cdot\mathbf{G}_\alpha, \nabla\cdot\mathbf{u}_\alpha)\\
&= (\nabla\cdot(a_1^{-1}\mathbf{v})\nabla\cdot\mathbf{u}_\alpha, \nabla\cdot\mathbf{u}_\alpha) - (\nabla(a_1^{-1})\cdot(\mathbf{v}\cdot\nabla\mathbf{u}_\alpha), \nabla\cdot\mathbf{u}_\alpha)\\
&\quad - (a_1^{-1}(\nabla\mathbf{v}^T:\nabla\mathbf{u}_\alpha), \nabla\cdot\mathbf{u}_\alpha) - ((\nabla\cdot\mathbf{u}_\alpha)\mathbf{v}\cdot\nabla(a_1^{-1}), \nabla\cdot\mathbf{u}_\alpha)\\
&\quad - (a_1^{-1}(\nabla\cdot\mathbf{u}_\alpha)\nabla\cdot\mathbf{v}, \nabla\cdot\mathbf{u}_\alpha) + ((\mathbf{v}\cdot\nabla\rho_\alpha)\mathbf{v}\cdot\nabla(a_1^{-2}), \nabla\cdot\mathbf{u}_\alpha)\\
&\quad + (a_1^{-2}\nabla(\mathbf{v}\cdot\nabla\rho_\alpha)\cdot\mathbf{v}, \nabla\cdot\mathbf{u}_\alpha) + (a_1^{-2}(\mathbf{v}\cdot\nabla\rho_\alpha)\nabla\cdot\mathbf{v}, \nabla\cdot\mathbf{u}_\alpha)\\
&\quad + (\nabla\cdot(a_2\nabla\rho_\alpha), \rho_{t,\alpha}) - (c(|\Omega|^{-1}\int_\Omega a_1 d\mathbf{x})\Delta^2\rho_\alpha, \rho_{t,\alpha})\\
&\quad + (\nabla\cdot\mathbf{G}_\alpha, \nabla\cdot\mathbf{u}_\alpha) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(B.12)}\\
&\le C(|D(a_1^{-1})|_{L^\infty}|\mathbf{v}|_{L^\infty} + |a_1^{-1}|_{L^\infty}|\nabla\cdot\mathbf{v}|_{L^\infty})\|\nabla\cdot\mathbf{u}_\alpha\|_0^2\\
&\quad + C(|D(a_1^{-1})|_{L^\infty}|\mathbf{v}|_{L^\infty} + |a_1^{-1}|_{L^\infty}|D\mathbf{v}|_{L^\infty})\|\nabla\cdot\mathbf{u}_\alpha\|_0\|D\mathbf{u}_\alpha\|_0\\
&\quad + C(|\mathbf{v}|_{L^\infty}^2|D(a_1^{-2})|_{L^\infty} + |a_1^{-2}|_{L^\infty}|D\mathbf{v}|_{L^\infty}|\mathbf{v}|_{L^\infty})\|\nabla\rho_\alpha\|_0\|\nabla\cdot\mathbf{u}_\alpha\|_0\\
&\quad + C|a_1^{-2}|_{L^\infty}|\mathbf{v}|_{L^\infty}^2\|\nabla\rho_\alpha\|_1\|\nabla\cdot\mathbf{u}_\alpha\|_0 - (a_2\nabla\rho_\alpha, \nabla\rho_{t,\alpha})\\
&\quad - (c(|\Omega|^{-1}\int_\Omega a_1 d\mathbf{x})\Delta\rho_\alpha, \Delta\rho_{t,\alpha}) + \|\nabla\cdot\mathbf{G}_\alpha\|_0\|\nabla\cdot\mathbf{u}_\alpha\|_0\\
&\le C(1 + |D(a_1^{-1})|_{L^\infty}^2|\mathbf{v}|_{L^\infty}^2 + |a_1^{-1}|_{L^\infty}^2|D\mathbf{v}|_{L^\infty}^2)\|\nabla\cdot\mathbf{u}_\alpha\|_0^2\\
&\quad + C(|\mathbf{v}|_{L^\infty}^4|D(a_1^{-2})|_{L^\infty}^2 + |a_1^{-2}|_{L^\infty}^2|D\mathbf{v}|_{L^\infty}^2|\mathbf{v}|_{L^\infty}^2)\|\nabla\cdot\mathbf{u}_\alpha\|_0^2\\
&\quad + C\|\nabla\rho_\alpha\|_0^2 + C|a_1^{-2}|_{L^\infty}^2|\mathbf{v}|_{L^\infty}^4\|\nabla\cdot\mathbf{u}_\alpha\|_0^2 + C\|\Delta\rho_\alpha\|_0^2 + C\|D\mathbf{u}_\alpha\|_0^2\\
&\quad - \frac{1}{2}\frac{d}{dt}(c(|\Omega|^{-1}\int_\Omega a_1 d\mathbf{x})\Delta\rho_\alpha, \Delta\rho_\alpha) + \frac{1}{2}(c(|\Omega|^{-1}\int_\Omega (a_1)_t d\mathbf{x})\Delta\rho_\alpha, \Delta\rho_\alpha)
\end{aligned}
$$

$$
-\frac{1}{2}\frac{d}{dt}(a_2\nabla\rho_\alpha,\nabla\rho_\alpha)+\frac{1}{2}((a_2)_t\nabla\rho_\alpha,\nabla\rho_\alpha)+C\|D\mathbf{G}_\alpha\|_0^2 \tag{B.13}
$$

where we used Cauchy's inequality $fg \le \frac{1}{2}(f^2+g^2)$, and where for some of the terms, we let $g=1$. We also used the fact that $\|\nabla\rho_\alpha\|_1^2 \le C\|\Delta\rho_\alpha\|_0^2$ when $\Omega=\mathbb{T}^N$ (a proof appears in [3]). And we used equation (B.8) to substitute for $\nabla\cdot\mathbf{u}_\alpha$.

Next, we estimate the term $\|D\mathbf{G}_\alpha\|_0^2$ in (B.12). We apply the D^γ differentiation operator, where the multi-index $|\gamma|=1$, to equation (B.10) for $\mathbf{G}_\alpha$, which yields

$$
\begin{aligned}
D^\gamma(\mathbf{G}_\alpha) &= \mathbf{F}_{\alpha+\gamma}-[(a_1^{-1}\mathbf{v}\cdot\nabla\mathbf{u})_{\alpha+\gamma}-a_1^{-1}\mathbf{v}\cdot\nabla\mathbf{u}_{\alpha+\gamma}]\\
&\quad+(a_1^{-1})_\gamma\mathbf{v}\cdot\nabla\mathbf{u}_\alpha+a_1^{-1}\mathbf{v}_\gamma\cdot\nabla\mathbf{u}_\alpha\\
&\quad-[(a_1^{-1}(\nabla\cdot\mathbf{u})\mathbf{v})_{\alpha+\gamma}-a_1^{-1}(\nabla\cdot\mathbf{u}_{\alpha+\gamma})\mathbf{v}]\\
&\quad+(a_1^{-1})_\gamma(\nabla\cdot\mathbf{u}_\alpha)\mathbf{v}+a_1^{-1}(\nabla\cdot\mathbf{u}_\alpha)\mathbf{v}_\gamma\\
&\quad+[(a_1^{-2}(\mathbf{v}\cdot\nabla\rho)\mathbf{v})_{\alpha+\gamma}-a_1^{-2}(\mathbf{v}\cdot\nabla\rho_{\alpha+\gamma})\mathbf{v}]\\
&\quad-(a_1^{-2})_\gamma(\mathbf{v}\cdot\nabla\rho_\alpha)\mathbf{v}-a_1^{-2}(\mathbf{v}_\gamma\cdot\nabla\rho_\alpha)\mathbf{v}-a_1^{-2}(\mathbf{v}\cdot\nabla\rho_\alpha)\mathbf{v}_\gamma\\
&\quad-[(a_2\nabla\rho)_{\alpha+\gamma}-a_2\nabla\rho_{\alpha+\gamma}]+(a_2)_\gamma\nabla\rho_\alpha
\end{aligned}
$$

For $|\gamma|=1$ and $|\alpha|=k-1$, where $0\le k-1\le r$, and by applying Lemma B.1 to the terms of the form $\|(fg)_{\alpha+\gamma}-fg_{\alpha+\gamma}\|_0^2$, we obtain the estimate

$$
\begin{aligned}
\|D^\gamma(\mathbf{G}_\alpha)\|_0^2 &\le C\|\mathbf{F}_{\alpha+\gamma}\|_0^2+C\|(a_1^{-1}\mathbf{v}\cdot\nabla\mathbf{u})_{\alpha+\gamma}-a_1^{-1}\mathbf{v}\cdot\nabla\mathbf{u}_{\alpha+\gamma}\|_0^2\\
&\quad+C\|(a_1^{-1})_\gamma\mathbf{v}\cdot\nabla\mathbf{u}_\alpha\|_0^2+C\|a_1^{-1}\mathbf{v}_\gamma\cdot\nabla\mathbf{u}_\alpha\|_0^2\\
&\quad+C\|(a_1^{-1}(\nabla\cdot\mathbf{u})\mathbf{v})_{\alpha+\gamma}-a_1^{-1}(\nabla\cdot\mathbf{u}_{\alpha+\gamma})\mathbf{v}\|_0^2\\
&\quad+C\|(a_1^{-1})_\gamma(\nabla\cdot\mathbf{u}_\alpha)\mathbf{v}\|_0^2+C\|a_1^{-1}(\nabla\cdot\mathbf{u}_\alpha)\mathbf{v}_\gamma\|_0^2\\
&\quad+C\|(a_1^{-2}(\mathbf{v}\cdot\nabla\rho)\mathbf{v})_{\alpha+\gamma}-a_1^{-2}(\mathbf{v}\cdot\nabla\rho_{\alpha+\gamma})\mathbf{v}\|_0^2\\
&\quad+C\|(a_1^{-2})_\gamma(\mathbf{v}\cdot\nabla\rho_\alpha)\mathbf{v}\|_0^2+C\|a_1^{-2}(\mathbf{v}_\gamma\cdot\nabla\rho_\alpha)\mathbf{v}\|_0^2\\
&\quad+C\|a_1^{-2}(\mathbf{v}\cdot\nabla\rho_\alpha)\mathbf{v}_\gamma\|_0^2+C\|(a_2\nabla\rho)_{\alpha+\gamma}-a_2\nabla\rho_{\alpha+\gamma}\|_0^2\\
&\quad+C\|(a_2)_\gamma\nabla\rho_\alpha\|_0^2\\
&\le C\|\mathbf{F}\|_k^2+C(\|a_1^{-1}\|_{k_1}^2\|D\mathbf{v}\|_{k_1}^2+\|D(a_1^{-1})\|_{k_1}^2\|\mathbf{v}\|_{k_1}^2)\|D\mathbf{u}\|_{k-1}^2\\
&\quad+C(|D(a_1^{-1})|_{L^\infty}^2|\mathbf{v}|_{L^\infty}^2+|a_1^{-1}|_{L^\infty}^2|D\mathbf{v}|_{L^\infty}^2)\|D\mathbf{u}_\alpha\|_0^2\\
&\quad+C(\|a_1^{-1}\|_{k_1}^2\|D\mathbf{v}\|_{k_1}^2+\|D(a_1^{-1})\|_{k_1}^2\|\mathbf{v}\|_{k_1}^2)\|\nabla\cdot\mathbf{u}\|_{k-1}^2\\
&\quad+C(|D(a_1^{-1})|_{L^\infty}^2|\mathbf{v}|_{L^\infty}^2+|a_1^{-1}|_{L^\infty}^2|D\mathbf{v}|_{L^\infty}^2)\|\nabla\cdot\mathbf{u}_\alpha\|_0^2\\
&\quad+C(\|D(a_1^{-2})\|_{k_1}^2\|\mathbf{v}\|_{k_1}^4+\|a_1^{-2}\|_{k_1}^2\|D\mathbf{v}\|_{k_1}^2\|\mathbf{v}\|_{k_1}^2)\|\nabla\rho\|_{k-1}^2\\
&\quad+C(|D(a_1^{-2})|_{L^\infty}^2|\mathbf{v}|_{L^\infty}^4+|a_1^{-2}|_{L^\infty}^2|D\mathbf{v}|_{L^\infty}^2|\mathbf{v}|_{L^\infty}^2)\|\nabla\rho_\alpha\|_0^2\\
&\quad+C\|Da_2\|_{k_1}^2\|\nabla\rho\|_{k-1}^2+C|Da_2|_{L^\infty}^2\|\nabla\rho_\alpha\|_0^2\\
&\le C\|\mathbf{F}\|_k^2+C(\|a_1^{-1}\|_{k_1}^2\|D\mathbf{v}\|_{k_1}^2+\|D(a_1^{-1})\|_{k_1}^2\|\mathbf{v}\|_{k_1}^2)\|D\mathbf{u}\|_{k-1}^2\\
&\quad+C(\|D(a_1^{-2})\|_{k_1}^2\|\mathbf{v}\|_{k_1}^4+\|a_1^{-2}\|_{k_1}^2\|D\mathbf{v}\|_{k_1}^2\|\mathbf{v}\|_{k_1}^2)\|\nabla\rho\|_{k-1}^2\\
&\quad+\|Da_2\|_{k_1}^2\|\nabla\rho\|_{k-1}^2
\end{aligned} \tag{B.14}
$$

where $k_1=\max\{k-1,s_0\}$ and $s_0=[\frac{N}{2}]+1=2$ for $N=2$ or $N=3$. Here, we used the Sobolev inequality $|f|_{L^\infty}\le C\|f\|_{s_0}$. We also used the Sobolev calculus inequality $\|fg\|_s\le C\|f\|_s\,\|g\|_s$ for $s>\frac{N}{2}$ (see, e.g., [6]).

We integrate equation (B.12) with respect to time, and use estimate (B.14) on the right-hand side, and then add over $0 \le |\alpha| \le r$, where $r \ge 1$, which yields the estimate

$$\begin{aligned} &\|\nabla\cdot\mathbf{u}\|_r^2 + \|\nabla\rho\|_r^2 + \|\Delta\rho\|_r^2 \\ &\le C_2(\|\nabla\cdot\mathbf{u}_0\|_r^2 + \|\nabla\rho_0\|_r^2 + \|\Delta\rho_0\|_r^2) + C_2\int_0^t \|\mathbf{F}\|_{r+1}^2 d\tau \\ &\quad + C_2K_2\int_0^t (\|D\mathbf{u}\|_r^2 + \|\nabla\rho\|_r^2 + \|\Delta\rho\|_r^2) d\tau \end{aligned} \tag{B.15}$$

where $C_2 = \hat{C}_2(r, c, c_1, c_2)$, and where we define K_2, which is an upper bound for the coefficients in (B.12), (B.14), as follows:

$$\begin{aligned} K_2 = \max\Big\{&1,\ \|a_1^{-1}\|_{q,T}^2\|D\mathbf{v}\|_{q,T}^2,\ \|D(a_1^{-1})\|_{q,T}^2\|\mathbf{v}\|_{q,T}^2, \\ &\|D(a_1^{-2})\|_{q,T}^2\|\mathbf{v}\|_{q,T}^4,\ \|a_1^{-2}\|_{q,T}^2\|D\mathbf{v}\|_{q,T}^2\|\mathbf{v}\|_{q,T}^2, \\ &|a_1^{-2}|_{L^\infty,T}^2|\mathbf{v}|_{L^\infty,T}^4,\ \|Da_2\|_{q,T}^2, \quad |(a_1)_t|_{L^\infty,T},\ |(a_2)_t|_{L^\infty,T}\Big\} \end{aligned} \tag{B.16}$$

where $q = \max\{r, s_0\}$, where $r \ge 1$, and where $s_0 = [\frac{N}{2}]+1 = 2$ for $N = 2$ or $N = 3$. Here we have used the fact that $0 < c_1 < a_1(\mathbf{x}, t) < c_2$ and $0 < c_1 < a_2(\mathbf{x}, t) < c_2$. We also used the Sobolev inequality $|f|_{L^\infty} \le C\|f\|_{s_0}$.

Next, we obtain an estimate for $\nabla\times\mathbf{u}$. Applying the curl operator to equation (B.9) yields

$$\begin{aligned} \frac{\partial\nabla\times\mathbf{u}_\alpha}{\partial t} &= -a_1^{-1}\mathbf{v}\cdot\nabla(\nabla\times\mathbf{u}_\alpha) - (\nabla\cdot\mathbf{u}_\alpha)\nabla\times(a_1^{-1}\mathbf{v}) \\ &\quad - \nabla(\nabla\cdot\mathbf{u}_\alpha)\times(a_1^{-1}\mathbf{v}) + \nabla\times(a_1^{-2}(\mathbf{v}\cdot\nabla\rho_\alpha)\mathbf{v}) \\ &\quad - \nabla a_2\times\nabla\rho_\alpha - \nabla\times(Q_k\mathbf{g})_\alpha + \nabla\times\mathbf{G}_\alpha + \mathbf{H}_\alpha \end{aligned} \tag{B.17}$$

where

$$\mathbf{H}_\alpha = -[\nabla\times(a_1^{-1}\mathbf{v}\cdot\nabla\mathbf{u}_\alpha) - a_1^{-1}\mathbf{v}\cdot\nabla(\nabla\times\mathbf{u}_\alpha)] \tag{B.18}$$

and where we estimate $\|\mathbf{H}_\alpha\|_0^2$ as follows:

$$\begin{aligned} \|\mathbf{H}_\alpha\|_0^2 &= \|\nabla\times(a_1^{-1}\mathbf{v}\cdot\nabla\mathbf{u}_\alpha) - a_1^{-1}\mathbf{v}\cdot\nabla(\nabla\times\mathbf{u}_\alpha)\|_0^2 \\ &\le C(|a_1^{-1}|_{L^\infty}^2|D\mathbf{v}|_{L^\infty}^2 + |D(a_1^{-1})|_{L^\infty}^2|\mathbf{v}|_{L^\infty}^2)\|D\mathbf{u}_\alpha\|_0^2 \end{aligned} \tag{B.19}$$

From (B.17), and using the fact that $(\nabla\times(Q_k\mathbf{g})_\alpha, \nabla\times\mathbf{u}_\alpha)=0$, we obtain the estimate

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\times\mathbf{u}_\alpha\|_0^2\\
&=(\frac{\partial(\nabla\times\mathbf{u}_\alpha)}{\partial t},\nabla\times\mathbf{u}_\alpha)\\
&=-(a_1^{-1}\mathbf{v}\cdot\nabla(\nabla\times\mathbf{u}_\alpha),\nabla\times\mathbf{u}_\alpha)-((\nabla\cdot\mathbf{u}_\alpha)\nabla\times(a_1^{-1}\mathbf{v}),\nabla\times\mathbf{u}_\alpha)\\
&\quad-(\nabla(\nabla\cdot\mathbf{u}_\alpha)\times(a_1^{-1}\mathbf{v}),\nabla\times\mathbf{u}_\alpha)\\
&\quad+(\nabla\times(a_1^{-2}(\mathbf{v}\cdot\nabla\rho_\alpha)\mathbf{v}),\nabla\times\mathbf{u}_\alpha)-(\nabla a_2\times\nabla\rho_\alpha,\nabla\times\mathbf{u}_\alpha)\\
&\quad-(\nabla\times(Q_k\mathbf{g})_\alpha,\nabla\times\mathbf{u}_\alpha)+(\nabla\times\mathbf{G}_\alpha,\nabla\times\mathbf{u}_\alpha)+(\mathbf{H}_\alpha,\nabla\times\mathbf{u}_\alpha) \qquad \text{(B.20)}\\
&\le C(|a_1^{-1}|_{L^\infty}|\nabla\cdot\mathbf{v}|_{L^\infty}+|D(a_1^{-1})|_{L^\infty}|\mathbf{v}|_{L^\infty})\|\nabla\times\mathbf{u}_\alpha\|_0^2\\
&\quad+C(|D(a_1^{-1})|_{L^\infty}^2|\mathbf{v}|_{L^\infty}^2+|a_1^{-1}|_{L^\infty}^2|\nabla\times\mathbf{v}|_{L^\infty}^2)\|\nabla\times\mathbf{u}_\alpha\|_0^2\\
&\quad+C\|\nabla\cdot\mathbf{u}_\alpha\|_0^2-(\nabla(\nabla\cdot\mathbf{u}_\alpha)\times(a_1^{-1}\mathbf{v}),\nabla\times\mathbf{u}_\alpha)\\
&\quad+C(|a_1^{-2}|_{L^\infty}^2|\mathbf{v}|_{L^\infty}^2|D\mathbf{v}|_{L^\infty}^2+|D(a_1^{-2})|_{L^\infty}^2|\mathbf{v}|_{L^\infty}^4)\|\nabla\times\mathbf{u}_\alpha\|_0^2\\
&\quad+|a_1^{-2}|_{L^\infty}^2|\mathbf{v}|_{L^\infty}^4\|\nabla\times\mathbf{u}_\alpha\|_0^2+C\|\nabla\rho_\alpha\|_0^2+C\|\nabla\rho_\alpha\|_1^2\\
&\quad+C|Da_2|_{L^\infty}^2\|\nabla\rho_\alpha\|_0^2+C\|\nabla\times\mathbf{u}_\alpha\|_0^2+C\|D\mathbf{G}_\alpha\|_0^2+C\|\mathbf{H}_\alpha\|_0^2
\end{aligned}
$$

where we used the fact that $-(a_1^{-1}\mathbf{v}\cdot\nabla(\nabla\times\mathbf{u}_\alpha),\nabla\times\mathbf{u}_\alpha)=\frac{1}{2}((\nabla\cdot(a_1^{-1}\mathbf{v}))(\nabla\times\mathbf{u}_\alpha),\nabla\times\mathbf{u}_\alpha)$.

Next, we estimate the term $-(\nabla(\nabla\cdot\mathbf{u}_\alpha)\times(a_1^{-1}\mathbf{v}),\nabla\times\mathbf{u}_\alpha)$ from (B.20) above. When $|\alpha|=0$, we obtain the estimate

$$
-(\nabla(\nabla\cdot\mathbf{u}_\alpha)\times(a_1^{-1}\mathbf{v}),\nabla\times\mathbf{u}_\alpha)\le C|a_1^{-1}|_{L^\infty}^2|\mathbf{v}|_{L^\infty}^2\|\nabla\times\mathbf{u}\|_0^2+C\|\nabla\cdot\mathbf{u}\|_1^2 \qquad \text{(B.21)}
$$

When $|\alpha|\ge 1$, we substitute equation (B.8) for $\nabla\cdot\mathbf{u}_\alpha$, to obtain the estimate

$$
\begin{aligned}
&-(\nabla(\nabla\cdot\mathbf{u}_\alpha)\times(a_1^{-1}\mathbf{v}),\nabla\times\mathbf{u}_\alpha)\\
&=(\nabla\rho_{t,\alpha}\times(a_1^{-1}\mathbf{v}),\nabla\times\mathbf{u}_\alpha)\\
&=\frac{d}{dt}(\nabla\rho_\alpha\times(a_1^{-1}\mathbf{v}),\nabla\times\mathbf{u}_\alpha)-(\nabla\rho_\alpha\times((a_1^{-1})_t\mathbf{v}),\nabla\times\mathbf{u}_\alpha)\\
&\quad-(\nabla\rho_\alpha\times(a_1^{-1}\mathbf{v}_t),\nabla\times\mathbf{u}_\alpha)-(\nabla\rho_\alpha\times(a_1^{-1}\mathbf{v}),\nabla\times\mathbf{u}_{t,\alpha}) \qquad \text{(B.22)}\\
&\le\frac{d}{dt}(\nabla\rho_\alpha\times(a_1^{-1}\mathbf{v}),\nabla\times\mathbf{u}_\alpha)+C|(a_1^{-1})_t|_{L^\infty}^2|\mathbf{v}|_{L^\infty}^2\|\nabla\times\mathbf{u}_\alpha\|_0^2\\
&\quad+C|a_1^{-1}|_{L^\infty}^2|\mathbf{v}_t|_{L^\infty}^2\|\nabla\times\mathbf{u}_\alpha\|_0^2+C\|\nabla\rho_\alpha\|_0^2\\
&\quad-(\nabla\rho_\alpha\times(a_1^{-1}\mathbf{v}),\nabla\times\mathbf{u}_{t,\alpha})
\end{aligned}
$$

and then we integrate by parts once to estimate the term $-(\nabla\rho_\alpha\times(a_1^{-1}\mathbf{v}),\nabla\times\mathbf{u}_{t,\alpha})$ from (B.22) above as follows:

$$
\begin{aligned}
&-(\nabla\rho_\alpha\times(a_1^{-1}\mathbf{v}),\nabla\times\mathbf{u}_{t,\alpha})\\
&=(\nabla\rho_\alpha\times((a_1^{-1})_\gamma\mathbf{v}),\nabla\times\mathbf{u}_{t,\alpha-\gamma})+(\nabla\rho_\alpha\times(a_1^{-1}\mathbf{v}_\gamma),\nabla\times\mathbf{u}_{t,\alpha-\gamma})\\
&\quad+(\nabla\rho_{\alpha+\gamma}\times(a_1^{-1}\mathbf{v}),\nabla\times\mathbf{u}_{t,\alpha-\gamma}) \qquad \text{(B.23)}\\
&\le C(|D(a_1^{-1})|_{L^\infty}^2|\mathbf{v}|_{L^\infty}^2+|a_1^{-1}|_{L^\infty}^2|D\mathbf{v}|_{L^\infty}^2)\|\nabla\rho_\alpha\|_0^2\\
&\quad+C|a_1^{-1}|_{L^\infty}^2|\mathbf{v}|_{L^\infty}^2\|\nabla\rho_{\alpha+\gamma}\|_0^2+C\|\nabla\times\mathbf{u}_{t,\alpha-\gamma}\|_0^2
\end{aligned}
$$

where $|\gamma| = 1$. From (B.17), we obtain the estimate

$$
\begin{aligned}
\|\nabla \times \mathbf{u}_{t,\alpha-\gamma}\|_0^2 &\le C|a_1^{-1}|_{L^\infty}^2 |\mathbf{v}|_{L^\infty}^2 \|\nabla \times \mathbf{u}_{\alpha-\gamma}\|_1^2 \\
&\quad + C(|D(a_1^{-1})|_{L^\infty}^2 |\mathbf{v}|_{L^\infty}^2 + C|a_1^{-1}|_{L^\infty}^2 |\nabla \times \mathbf{v}|_{L^\infty}^2)\|\nabla \cdot \mathbf{u}_{\alpha-\gamma}\|_0^2 \\
&\quad + C|a_1^{-1}|_{L^\infty}^2 |\mathbf{v}|_{L^\infty}^2 \|\nabla \cdot \mathbf{u}_{\alpha-\gamma}\|_1^2 + C|a_1^{-2}|_{L^\infty}^2 |\mathbf{v}|_{L^\infty}^4 \|\nabla \rho_{\alpha-\gamma}\|_1^2 \\
&\quad + C(|D(a_1^{-2})|_{L^\infty}^2 |\mathbf{v}|_{L^\infty}^4 + |a_1^{-2}|_{L^\infty}^2 |D\mathbf{v}|_{L^\infty}^2 |\mathbf{v}|_{L^\infty}^2)\|\nabla \rho_{\alpha-\gamma}\|_0^2 \\
&\quad + C|Da_2|_{L^\infty}^2 \|\nabla \rho_{\alpha-\gamma}\|_0^2 + C\|\nabla \times (Q_k\mathbf{g})_{\alpha-\gamma}\|_0^2 + C\|D\mathbf{G}_{\alpha-\gamma}\|_0^2 \\
&\quad + C\|\mathbf{H}_{\alpha-\gamma}\|_0^2
\end{aligned} \tag{B.24}
$$

Note that if $|\alpha| = 1$, then we choose $\gamma = \alpha$.

When we estimate the term $C\|\nabla \times (Q_k\mathbf{g})\|_{r-1}^2 \le C\|Q_k\mathbf{g}\|_r^2$, which comes from adding inequality (B.24) over $1 \le |\alpha| \le r$, where $|\gamma| = 1$, we will use the fact that $\|Q_k\mathbf{f}\|_r^2 \le \|\mathbf{f}\|_r^2$, for any function $f \in H^r(\Omega)$, which follows by the definition of the projection operator Q_k in Lemma A.1. And we will use the definition (B.3) of $Q_k\mathbf{g}$.

Integrating equation (B.20) with respect to time, and using the estimates (B.14), (B.19), (B.21)-(B.24) on the right-hand side, and using the definition (B.3) of $Q_k\mathbf{g}$, and adding over $0 \le |\alpha| \le r$, where $r \ge 1$, yields

$$
\begin{aligned}
&\frac{1}{2}\|\nabla \times \mathbf{u}\|_r^2 \\
&\le \frac{1}{2}\|\nabla \times \mathbf{u}_0\|_r^2 + C\int_0^t (|a_1^{-1}|_{L^\infty}|\nabla \cdot \mathbf{v}|_{L^\infty} + |D(a_1^{-1})|_{L^\infty}|\mathbf{v}|_{L^\infty})\|\nabla \times \mathbf{u}\|_r^2 d\tau \\
&\quad + C\int_0^t (|D(a_1^{-1})|_{L^\infty}^2 |\mathbf{v}|_{L^\infty}^2 + |a_1^{-1}|_{L^\infty}^2 |\nabla \times \mathbf{v}|_{L^\infty}^2)\|\nabla \times \mathbf{u}\|_r^2 d\tau \\
&\quad + C\int_0^t (\|\nabla \rho\|_{r+1}^2 + \|\nabla \cdot \mathbf{u}\|_r^2 + \|\nabla \rho\|_r^2) d\tau \\
&\quad + \sum_{0 \le |\alpha| \le r} |a_1^{-1}|_{L^\infty}|\mathbf{v}|_{L^\infty}\|\nabla \rho_\alpha\|_0 \|\nabla \times \mathbf{u}_\alpha\|_0 \\
&\quad + \sum_{0 \le |\alpha| \le r} |a_1(\mathbf{x},0)^{-1}|_{L^\infty}|\mathbf{v}_0|_{L^\infty}\|\nabla(\rho_0)_\alpha\|_0 \|\nabla \times (\mathbf{u}_0)_\alpha\|_0 \\
&\quad + C\int_0^t (|(a_1^{-1})_t|_{L^\infty}^2 |\mathbf{v}|_{L^\infty}^2 + |a_1^{-1}|_{L^\infty}^2 |\mathbf{v}_t|_{L^\infty}^2)\|\nabla \times \mathbf{u}\|_r^2 d\tau \\
&\quad + C\int_0^t (|a_1^{-2}|_{L^\infty}^2 |D\mathbf{v}|_{L^\infty}^2 |\mathbf{v}|_{L^\infty}^2 + |D(a_1^{-2})|_{L^\infty}^2 |\mathbf{v}|_{L^\infty}^4)\|\nabla \times \mathbf{u}\|_r^2 d\tau \\
&\quad + C\int_0^t (|a_1^{-2}|_{L^\infty}^2 |\mathbf{v}|_{L^\infty}^4 + |a_1^{-1}|_{L^\infty}^2 |\mathbf{v}|_{L^\infty}^2)\|\nabla \times \mathbf{u}\|_r^2 d\tau \\
&\quad + C\int_0^t (|Da_2|_{L^\infty}^2 + |D(a_1^{-1})|_{L^\infty}^2 |\mathbf{v}|_{L^\infty}^2 + |a_1^{-1}|_{L^\infty}^2 |D\mathbf{v}|_{L^\infty}^2)\|\nabla \rho\|_r^2 d\tau \\
&\quad + C\int_0^t (|a_1^{-1}|_{L^\infty}^2 |\mathbf{v}|_{L^\infty}^2 \|\nabla \rho\|_{r+1}^2 + \|\nabla \times \mathbf{u}_t\|_{r-1}^2 + \|\nabla \times \mathbf{u}\|_r^2) d\tau \\
&\quad + C\sum_{0 \le |\alpha| \le r} \int_0^t (\|D\mathbf{G}_\alpha\|_0^2 + \|\mathbf{H}_\alpha\|_0^2) d\tau
\end{aligned}
$$

$$
\begin{aligned}
&\le (\frac{1}{2}+\epsilon)\|\nabla\times\mathbf{u}_0\|_r^2 + \frac{1}{4\epsilon}|a_1(\mathbf{x},0)^{-1}|_{L^\infty}^2|\mathbf{v}_0|_{L^\infty}^2\|\nabla\rho_0\|_r^2 \\
&\quad + \epsilon\|\nabla\times\mathbf{u}\|_r^2 + \frac{1}{4\epsilon}|a_1^{-1}|_{L^\infty}^2|\mathbf{v}|_{L^\infty}^2\|\nabla\rho\|_r^2 \\
&\quad + CK_3\int_0^t(\|D\mathbf{u}\|_r^2+\|\nabla\rho\|_r^2+\|\Delta\rho\|_r^2)d\tau + C\int_0^t\|\mathbf{F}\|_{r+1}^2 d\tau
\end{aligned} \tag{B.25}
$$

where we used Cauchy's inequality with ϵ, namely $fg \le \frac{1}{4\epsilon}f^2+\epsilon g^2$, where we choose $\epsilon = 1/4$, and where we define K_3, which is an upper bound for the coefficients, as follows

$$
\begin{aligned}
K_3 = \max\Big\{&1,\ \|a_1^{-1}\|_{q,T}^2\|D\mathbf{v}\|_{q,T}^2,\ \|D(a_1^{-1})\|_{q,T}^2\|\mathbf{v}\|_{q,T}^2,\ \|Da_2\|_{q,T}^2, \\
&\|D(a_1^{-2})\|_{q,T}^2\|\mathbf{v}\|_{q,T}^4,\ \|a_1^{-2}\|_{q,T}^2\|D\mathbf{v}\|_{q,T}^2\|\mathbf{v}\|_{q,T}^2, \\
&\|a_1^{-1}\|_{q,T}^2\|\mathbf{v}\|_{q,T}^2,\ \|a_2\|_{q,T}^2,\ \|a_1^{-2}\|_{q,T}^2\|\mathbf{v}\|_{q,T}^4,\ |a_1^{-1}|_{L^\infty,T}^2|\mathbf{v}|_{L^\infty,T}^2, \\
&|a_1^{-2}|_{L^\infty,T}^2|\mathbf{v}|_{L^\infty,T}^4,\ |(a_1^{-1})_t|_{L^\infty,T}^2|\mathbf{v}|_{L^\infty,T}^2,\ |a_1^{-1}|_{L^\infty,T}^2|\mathbf{v}_t|_{L^\infty,T}^2\}
\end{aligned} \tag{B.26}
$$

where $q = \max\{r, s_0\}$, where $r \ge 1$, and where $s_0 = [\frac{N}{2}]+1 = 2$ for $N = 2$ or $N = 3$.

After multiplying estimate (B.25) by β, where $0 < \beta < 1$ is a constant, and then adding the resulting inequality to the estimate (B.15) for $\nabla\cdot\mathbf{u}$, and using the fact that $\epsilon = 1/4$, and using the fact that $\|D\mathbf{u}\|_r^2 = \|\nabla\cdot\mathbf{u}\|_r^2 + \|\nabla\times\mathbf{u}\|_r^2$, which follows from the identity $\Delta\mathbf{u}_\alpha = \nabla(\nabla\cdot\mathbf{u}_\alpha) - \nabla\times(\nabla\times\mathbf{u}_\alpha)$, we obtain

$$
\begin{aligned}
&\frac{\beta}{4}\|D\mathbf{u}\|_r^2 + \|\nabla\rho\|_r^2 + \|\Delta\rho\|_r^2 \\
&= \beta(\frac{1}{2}-\epsilon)(\|\nabla\times\mathbf{u}\|_r^2+\|\nabla\cdot\mathbf{u}\|_r^2) + \|\nabla\rho\|_r^2 + \|\Delta\rho\|_r^2 \\
&\le C_3(\|\nabla\times\mathbf{u}_0\|_r^2+\|\nabla\cdot\mathbf{u}_0\|_r^2+\|\nabla\rho_0\|_r^2+\|\Delta\rho_0\|_r^2) \\
&\quad + \frac{\beta}{4\epsilon}|a_1(\mathbf{x},0)^{-1}|_{L^\infty}^2|\mathbf{v}_0|_{L^\infty}^2\|\nabla\rho_0\|_r^2 + \frac{\beta}{4\epsilon}|a_1^{-1}|_{L^\infty}^2|\mathbf{v}|_{L^\infty}^2\|\nabla\rho\|_r^2 \\
&\quad + C_3K_4\int_0^t(\|D\mathbf{u}\|_r^2+\|\nabla\rho\|_r^2+\|\Delta\rho\|_r^2)d\tau + C_3\int_0^t\|\mathbf{F}\|_{r+1}^2 d\tau
\end{aligned} \tag{B.27}
$$

where $C_3 = \hat{C}_3(r, c, c_1, c_2)$, and where we define

$$
\begin{aligned}
K_4 = \max\Big\{&1,\ \|a_1^{-1}\|_{q+1,T}^2\|\mathbf{v}\|_{q+1,T}^2,\ \|a_2\|_{q+1,T}^2,\quad \|a_1^{-2}\|_{q+1,T}^2\|\mathbf{v}\|_{q+1,T}^4, \\
&\|(a_1^{-1})_t\|_{2,T}^2\|\mathbf{v}\|_{2,T}^2,\ \|a_1^{-1}\|_{2,T}^2\|\mathbf{v}_t\|_{2,T}^2,\ \|(a_1)_t\|_{2,T},\quad \|(a_2)_t\|_{2,T}\Big\}
\end{aligned} \tag{B.28}
$$

where $q = \max\{r, s_0\}$, where $r \ge 1$, and where $s_0 = [\frac{N}{2}]+1 = 2$ for $N = 2$ or $N = 3$. Here, we used the Sobolev inequality $|f|_{L^\infty} \le C\|f\|_{s_0}$. Note that $K_2 \le K_4$ and $K_3 \le K_4$.

Next, using the fact that $0 < c_1 < a_1(\mathbf{x}, t) < c_2$ where $c_1 < 1$, and using the fact that $|\mathbf{v}(\mathbf{x}, t)| < c_3$, where $c_3 > 1$, we define $\beta = c_1^2/(2c_3^2)$ (so that we have $\beta < 1$), and we have already defined $\epsilon = 1/4$. We obtain the following estimate for one of the terms from (B.27):

$$
\frac{\beta}{4\epsilon}|a_1^{-1}|_{L^\infty}^2|\mathbf{v}|_{L^\infty}^2\|\nabla\rho\|_r^2 = \frac{c_1^2}{2c_3^2}|a_1^{-1}|_{L^\infty}^2|\mathbf{v}|_{L^\infty}^2\|\nabla\rho\|_r^2 \le \frac{1}{2}\|\nabla\rho\|_r^2
$$

Similarly, we obtain the estimate

$$\frac{\beta}{4\epsilon}|a_1(\mathbf{x},0)^{-1}|_{L^\infty}^2|\mathbf{v}_0|_{L^\infty}^2\|\nabla\rho_0\|_r^2 \le \frac{1}{2}\|\nabla\rho_0\|_r^2$$

Using these estimates in the right-hand side of (B.27) and then moving the term $\frac{1}{2}\|\nabla\rho\|_r^2$ to the left-hand side, and applying Gronwall's inequality yields the desired estimate

$$\begin{aligned}\|D\mathbf{u}\|_r^2 + \|\nabla\rho\|_r^2 + \|\Delta\rho\|_r^2 &\le C_4(1 + C_4K_4Te^{C_4K_4T})(\|D\mathbf{u}_0\|_r^2 + \|\nabla\rho_0\|_{r+1}^2) \\ &\quad + C_4(1 + C_4K_4Te^{C_4K_4T})\int_0^t \|\mathbf{F}\|_{r+1}^2 d\tau\end{aligned} \tag{B.29}$$

where $C_4 = \hat{C}_4(r, c, c_1, c_2, c_3)$. From (B.7), we obtain the L^2 estimate

$$\begin{aligned}\|\mathbf{u}\|_0^2 + \|\rho\|_0^2 + \|\nabla\rho\|_0^2 &\le C_5(1 + C_5K_4Te^{C_5K_4T})(\|\mathbf{u}_0\|_0^2 + \|\rho_0\|_0^2 + \|\nabla\rho_0\|_0^2) \\ &\quad + C_5(1 + C_5K_4Te^{C_5K_4T})\int_0^t (\|D\mathbf{u}\|_0^2 + \|\mathbf{F}\|_0^2) d\tau\end{aligned}$$

where $C_5 = \hat{C}_5(c, c_1, c_2)$, and where we used the fact that $K_1 \le CK_4$, where K_1 was defined in (B.6). The preceding two estimates are the desired result. □

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