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BOUNDED SOLUTIONS: DIFFERENTIAL VS DIFFERENCE EQUATIONS

JEAN MAWHIN

Abstract. We compare some recent results on bounded solutions (over $\mathbb{Z}$) of nonlinear difference equations and systems to corresponding ones for nonlinear differential equations. Bounded input-bounded output problems, lower and upper solutions, Landesman-Lazer conditions and guiding functions techniques are considered.

1. Introduction

In this paper, we survey some recent results on bounded solutions (over $\mathbb{Z}$) of nonlinear difference equations or systems, and compare them to the corresponding situations for bounded solutions (over $\mathbb{R}$) of nonlinear differential equations or systems.

We first give some maximum and anti-maximum principles for bounded solutions of linear differential equations of the form

$$u'(t) + \lambda u(t) = f(t)$$

and of corresponding linear difference equations of the form

$$\Delta u_m + \lambda u_m = f_m \quad (m \in \mathbb{Z}).$$

Then we compare Landesman-Lazer conditions for bounded solutions of Duffing's differential equations

$$x'' + cx' + g(x) = p(t),$$

with those for bounded solutions of Duffing's difference equations

$$\Delta^2 x_{m-1} + c\Delta x_m + g(x_m) = p_m \quad (m \in \mathbb{Z})$$

or

$$\Delta^2 x_{m-1} + c\Delta x_{m-1} + g(x_m) = p_m \quad (m \in \mathbb{Z}).$$

Finally, we compare the method of guiding functions for systems of ordinary differential equations

$$x' = f(t, x)$$

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and for systems of difference equations

$$\Delta x_m = f_m(x_m),$$

or corresponding discrete dynamical systems

$$x_{m+1} = g_m(x_m).$$

2. Bounded input–bounded output problem for first order linear equations

2.1. **Bounded solutions of linear ordinary differential equations.** The *bounded input-bounded output (BIBO)* problem for the linear ordinary differential equation

$$u'(t) + \lambda u(t) = f(t) \tag{2.1}$$

consists in finding conditions upon λ under which, for each $f \in L^\infty(\mathbb{R})$, (2.1) has a unique solution $u \in AC(\mathbb{R}) \cap L^\infty(\mathbb{R})$. We denote the usual norm of $v \in L^\infty(\mathbb{R})$ by $|u|_\infty$. Such a solution is simply called a *bounded solution* of (2.1). The BIBO problem was essentially solved as follows by Perron in 1930 [14]. If $\lambda = 0$, we have no uniqueness for $f \equiv 0$, and no existence for $f(t) \equiv 1$. If $\lambda \neq 0$, the homogeneous problem

$$u'(t) + \lambda u(t) = 0 \tag{2.2}$$

only has the trivial bounded solution. For $\lambda > 0$,

$$u(t) = \int_{-\infty}^{t} e^{-\lambda(t-s)} f(s)\, ds \tag{2.3}$$

is a bounded solution of (2.1), and hence the unique one. For $\lambda < 0$,

$$u(t) = -\int_{t}^{+\infty} e^{-\lambda(t-s)} f(s)\, ds \tag{2.4}$$

is a bounded solution of (2.1), and hence the unique one. We summarize the results in the following

Proposition 2.1. *Equation* (2.1) *has a unique solution* $u \in AC(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *for each* $f \in L^\infty(\mathbb{R})$ *if and only if* $\lambda \in \mathbb{R} \setminus \{0\}$.

2.2. **A maximum principle for bounded solutions of differential equations.** The following definition is modelled upon the one given in [5] in a different context.

Definition 2.2. Given $\lambda \in \mathbb{R} \setminus \{0\}$, the linear operator $d/dt + \lambda I : AC(\mathbb{R}) \cap L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ satisfies a *maximum principle* (MP) if, for each $f \in L^\infty(\mathbb{R})$, (2.1) has a unique solution u and if $f(t) \geq 0$ ($t \in \mathbb{R}$) implies that $\lambda u(t) \geq 0$ ($t \in \mathbb{R}$). The MP is *strong* if, furthermore, $f(t) \geq 0$ ($t \in \mathbb{R}$) and $\int_{\mathbb{R}} f > 0$ imply that $\lambda u(t) > 0$ ($t \in \mathbb{R}$)).

A direct consideration of formulas (2.3) and (2.4) immediately implies the following

Proposition 2.3. *If* $f \in L^\infty(\mathbb{R})$, *the BIBO problem for* (2.1) *has a MP if and only if* $\lambda \in \,]-\infty, 0[\cup]0, +\infty[$, *and the MP is not strong.*

2.3. **Bounded solutions of linear difference equations.** Let $l^\infty(\mathbb{Z}) = \{u = (u_m)_{m\in\mathbb{Z}} : \sup_{m\in\mathbb{Z}} |u_m| < \infty\}$. Endowed with the norm $|u|_\infty := \sup_{m\in\mathbb{Z}} |u_m|$, $l^\infty(\mathbb{Z})$ is a Banach space. We denote by $\Delta u_m = u_{m+1} - u_m$ $(m \in \mathbb{Z})$ the forward difference operator acting on sequences $(u_m)_{m\in\mathbb{Z}}$. The bounded input-bounded output (BIBO) problem we address is to find the values of λ such that, for each $(f_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$, the linear difference equation

$$\Delta u_m + \lambda u_m = f_m \quad (m \in \mathbb{Z}) \tag{2.5}$$

has a unique solution $(u_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$. We refer those solutions as *bounded solutions*.

Easy computations show that, for $\lambda = 0$, existence or uniqueness may fail. Namely, for $f_m = 0$ $(m \in \mathbb{Z})$, any constant sequence is a solution in $l^\infty(\mathbb{Z})$, and, for $f_m = 1$ $(m \in \mathbb{Z})$, the solutions given by $u_m = u_0 + m$ $(m \in \mathbb{Z})$ are all unbounded. Similarly, for $\lambda = 2$, any alternating sequence $(-1)^m c$ is a solution of (2.5) with $f_m = 0$ $(m \in \mathbb{Z})$, and, for $f_m = (-1)^m$ $(m \in \mathbb{Z})$ none of the solutions $u_m = (-1)^m u_0 + m(-1)^{m+1}$ $(m \in \mathbb{Z})$ is bounded.

Now, for $\lambda \in \mathbb{R} \setminus \{0, 2\}$, it is easy to see that the homogeneous difference equation

$$\Delta u_m + \lambda u_m = 0 \tag{2.6}$$

only has the trivial solution in $l^\infty(\mathbb{Z})$. On the other hand, if $\lambda \in]0, 2[$,

$$u_m = \sum_{k=-\infty}^{m-1} (1-\lambda)^{m-k-1} f_k \quad (m \in \mathbb{Z}) \tag{2.7}$$

is a solution of (2.5) belonging to $l^\infty(\mathbb{Z})$, and hence the unique one. Similarly, if $\lambda \in]-\infty, 0[\cup]2, +\infty[$,

$$u_m = -\sum_{k=m}^{\infty} (1-\lambda)^{m-k-1} f_k \quad (m \in \mathbb{Z}) \tag{2.8}$$

is the unique solution of (2.5) belonging to $l^\infty(\mathbb{Z})$. We summarize the results in the following proposition.

Proposition 2.4. *Equation* (2.5) *has a unique solution* $(u_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ *for each* $(f_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ *if and only if* $\lambda \in \mathbb{R} \setminus \{0, 2\}$.

2.4. **A maximum principle for bounded solutions of difference equations.** The following definition is modelled upon the one given in [5] in a different context.

Definition 2.5. Given $\lambda \in \mathbb{R} \setminus \{0\}$, the linear operator $\Delta + \lambda I : l^\infty(\mathbb{Z}) \to l^\infty(\mathbb{Z})$ satisfies a *maximum principle* (MP) if for each $(f_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$, the equation (2.5) has a unique solution and if $f_m \geq 0$ $(m \in \mathbb{Z})$ implies that $\lambda u_m \geq 0$ $(m \in \mathbb{Z})$. The maximum principle is said to be *strong* if, in addition, $f_m \geq 0$ $(m \in \mathbb{Z})$, and $\sup_{m\in\mathbb{Z}} f_m > 0$ imply that $\lambda u_m > 0$ $(m \in \mathbb{Z})$).

Notice that, in the more classical terminology modelled on the one for second order elliptic operators, the above definition corresponds to a *maximum principle* when $\lambda < 0$, and to an *anti-maximum principle* in the sense of Clément-Pelletier [6] when $\lambda > 0$. The following result can be read directly upon formulas (2.7) and (2.8).

Proposition 2.6. *The BIBO problem for* (2.5) *has a MP if and only if* $\lambda \in]-\infty, 0[\cup]0, 1]$*, and this MP is not strong;*

2.5. **BIBO problem: linear differential vs linear difference equations.** It follows from Propositions 2.3 and 2.6 that the ranges of values for which a maximum principle hold are different in the differential and the difference cases. The following simple propositions help to understand the reason of this difference. Given a linear operator L between Banach spaces, let $\sigma(L)$ denotes its (complex) spectrum and $\mathcal{R}(L) = \mathbb{C} \setminus \sigma(L)$ denote its resolvent set. The following propositions are analogous to those proved in [5] is a different context.

Proposition 2.7. *If the BIBO problem for $L + \lambda I$, with $L = \Delta$ or d/dt has a MP for some $\lambda \neq 0$, then*

$$|u|_\infty \le \frac{|f|_\infty}{|\lambda|}. \tag{2.9}$$

Proof. If $u \in L^\infty(\mathbb{R})$ is the solution of (2.1) and $v = \frac{|f|_\infty}{\lambda} \in L^\infty(\mathbb{R})$ the solution of

$$Lv + \lambda v = |f|_\infty,$$

then $v - u \in L^\infty(\mathbb{R})$ is the solution of

$$L(v-u) + \lambda(v-u) = |f|_\infty - f$$

and the MP implies that $\lambda(v-u) \ge 0$, i.e. that

$$\lambda u \le |f|_\infty.$$

Similarly, we have

$$L(v+u) + \lambda(v+u) = |f|_\infty + f$$

and hence, by the MP, $\lambda(v+u) \ge 0$, i.e. $\lambda u \ge -|f|_\infty$. □

In the ordinary differential equation case, the estimate (2.9) can also be obtained directly for any $\lambda \in \mathbb{R} \setminus \{0\}$. Indeed, it follows from (2.3) that if $\lambda > 0$, then

$$|u(t)| \le |f|_\infty \int_{-\infty}^{t} e^{-\lambda(t-s)}\, ds = \frac{1}{\lambda}|f|_\infty.$$

Similarly, if $\lambda < 0$, we get

$$|u(t)| \le |f|_\infty \int_{t}^{+\infty} e^{-\lambda(t-s)}\, ds = -\frac{1}{\lambda}|f|_\infty.$$

In the DE case, the following estimates can be obtained directly from the formulas (2.7) and (2.8)

$$|u|_\infty \le \frac{|f|_\infty}{|\lambda|} \quad \text{if } \lambda < 0, \quad |u|_\infty \le \frac{|f|_\infty}{|\lambda|} \quad \text{if } 0 < \lambda \le 1,$$
$$|u|_\infty \le \frac{|f|_\infty}{2-\lambda} \quad \text{if } 1 < \lambda < 2, \quad |u|_\infty \le \frac{|f|_\infty}{\lambda - 2} \quad \text{if } 2 < \lambda.$$

Proposition 2.8. *If the BIBO problem for $L + \lambda I$, with $L = \Delta$ or d/dt has a MP for some $\lambda \neq 0$, then*

$$\mathcal{R}(L) \supset \{\mu \in \mathbb{C} : |\mu - \lambda| < |\lambda|\}. \tag{2.10}$$

Proof. We have, for $\mu \in \mathbb{C}$,

$$\begin{aligned} Lu + \mu u = f &\Leftrightarrow Lu + \lambda u + (\mu - \lambda)u = f \\ &\Leftrightarrow u + (\mu - \lambda)(L+\lambda)^{-1}u = (L+\lambda)^{-1} f, \end{aligned}$$

and, using Proposition 2.7,

$$|(\mu - \lambda)(L + \lambda)^{-1} u|_\infty \leq |\mu - \lambda| \frac{|u|_\infty}{|\lambda|},$$

so that, for $\frac{|\mu-\lambda|}{|\lambda|} < 1$, equation $Lu + \mu u = f$ has a unique bounded solution. $\square$

It is easy to check that, for the BIBO problem in the ordinary differential equation case, the spectrum $\sigma(L)$ of $L : AC(\mathbb{R}) \cap L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ is equal to $i\mathbb{R}$.

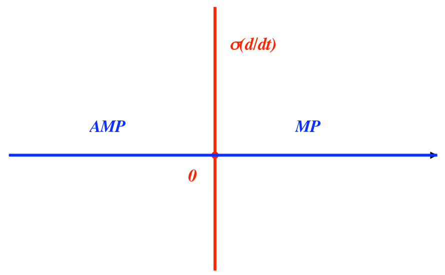

Figure 1. ODE spectrum

Therefore, for any $\lambda \in \mathbb{R}$, the set $\{\mu \in \mathbb{C} : |\mu - \lambda| < |\lambda|\}$ is always contained in the resolvent set $\mathcal{R}(L)$.

Similarly, for the BIBO problem in the difference equation case, the spectrum $\sigma(L)$ of $L : l^\infty(\mathbb{Z}) \to l^\infty(\mathbb{Z})$ is the circle $\{1 + e^{i\theta} : \theta \in [0, 2\pi]\}$. Hence, for any $\lambda < 0$, the set $\{\mu \in \mathbb{C} : |\mu - \lambda| < |\lambda|\}$ is contained in $\mathcal{R}(L)$, but, for $\lambda > 0$, this is only true for $\lambda \in]0, 1]$. This, together with Proposition 2.8, sheds some light on the fact that the maximum principle for the BIBO problem in the difference case only holds for $\lambda \in]-\infty, 0[\cup]0, 1]$. Notice also that the estimate $|u|_\infty \leq \frac{|f|_\infty}{|\lambda|}$ only holds for those values of λ.

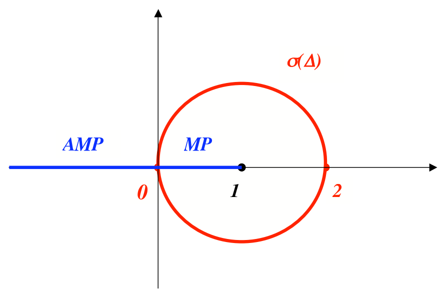

Figure 2. DE spectrum

3. Bounded input–bounded output problems for some Duffing's equations

3.1. Linear equations. It is a standard result that the second order linear ordinary differential equation

$$x'' + cx' + ax = f(t) \tag{3.1}$$

has a unique solution $x \in AC^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for any $f \in L^\infty(\mathbb{R})$ if and only if $a < 0$.

3.2. **Duffing's equations.** Duffing's differential equations are nonlinear second order differential equations of the form

$$x'' + cx' + g(x) = p(t), \tag{3.2}$$

where $c \in \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ and $p : \mathbb{R} \to \mathbb{R}$ are continuous.

Correspondingly, we call *Duffing difference equations* the second order nonlinear difference equations of the form

$$\Delta^2 x_{m-1} + c\Delta x_m + g(x_m) = p_m \quad (m \in \mathbb{Z}) \tag{3.3}$$

or

$$\Delta^2 x_{m-1} + c\Delta x_{m-1} + g(x_m) = p_m \quad (m \in \mathbb{Z}) \tag{3.4}$$

where

$$\Delta^2 x_{m-1} = x_{m+1} - 2x_m + x_{m-1} \quad (m \in \mathbb{Z}),$$

$g \in C(\mathbb{R}, \mathbb{R})$, and $c \in \mathbb{R}$.

The bounded input-bounded output (BIBO) problem for (3.2) consists, for given g, in determining the inputs $p \in L^\infty(\mathbb{R})$ for which equation (3.2) has at least one solution $u \in AC^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. This problem was first considered by Ahmad [1], and then by Ortega [12], Ortega-Tineo [13], and Mawhin-Ward [10].

Similarly, the bounded input-bounded output (BIBO) problem for (3.3) or (3.4) consists, for given g, in determining the inputs $(p_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ for which (3.3) or (3.4) has at least one solution $(x_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$. See [3, 9].

3.3. **Bounded lower and upper solutions.** We develop a method of lower and upper solutions for the bounded solutions of (3.3) and (3.4). We first need a limiting lemma [9].

Lemma 3.1. *Let $f_m \in C(\mathbb{R}, \mathbb{R})$ $(m \in \mathbb{Z})$, $c \in \mathbb{R}$ Assume that, for each $n \in \mathbb{N}^*$, there exists $(x_m^n)_{-n-1\le m\le n+1}$ such that*

$$\Delta^2 x_{m-1}^n + c\Delta x_m^n + f_m(x_m^n) = 0 \quad (-n \le m \le n)$$

and such that $\alpha_m \le x_m^n \le \beta_m$ $(|m| \le n+1)$ for some $(\alpha_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$, $(\beta_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$. Then there exists $(\widehat{x}_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ such that

$$\Delta^2 \widehat{x}_{m-1} + c\Delta \widehat{x}_m + f_m(\widehat{x}_m) = 0, \ \alpha_m \le \widehat{x}_m \le \beta_m \quad (m \in \mathbb{Z}).$$

The same result for

$$\Delta^2 \widehat{x}_{m-1} + c\Delta \widehat{x}_{m-1} + f_m(\widehat{x}_m) = 0 \ (m \in \mathbb{Z}).$$

The proof is based upon Borel-Lebesgue lemma and Cantor diagonalization process.

We now define the concept of bounded lower and upper solutions for second order difference equations [9]. Let $f_m \in C(\mathbb{R}, \mathbb{R})$ $(m \in \mathbb{Z})$, $c \in \mathbb{R}$.

Definition 3.2. $(\alpha_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ (resp. $(\beta_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$) is a *bounded lower solution* (resp. *upper solution*) for

$$\Delta^2 x_{m-1} + c\Delta x_m + f_m(x_m) = 0 \quad (m \in \mathbb{Z})$$

if

$$\Delta^2 \alpha_{m-1} + c\Delta \alpha_m + f_m(\alpha_m) \ge 0$$
$$(\text{resp.} \quad \Delta^2 \beta_{m-1} + c\Delta \beta_m + f_m(\beta_m) \le 0) \quad (m \in \mathbb{Z})$$

A similar definition holds for

$$\Delta^2 x_{m-1} + c\Delta x_{m-1} + f_m(x_m) = 0 \quad (m \in \mathbb{Z}).$$

We have the associated existence theorem.

Theorem 3.3. *If $c \geq 0$ (resp. $c \leq 0$) and*

$$\Delta^2 x_{m-1} + c\Delta x_m + f_m(x_m) = 0 \quad (m \in \mathbb{Z})$$

$$(\textit{resp.} \quad \Delta^2 x_{m-1} + c\Delta x_{m-1} + f_m(x_m) = 0 \quad (m \in \mathbb{Z}))$$

has a lower solution $(\alpha_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ and an upper solution $(\beta_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ such that $\alpha_m \leq \beta_m$ $(m \in \mathbb{Z})$, then it has a solution $(x_m)_{m\in\mathbb{Z}}$ such that $\alpha_m \leq x_m \leq \beta_m (m \in \mathbb{Z})$

Proof. The proof is based upon the existence theorem for lower and upper solutions for the Dirichlet problem

$$\Delta^2 x_{m-1} + c\Delta x_m + f_m(x_m) = 0 \quad (-n \leq m \leq n)$$

$$x_{-n-1} = \alpha_{-n-1}, \quad x_{n+1} = \alpha_{n+1}$$

for each n and the limiting Lemma 3.1. ◻

An important special case is that of constant lower and upper solutions.

Corollary 3.4. *If $c \geq 0$ and if $\exists \alpha \leq \beta$ such that $f_m(\beta) \leq 0 \leq f_m(\alpha)$ $(m \in \mathbb{Z})$, then*

$$\Delta^2 x_{m-1} + c\Delta x_m + f_m(x_m) = 0 \quad (m \in \mathbb{Z})$$

has a solution $(x_m)_{m\in\mathbb{Z}}$ such that $\alpha \leq x_m \leq \beta$ $(m \in \mathbb{Z})$.

Example 3.5. If $c \geq 0$ and $a > 0$, then for each $(p_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$

$$\Delta^2 x_{m-1} + c\Delta x_m - a x_m = p_m \quad (m \in \mathbb{Z})$$

has a unique solution $(u_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$.

Similar results hold if $c \leq 0$ for the equations

$$\Delta^2 x_{m-1} + c\Delta x_{m-1} + f_m(x_m) = 0 \quad (m \in \mathbb{Z})$$

$$\Delta^2 x_{m-1} + c\Delta x_{m-1} - a x_m = p_m \quad (m \in \mathbb{Z}).$$

In the ordinary differential equation case, a similar result holds for all $c \in \mathbb{R}$ for the equations

$$x'' + cx' + f(t, x) = 0$$

$$x'' + cx' - ax = p(t) \quad (a > 0, \quad p \in L^\infty(\mathbb{R}))$$

(see [4, 11]).

3.4. **Second order linear equations.** The following result can be proved like Proposition 2.4.

Proposition 3.6. *If $c \notin \{-2, 0\}$*

$$\Delta x_{m-1} + c x_m = h_m \quad (m \in \mathbb{Z})$$

has a unique solution $(x_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ for each $(h_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$.

Before dealing with second order difference equations, we introduce some notions and results for sequences with bounded primitive. The corresponding concepts for functions upon $\mathbb{R}$ were introduced in [12].

Definition 3.7. The Δ-primitive $(H_m^\Delta)_{m\in\mathbb{Z}}$ of $(h_m)_{m\in\mathbb{Z}}$ is any sequence $(H_m^\Delta)_{m\in\mathbb{Z}}$ such that $\Delta H_m^\Delta = h_m$ $(m \in \mathbb{Z})$.

Such a Δ-primitive is for example given by

$$H_m^\Delta = \begin{cases} \sum_{k=0}^{m-1} h_k & \text{if } m \geq 1 \\ 0 & \text{if } m = 0 \\ -\sum_{k=m}^{-1} h_k & \text{if } m \leq -1 \end{cases} \qquad (m \in \mathbb{Z})$$

We define the space $BP(\mathbb{Z})$ as the set

$$\{(h_m)_{m\in\mathbb{Z}} : (H_m^\Delta)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})\}.$$

It is easy to check that $BP(\mathbb{Z}) \subsetneq l^\infty(\mathbb{Z})$. The situation is different in the continuous case, where $BP(\mathbb{R}) \not\subset BC(\mathbb{R})$, and $BC(\mathbb{R}) \not\subset BP(\mathbb{R})$.

We have now the following result for the BIBO problem for some linear second order difference equations.

Proposition 3.8. *If $c \notin \{-2, 0\}$,*

$$\Delta^2 x_{m-1} + c\Delta x_m = h_m \quad (m \in \mathbb{Z})$$

has a solution $(x_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ if and only if $h \in BP(\mathbb{Z})$.

Proposition 3.9. *If $c \notin \{0, 2\}$,*

$$\Delta^2 x_{m-1} + c\Delta x_{m-1} = h_m \quad (m \in \mathbb{Z})$$

has a solution $(x_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ if and only if $h \in BP(\mathbb{Z})$.

The corresponding results for ordinary differential equations were proved by Ortega in [12].

Proposition 3.10. *If $c \neq 0$, equation*

$$x'' + cx' = h(t)$$

has a solution $x \in AC(\mathbb{R}) \cap L^\infty(\mathbb{R})$ if and only if $h \in BP(\mathbb{R})$.

We now introduce concepts of generalized mean values to bounded sequences.

Definition 3.11. The *lower (resp. upper) mean value* of $(p_j)_{j\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ is the real number defined by

$$\widehat{p} := \lim_{n\to\infty} \inf_{m-k\geq n} \Big(\frac{1}{m-k}\sum_{j=k+1}^{m} p_j\Big)$$

$$\Big(\text{resp.} \quad \widetilde{p} := \lim_{n\to\infty} \sup_{m-k\geq n} \Big(\frac{1}{m-k}\sum_{j=k+1}^{m} p_j\Big)\Big)$$

Lemma 3.12. *The following statements are equivalent :*

(i) $\alpha < \widehat{p} \leq \widetilde{p} < \beta$.
(ii) *there exists $(p_m^*)_{m\in\mathbb{Z}} \in BP(\mathbb{Z})$, $(p_m^{**})_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ such that $p_m = p_m^* + p_m^{**}$ $(m \in \mathbb{Z})$ and $\alpha < \inf_{k\in\mathbb{Z}} p_k^{**} \leq \sup_{k\in\mathbb{Z}} p_k^{**} < \beta$.*

Corollary 3.13. *If $\widehat{p} = \widetilde{p} = 0$, then, for each $\epsilon > 0$ there exists $(p_m^*)_{m\in\mathbb{Z}} \in BP(\mathbb{Z})$, $(p_m^{**})_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ such that $p_m = p_m^* + p_m^{**}$ $(m \in \mathbb{Z})$, $\sup_{k\in\mathbb{Z}} |p_k^{**}| < \epsilon$.*

In the continuous case those results and concepts are due to Ortega-Tineo [13].

3.5. **Duffing difference equations.** We can now prove the following result for the existence of bounded solutions of Duffing difference equations.

Theorem 3.14. *Assume that the following conditions hold.*

(1) $c > 0$, $g \in C(\mathbb{R}, \mathbb{R})$, $(p_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$
(2) *There exists* $r_0 > 0$ *and* $\delta_- < \delta_+$ *such that*
$$g(y) \geq \delta_+ \quad \text{for } y \leq -r_0, \quad g(y) \leq \delta_- \quad \text{for } y \geq r_0.$$
(3) $\delta_- < \widehat{p} \leq \widetilde{p} < \delta_+$.

Then
$$\Delta^2 x_{m-1} + c\Delta x_m + g(x_m) = p_m \quad (m \in \mathbb{Z})$$
has at least one solution $(x_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$.

Proof. Write $p_m = p_m^* + p_m^{**}$ ($m \in \mathbb{Z}$) with $(p_m^*)_{m\in\mathbb{Z}} \in BP(\mathbb{Z})$, $(p_m^{**})_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ and $\delta_- < \inf_{k\in\mathbb{Z}} p_k^{**} \leq \sup_{k\in\mathbb{Z}} p_k^{**} < \delta_+$. By Proposition 3.8,
$$\Delta^2 x_{m-1} + c\Delta x_m = p_m^* \quad (m \in \mathbb{Z})$$
has a solution $(u_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$. Letting $x_m = u_m + z_m$ ($m \in \mathbb{Z}$), we obtain the equivalent problem
$$\Delta^2 z_{m-1} + c\Delta z_m + g(u_m + z_m) - p_m^{**} = 0 \quad (m \in \mathbb{Z}). \tag{3.5}$$
Then $\alpha = -r_0 - \sup_{k\in\mathbb{Z}} u_k$ is a lower solution and $\beta = r_0 - \inf_{k\in\mathbb{Z}} u_k$ an upper solution for (3.5), and we conclude using Corollary 3.4. □

3.6. **Landesman-Lazer condition.** Theorem 3.14 gives existence conditions of the Landesman-Lazer type.

Corollary 3.15. *If* $c > 0$, $g \in C(\mathbb{R}, \mathbb{R})$, $(p_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$, *and*
$$\overline{\lim}_{y\to+\infty} g(y) < \widehat{p} \leq \widetilde{p} < \underline{\lim}_{y\to-\infty} g(y) \tag{3.6}$$
then
$$\Delta^2 x_{m-1} + c\Delta x_m + g(x_m) = p_m \quad (m \in \mathbb{Z})$$
has a solution $(x_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$.

Remark 3.16. If, for all $x \in \mathbb{R}$,
$$-\infty < \overline{\lim}_{y\to+\infty} g(y) < g(x) < \underline{\lim}_{y\to-\infty} g(y) < +\infty$$
then $(p_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ and (3.6) is necessary for the existence of a bounded solution.

Similar results hold for
$$\Delta^2 x_{m-1} + c\Delta x_{m-1} + g(x_m) = p_m \quad (c < 0) \quad (m \in \mathbb{Z})$$
In the ordinary differential equation case, similar results hold for
$$x'' + cx' + g(x) = p(t) \quad (c \neq 0)$$
(see [10]).

Example 3.17. 1. If $c > 0$, $b > 0$,
$$\Delta^2 x_{m-1} + c\Delta x_m - b\frac{x_m}{1 + |x_m|} = p_m \quad (m \in \mathbb{Z})$$
has a solution $(x_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ if and only if $(p_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ and $-b < \widehat{p} \leq \widetilde{p} < b$.

2. If $c > 0$, $b > 0$, and $0 \le a < 1$,

$$\Delta^2 x_{m-1} + c\Delta x_m - b\frac{x_m}{1+|x_m|^a} = p_m \quad (m \in \mathbb{Z})$$

has a solution $(x_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ if and only if $(p_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$.

It remains an open problem to prove or disprove that if $c > 0$ and $b > 0$,

$$\Delta^2 x_{m-1} + c\Delta x_m + \frac{bx_m}{1+|x_m|} = p_m \quad (m \in \mathbb{Z})$$

has a solution $(x_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ if and only if $(p_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ and $-b < \widehat{p} \le \widetilde{p} < b$.

Similarly it is an open problem to prove or disprove that if $c > 0$, $b > 0$, and $0 \le a < 1$,

$$\Delta^2 x_{m-1} + c\Delta x_m + \frac{bx_m}{1+|x_m|^a} = p_m \quad (m \in \mathbb{Z})$$

has a solution $(x_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ if and only if $(p_m)_{m\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$.

The corresponding results are true in the ordinary differential equation case [1, 12, 13].

4. Guiding functions for bounded solutions of systems of difference equations

4.1. **Guiding functions for ordinary differential equations.** Consider the system

$$x' = f(t, x) \tag{4.1}$$

where $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$.

Definition 4.1. A guiding function for (4.1) is a function $V \in C^1(\mathbb{R}^n, \mathbb{R})$ such that, for some $\rho_0 > 0$,

$$\langle \nabla V(x), f(t, x)\rangle \le 0$$

when $\|x\| \ge \rho_0$.

The following theorem was first proved by Krasnosel'skii-Perov in 1958 [8]. A simpler proof has been given by Alonso-Ortega in 1995 [2].

Theorem 4.2. *If (4.1) has a guiding function V such that $\lim_{\|x\|\to\infty} V(x) = +\infty$, then (4.1) has a solution x bounded over $\mathbb{R}$.*

A natural question is to know if a corresponding result holds for a difference system

$$x_{n+1} - x_n = f_n(x_n) \quad (n \in \mathbb{Z})$$

or, equivalently for a discrete dynamical system

$$x_{n+1} = g_n(x_n) \quad (n \in \mathbb{Z}).$$

4.2. **Guiding function for difference equations.** Let us consider the system

$$x_{m+1} = g_m(x_m) \quad (m \in \mathbb{Z}) \tag{4.2}$$

where $g_m \in C(\mathbb{R}^n, \mathbb{R}^n)$ $(m \in \mathbb{Z})$.

Definition 4.3. A guiding function for (4.2) is a function $V \in C(\mathbb{R}^n, \mathbb{R})$, such that, for some $\rho_0 > 0$, $V(g_m(x)) \le V(x)$ when $\|x\| \ge \rho_0$ $(m \in \mathbb{Z})$.

The result corresponding to Theorem 4.2 would be : if $x_{m+1} = g_m(x_m)$ $(m \in \mathbb{Z})$ has a guiding function V such that $\lim_{\|x\|\to\infty} V(x) = +\infty$, then it has a bounded solution.

The following example, given in [3], shows that this result is *false*. Consider the maps $g_m \in C(\mathbb{R}, \mathbb{R})$ defined by

$$g_m(x) = \begin{cases} 1 & \text{if } x \le -2, \\ mx + 2m + 1 & \text{if } -2 < x < -1, \\ m+1 & \text{if } -1 \le x \le 1, \\ -mx + 2m + 1 & \text{if } 1 < x < 2, \\ 1 & \text{if } x \ge 2. \end{cases} \quad (m \in \mathbb{Z})$$

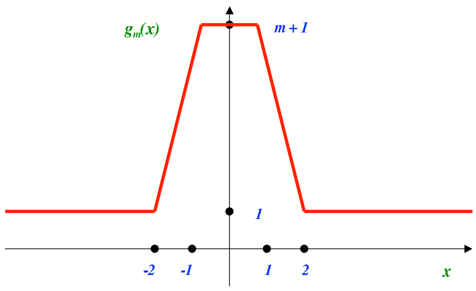

FIGURE 3. Graph of $g_m(x)$

Notice that $g_0(x) = 1$ $(x \in \mathbb{R})$, and hence $x_1 = g_0(x_0) = 1$, $x_2 = g_1(1) = 2$, $x_3 = g_2(3) = 1$, $x_4 = g_3(1) = 4$, $\dots$, $x_{2k-1} = 1$, $x_{2k} = 2k$ $(k \in \mathbb{N}_0,\ x_0 \in \mathbb{R})$. Hence, all the solutions of

$$x_{m+1} = g_m(x_m) \quad (m \in \mathbb{Z}) \tag{4.3}$$

are unbounded in the future, and no bounded solution exists. On the other hand, $V(x) = |x|$ with $\rho_0 = 3$ is a coercive guiding function for (4.3).

But the following existence theorem can be proved [3]. It uses another limiting lemma, due, for ordinary differential equations, to Krasnosel'skii [7], and whose proof is similar to that of Lemma 3.1.

Lemma 4.4. *Assume that $g_m \in C(\mathbb{R}^n, \mathbb{R}^n)$ $(m \in \mathbb{Z})$ and that there exists $\rho > 0$ such that, for each $k \in \mathbb{N}^*$*

$$x_{m+1} = g_m(x_m) \quad (-k \le m \le k)$$

has a solution $(x_m^k)_{-k \le m \le k+1}$, satisfying

$$\max_{-k \le m \le k+1} \|x_m^k\| \le \rho.$$

Then there exists a solution $(\widehat{x}_m)_{m\in\mathbb{Z}}$ of (4.2) such that $\sup_{m\in\mathbb{Z}} \|\widehat{x}_m\| \le \rho$.

Theorem 4.5. *Let $g_m \in C(\mathbb{R}^n, \mathbb{R}^n)$ $(m \in \mathbb{Z})$. If (4.2) has a guiding function V with constant ρ_0 such that $\lim_{\|x\|\to\infty} V(x) = +\infty$ and such that*

$$\sup_{m\in\mathbb{Z}} \max_{\|x\|\le\rho_0} \|g_m(x)\| < \infty, \tag{4.4}$$

then (4.2) has a solution $(x_m)_{m\in\mathbb{Z}} \in (l^\infty(\mathbb{Z}))^n$.

Proof. Take $\rho_1 > \max\{\rho_0, \sup_{m\in\mathbb{Z}} \max_{\|x\|\le\rho_0} \|g_m(x)\|\}$. Define

$$V_1 := \max_{\|x\|\le\rho_1} V(x).$$

Take $\rho_2 > \rho_1$ such that

$$B_{\rho_0} \subset B_{\rho_1} \subset S_1 := \{x \in \mathbb{R}^n : V(x) \le V_1\} \subset B_{\rho_2}.$$

Then it is easy to show that S_1 is positively invariant under the flow (4.2). For $n \in \mathbb{N}$ fixed and $(x^n)_{m\ge -n}$ the solution such that $x^n_{-n} = 0$ is such that

$$x^n_m \in S_1 \subset B_{\rho_2} \quad (m \ge -n,\ n \in \mathbb{N}).$$

Finally, use Lemma 4.4 to obtain a solution $(x_m)_{m\in\mathbb{Z}} \in (l^\infty(\mathbb{Z}))^n$. $\square$

Remark 4.6. Inequality (4.4) trivially holds if $g_m = g$ $(m \in \mathbb{Z})$.

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Jean Mawhin

Département de mathématique, Université Catholique de Louvain, B-1348 Louvain-la-Neuve, Belgium

E-mail address: jean.mawhin@uclouvain.be