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INFINITELY MANY PERIODIC SOLUTIONS OF NONLINEAR WAVE EQUATIONS ON S^n

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Abstract. The existence of time periodic solutions of nonlinear wave equations

$$u_{tt} - \Delta_n u + (\frac{n-1}{2})^2 u = g(u) - f(t,x)$$

on n-dimensional spheres is considered. The corresponding functional of the equation is studied by the convexity in suitable subspaces, minimax arguments for almost symmetric functional, comparison principles and Morse theory. The existence of infinitely many time periodic solutions is obtained where $g(u) = |u|^{p-2}u$ and the non-symmetric perturbation f is not small.

1. Introduction

This article is focused on the nonlinear wave equation

$$Au = g(u) - f(t,x), \quad (t,x) \in S^1 \times S^n,\ n > 1, \tag{1.1}$$

where $Au = u_{tt} - \Delta_n u + (\frac{n-1}{2})^2 u$, $g(u) = |u|^{p-2}u$ and $f(t,x)$ is 2π-periodic function in t.

The main difficulty of Problem (1.1) is the lack of compactness. When n is odd, the null space of A is infinite dimensional, and the component of u in this eigenspace is very difficult to control. This fact makes the problem much harder than an elliptic equation $\Delta u = g(x,u)$, or than a Hamiltonian system in which every eigenspace is finite dimensional. The associated functional of Equation (1.1) is indefinite in a very strong sense. In particular, it is not bounded from above or from below, and it does not satisfy the Palais-Smale compactness condition in any reasonable space.

In the case of $n = 1$, Bahri, Brezis, Coron, Nirenberg Rabinowitz and Tanaka [4, 7, 8, 9, 11, 16] have proved the existence of nontrivial periodic solutions of (1.1) under reasonable assumptions on g(u) at $u = 0$ and at u infinity, and the monotonicity of g. For $n > 1$, Benci and Fortunato [6] proved by using the dual variational method that the wave equation (1.1) possesses infinitely many 2π-periodic solutions in L^p in the case $g(u) = |u|^{p-2}u$, $2 < p < 2 + \frac{2}{n}$ and $f = 0$. The existence of a nontrivial periodic solution in the case of $g(0) = 0$ and $f = 0$, and the existence of multiple, in some cases infinitely many, time periodic solutions for several classes

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of nonlinear terms which satisfy symmetry and some growth conditions were established in Zhou [20, 21]. These conditions include time translation invariance or oddness; $f = 0$ and $g(u) \sim |u|^{p-2}u$ as $u \to \infty$, $(2 < p < \frac{2(n+1)}{n-1})$.

In this paper, we are going to study the effect of perturbations which are not small, destroy the symmetry with $f \neq 0$, and show how multiple solutions persist despite these nonsymmetric perturbations. Our main result is the following

Theorem 1.1. *Suppose that*

$$2 < p < \frac{7n + 1 + \sqrt{25n^2 - 2n + 9}}{2(3n - 1)}.$$

Then for any $f(t, x) \in L^{p/(p-1)}(S^1 \times S^n)$, 2π-periodic in t, the non-linear wave equation (1.1) *has infinitely many periodic weak solutions in $L^p(S^1 \times S^n) \cap H(S^1 \times S^n)$.*

Remark 1.2. By a weak solution of (1.1), we mean a function $u(t, x)$ satisfying

$$\int_{S^1 \times S^n} [u(\phi_{tt} - \Delta_n \phi + (\frac{n-1}{2})^2 \phi) + g(u)\phi - f\phi]\, dx\, dt = 0$$

for all $\phi \in C^\infty(S^1 \times S^n)$.

Remark 1.3. In general we cannot expect the equation (1.1) to have nontrivial solution if g in (1.1) is not super-linear [20].

In [20], the existence result is proved for the case g is an odd function and for $2 < p < \frac{1}{2}(1 + (\frac{9n-1}{n-1})^{1/2})$, where finite-dimensional approximations are used to overcome the lack of compactness mentioned above. Using, however, Tanaka's idea [16], we get around these difficulties by introducing a new functional $I(u) = \max_{v \in N} F(u + v)$ where N is the kernel space of the wave operator A, u is in the orthogonal complement of N and $F(u)$ (3.1) is the associated functional of the wave equation (1.1). Then since the nonlinear term $g(u) = |u|^{p-2}u$ is monotone ($g'(t) > 0$ for $t \neq 0$), we can use Lyapunov-Schmitt argument (Lemma 3.1) along with a compact embedding theorem (Theorem 2.1) to show that $I(u)$ has the desired compactness properties. And it is easy to see that each critical point of $I(u)$ corresponds to a unique critical point of $F(u)$. We are able to make a slight improvement on p compared to the result in [20].

If $f(t, x) \equiv 0$, the equation (1.1) has a natural symmetry, i.e., the functional $F(u)$ is symmetric and it is easier to handle. We will address the case where $f(t, x)$ is not identically 0 as a perturbation from symmetry by using the ideas from [12] where elliptic equations and Hamiltonian systems are discussed. The situation for the wave equation is more complicated since the operator A has infinitely many positive and infinitely many negative eigenvalues. The idea is based on some topological linking theorems. The key in this argument is to estimate the size of some explicitly constructed critical values. To do this, a symmetric comparison functional $K(u)$, defined only on the positive eigenspace, is introduced ([16, 2, 3]). Using the symmetry the critical values of $K(u)$ are constructed, and the relations between critical values of $I(u)$ and $K(u)$ is established. Then the estimate of Morse index at the critical points of $K(u)$ as in [5, 16] will lead us to the needed estimate for construction of critical points of $I(u)$.

For more general nonlinearity where g is not an odd function, we believe that same variational scheme can be applied. However, since the resulting functional is not symmetric anymore, S^1-action (instead of Z_2-action) should be considered and the analysis will be more complicated. We are working on a case where g is superlinear; i.e., $g(\xi)/\xi \to \infty$ as $|\xi| \to \infty$.

2. Preliminaries and notation

Let A be the linear wave operator such that

$$Au = u_{tt} - \Delta_n u + (\frac{n-1}{2})^2 u,$$

where $(t, x) \in S^1 \times S^n$, $n > 1$. It is well known that the eigenvalues of A are

$$\lambda(\ell, j) = (\ell + \frac{n-1}{2} - j)(\ell + \frac{n-1}{2} + j), \quad \ell, j = 0, 1, 2, \dots, \tag{2.1}$$

and the corresponding eigenfunctions in $L^2(S^1 \times S^n)$ are

$$\phi_{\ell,m}(x)\, sin\, jt, \quad \phi_{\ell,m}(x)\, cos\, jt, \quad m = 1, 2, \dots, M(\ell, n),$$

where $\phi_{\ell,m}(x)$, $m = 1, 2, \dots, M(\ell, n)$, are spherical harmonics of degree ℓ on S^n and

$$M(\ell, n) = \frac{(2\ell + n - 1)\Gamma(\ell + n - 1)}{\Gamma(\ell + 1)\,\Gamma(n)} = O(\ell^{n-1}).$$

Then $u \in L^2(S^1 \times S^n)$ can be written as

$$u = \sum_{\ell,j,m} u_{\ell,j,m} e^{ijt} \phi_{\ell,m}(x),$$

where $u_{\ell,j,m}$ are the Fourier coefficients. Note that

$$(Au, u)_{L^2} = \sum_{\ell,j,m} \lambda(\ell, j) |u_{\ell,j,m}|^2.$$

So the Sobolev space we will work on is defined as

$$H = \{u \in L^2(S^1 \times S^n) : \|u\|_H^2 = \sum_{\ell,j,m} |\lambda(\ell, j)||u_{\ell,j,m}|^2 + \sum_{\lambda(\ell,j)=0} |u_{\ell,j,m}|^2 < \infty\}.$$

Clearly H is a Hilbert space with the inner product

$$\langle u, v\rangle_H = \sum_{\ell,j,m} |\lambda(\ell, j)| u_{\ell,j,m} \bar{v}_{\ell,j,m} + \sum_{\lambda(\ell,j)=0} u_{\ell,j,m} \bar{v}_{\ell,j,m}.$$

We decompose H into invariant subspaces:

$$\begin{gathered} N = \{u \in H : u_{\ell,j,m} = 0 \text{ for } \lambda(\ell, j) \neq 0\}, \\ E^+ = \{u \in H : u_{\ell,j,m} = 0 \text{ for } \lambda(\ell, j) \leq 0\}, \\ E^- = \{u \in H : u_{\ell,j,m} = 0 \text{ for } \lambda(\ell, j) \geq 0\}. \end{gathered}$$

As can be seen from the expression of the eigenvalues, if the space S^n is odd dimensional; i.e., n odd, the kernel N of the operator A is infinite dimensional and $\|u\|_H = \|u\|_{L^2}$ for $u \in N$. Consequently, we only have a compact embedding of the type $E \hookrightarrow L^p$, $(p > 2)$ for $E = E^+ \oplus E^-$ the orthogonal complement of N.

Theorem 2.1 ([20]). *For any $2 \leq p < \frac{2n+2}{n-1}$, $E \hookrightarrow L^p$ is compact.*

Remark 2.2. Unlike the 1-dimensional case where the existence result is obtained for all of $2 < p < \infty$ [16, 20], the above embedding theorem 2.1 presents a crucial restriction on p for any existence results of wave equations on S^n, $n > 1$. Note that in 1-dimension the compact embedding $E \hookrightarrow L^p$ works for all of $2 < p < \infty$ [9, 18, 20].

Remark 2.3. If n is even, then $N = \emptyset$ and $H = E$, and hence problems are much easier to handle.

3. Variational Scheme

We now set up a variational formulation for the wave equation (1.1) as in [16]. The functional corresponding to the equation (1.1) for $u \in H$ is given by

$$F(u) = \frac{1}{2}\langle Lu, u\rangle_H - \int_\Omega (\frac{1}{p}|u|^p - f\cdot u)dt\,dx, \tag{3.1}$$

where $\Omega = S^1 \times S^n$, and L is the continuous self-adjoint operator in H associated with the operator A, i.e.,

$$\langle Lu, v\rangle_H = (Au, v) = \sum_{\ell,j,m} \lambda(\ell, j) u_{\ell,j,m} \bar{v}_{\ell,j,m}.$$

Using the Hilbert Space norm defined above, for $u = u^+ + u^- \in E$, $u^+ \in E^+$, $u^- \in E^-$ and $v \in N$, $F(u)$ can be written as

$$F(u + v) = \frac{1}{2}\|u^+\|_E^2 - \frac{1}{2}\|u^-\|_E^2 - \frac{1}{p}\|u + v\|_p^p + (f, u + v), \tag{3.2}$$

which is in $C^2(E \oplus N, \mathbb{R})$. Because of the compact embedding Theorem 2.1 on E, we instead work with the functional $I(u)$ on E,

$$I(u) = \max_{v\in N} F(u + v) = \frac{1}{2}\|u^+\|_E^2 - \frac{1}{2}\|u^-\|_E^2 - Q(u), \tag{3.3}$$

where

$$Q(u) = \min_{v\in N}[\frac{1}{p}\|u + v\|_p^p - (f, u + v)], \tag{3.4}$$

The functional $Q(u)$ has the following properties.

Lemma 3.1. (i) *For all $u \in L^{p+1}$, there exists a unique $v(u) \in N$ such that*

$$Q(u) = \frac{1}{p}\|u + v(u)\|_p^p - (f, u + v(u)). \tag{3.5}$$

(ii) *The map $v : L^p \to N$ is continuous.*
(iii) *$Q : E \to \mathbb{R}$ is in C^1 and for all $u, h \in E$,*

$$\langle Q'(u), h\rangle = (|u + v(u)|^{p-2}(u + v(u)) - f, h). \tag{3.6}$$

Moreover, $Q' : E \to E^*$ is compact and there are constants $C_1, C_2 > 0$ depending on $\|f\|_{p/(p-1)}$ such that for all $u \in E$,

$$\|Q'\|_{E^*} \le C_1(|Q(u)|^{\frac{p-1}{p}} + 1), \tag{3.7}$$

$$|\langle Q'(u), u\rangle - pQ(u)| \le C_2(|Q(u)|^{1/p} + 1). \tag{3.8}$$

The proofs of the results in this section and the next section can be done as in [16, 12] with slight modifications for n-dimension, so we omit most of them. For later use we introduce $Q_0 \in C^1(E, \mathbb{R})$ defined by

$$Q_0(u) = \min_{v \in N} \frac{1}{p}\|u+v\|_p^p = \frac{1}{p}\|u+v_0(u)\|_p^p, \tag{3.9}$$

where $v_0(u)$ can be given uniquely as in Lemma 3.1. $Q(u)$ and $Q_0(u)$ have the following relations.

Lemma 3.2. *There is a constant $C > 0$ depending on $\|f\|_{p/(p-1)}$ such that for $u \in E$,*

$$|Q(u)| \le C(Q_0(u) + 1), \tag{3.10}$$

$$|Q(u) - Q_0(u)| \le C(Q_0(u)^{1/p} + 1). \tag{3.11}$$

We will show that there is an unbounded sequence $\{u_k\}$ of critical points of $I(u)$. Then it is easy to see that $u_k + v(u_k)$ are critical points of $F(u)$.

Modified functional. Now we will follow the procedures of Rabinowitz [12] as in [16] in constructing critical values for functionals that are not symmetric. The procedure requires an estimate on the deviation from symmetry of I of the form

$$|I(u) - I(-u)| \le \beta_1(|I(u)|^\mu + 1) \quad \text{for } u \in E$$

that I does not satisfy. We introduce a modified functional $J(u)$: Let $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ be such that $\chi(\tau) = 1$ for $\tau \le 1, \chi(\tau) = 0$ for $\tau \ge 2$ and $-2 \le \chi'(\tau) \le 0, 0 \le \chi(\tau) \le 1$, for $\tau \in \mathbb{R}$. For $u = u^+ + u^- \in E^+ \oplus E^- = E$ and $a = max\{1, \frac{12}{p-1}\}$, let

$$\Phi(u) = a(I(u)^2 + 1)^{1/2}, \quad \psi(u) = \chi(\Phi(u)^{-1}Q_0(u)).$$

Define

$$J(u) = \frac{1}{2}\|u^+\|_E^2 - \frac{1}{2}\|u^-\|_E^2 - Q_0(u) - \psi(u)(Q(u) - Q_0(u)).$$

The functional $J(u) \in C^1(E, \mathbb{R})$ satisfies the following conditions.

Proposition 3.3. (i) *there is $\alpha = \alpha(\|f\|_{p/(p-1)}) > 0$ such that for $u \in E$,*

$$|J(u) - J(-u)| \le \alpha(|J(u)|^{1/p} + 1),$$

(ii) *there is $M_0 > 0$ such that $J(u) \ge M_0$ and $\|J'(u)\|_{E^*} \le 1$ implies $J(u) = I(u)$.*
(iii) *If $J'(u) = 0$ and $J(u) \ge M_0$ for $u \in E$, then $I(u) = J(u)$ and $I'(u) = 0$.*
(iv) *$J(u)$ satisfies (P.S.) on the set $\{u : J(u) \ge M_0\}$.*

Using the above proposition, we can show that large critical values of $J(u)$ are also critical values of $I(u)$.

Construction of critical values (Rabinowitz's process). We rearrange the positive eigenvalues of the wave operator A as $0 < \mu_1 \le \mu_2 \le \mu_3 \le \dots$, and let $e_1, e_2, e_3, \dots$ be the corresponding orthonormal eigenfunctions. Then the positive eigenspace E^+ can be written as

$$E^+ = \overline{\text{span}}\{e_j : j \in N\}.$$

Define

$$E_k^+ = \overline{\text{span}}\{e_j : 1 \le j \le k\}.$$

Since, for $u \in E_k^+$, $\|u\|_E \le \mu_k^{1/2} \|u\|_{L^2}$, for $u = u^+ + u^- \in E_k^+ \oplus E^-$, by Lemma 3.1 and Lemma 3.2, we have

$$J(u) \le \frac{1}{2}\|u^+\|_E^2 - c\mu_k^{-p/2}\|u^+\|_E^p - \frac{1}{2}\|u^-\|_E^2 + C. \tag{3.12}$$

Hence there is an $R_k > 0$ such that $J(u) \le 0$ for all $u \in E_k^+ \oplus E^-$ with $\|u\|_E \ge R_k$. We may assume that $R_k < R_{k+1}$ for each $k \in \mathbb{N}$.

Now we construct minimax values following [12]. Let B_R denote the closed unit ball of radius R in E, $D_k = B_{R_k} \cap (E_k^+ \oplus E^-)$ and

$$\Gamma_k = \{\gamma \in C(D_k, E) : \gamma \text{satisfies conditions } (\gamma 1)\text{-}(\gamma 3) \text{ below}\},$$

(γ1) γ is odd in D_k,
(γ2) $\gamma(u) = u$ for all $u \in \partial D_k$,
(γ3) $\gamma(u) = \alpha^+(u)u^+ + \alpha^-(u)u^- + \kappa(u)$, where $\alpha^+ \in C(D_k, [0,1])$ and $\alpha^- \in C(D_k, [1, \bar\alpha])$ are even functionals ($\bar\alpha \ge 1$ depends on γ) and κ is a compact operator such that on ∂D_k, $\alpha(u) = \alpha^+(u) + \alpha^-(u) = 1$ and $\kappa(u) = 0$.

Define

$$b_k = \inf_{\gamma\in\Gamma} \sup_{u\in D_k} J(\gamma(u)), \quad k \in \mathbb{N}.$$

If $f \equiv 0$ and J is even, it can be shown as in [1] that the numbers b_k are critical values of J. If f is not identically 0, that need not be the case. However we will use these numbers as the basis for a comparison argument. To construct a sequence of critical values of J, we must define another set of minimax values. Let

$$U_k = D_{k+1} \cap \{u \in E : \langle u, e_{k+1}\rangle \ge 0\},$$
$$\Lambda_k = \{\lambda \in C(U_k, E) : \lambda \text{ satisfies } (\lambda 1)\text{-}(\lambda 3) \text{ below}\},$$

where

(λ1) $\lambda|_{D_k} \in \Gamma_k$,
(λ2) $\lambda(u) = u$ on $\partial U_k \setminus D_k$,
(λ3) $\lambda(u) = \tilde\alpha^+(u)u^+ + \tilde\alpha^-(u)u^- + \tilde\kappa(u)$, where $\tilde\alpha^+ \in C(U_k, [0,1])$ and $\tilde\alpha^- \in C(U_k, [1, \tilde\alpha])$ are even functionals ($\tilde\alpha \ge 1$ depends on λ) and $\tilde\kappa$ is a compact operator such that $\tilde\alpha(u) = 1$ and $\tilde\kappa(u) = 0$ on $\partial U_k \setminus D_k$.

Now define

$$c_k = \inf_{\lambda\in\Lambda} \sup_{u\in U_k} J(\lambda(u)) \quad k \in \mathbb{N}.$$

By definition of b_k and c_k we easily see that $c_k \ge b_k$. The key to this construction is that we have the following existence result.

Proposition 3.4. *Suppose $c_k > b_k \ge M_0$. Let $\delta \in (0, c_k - b_k)$ and*

$$\Lambda_k(\delta) = \{\lambda \in \Lambda_k; J(\lambda) \le b_k + \delta \text{ on } D_k\}.$$

Then

$$c_k(\delta) = \inf_{\lambda\in\Lambda_k(\delta)} \sup_{u\in U_k} J(\lambda(u)) \; (\ge c_k)$$

is a critical value of $I(u)$.

Proof. By (iii) of Proposition 3.3, it is sufficient to show that $c_k(\delta)$ is a critical value of $J(u)$. First note that by definition of b_k and $\Lambda_k, \Lambda_k(\delta) \ne \emptyset$. Choose $\bar\varepsilon = \frac{1}{2}(c_k - b_k - \delta) > 0$. Now suppose that $c_k(\delta)$ is not a critical value of J. Then by a version of Deformation Lemma 3.5 below there exist $\varepsilon \in (0, \bar{\varepsilon}]$ and η as in the lemma. Choose $H \in \Lambda_k(\delta)$ such that

$$\max_{U_k} J(H(u)) \le c_k(\delta) + \varepsilon.$$

Let $\bar{H} = \eta(1, H)$. We need to show $\bar{H} \in \Lambda_k$. Clearly $\bar{H} \in C(U_k, E)$. (λ_1) and (λ_2) easily follow from the choice of H and *(iv)* of Lemma 3.5. Since H satisfies (λ_3), so does $\bar{H}$ by the deformation Lemma 3.5. Moreover on D_k, $J(H(u)) \le c_k(\delta) - \bar{\varepsilon}$ and hence $J(\bar{H}(u)) = J(H(u)) \le b_k + \delta$ on D_k, again by (iv) of Lemma 3.5. Therefore $\bar{H}(u) \in \Lambda_k(\delta)$ and by (v) of Lemma 3.5,

$$\max_{U_k} J(H(u)) \le c_k(\delta) - \varepsilon,$$

which contradicts to the definition of $c_k(\delta)$. Hence $c_k(\delta)$ is a critical value of J □

Lemma 3.5 (cf. [13, 14, 16])**.** *Suppose that $c > M_0$ is a regular value of $J(u)$, that is, $J'(u) \neq 0$ when $J(u) = c$. Then for any $\tilde{\varepsilon} > 0$, there exist an $\varepsilon \in (0, \tilde{\varepsilon}]$ and $\eta \in C([0, 1] \times E, E)$ such that*

(i) *$\eta(t, \cdot)$ is odd for all $t \in [0, 1]$ if $f(t, x) \equiv 0$;*
(ii) *$\eta(t, \cdot)$ is a homeomorphism of E onto E for all t;*
(iii) *$\eta(0, u) = u$for all $u \in E$;*
(iv) *$\eta(t, u) = u$ if $J(u) \notin [c - \tilde{\varepsilon}, c + \tilde{\varepsilon}]$;*
(v) *$J(\eta(1, u)) \le c - \varepsilon$ if $J(u) \le c + \varepsilon$;*
(vi) *$\eta(1, u)$ satisfies* ($\lambda 3$).

Therefore, to establish the existence of critical values, it suffices to show that there exists a subsequence $\{k_j\}$ such that

$$c_{k_j} > b_{k_j} \ge M_0 \quad \text{for } j \in \mathbb{N} \text{ and } b_{k_j} \to \infty \text{ as } j \to \infty. \tag{3.13}$$

This can be shown by the following, due to the almost symmetry of $J(u)$ ((i) of Proposition 3.3).

Proposition 3.6. *If $c_k = b_k$, for all $k \ge k_0$, then there exists a constant $\bar{C} > 0$ such that*

$$b_k \le \bar{C} k^{p/(p-1)} \quad \textit{for all } k \in \mathbb{N}. \tag{3.14}$$

Therefore, we need to show only that there exists a subsequence $\{k_j\}$, $\varepsilon > 0$ and $C_\varepsilon > 0$ satisfying the inequality

$$b_{k_j} > C_\varepsilon k_j^{p/(p-1-\varepsilon)} \quad \text{for all } j \in \mathbb{N}. \tag{3.15}$$

4. Bahri-Berestycki's max-min value β_k [2, 3]

To show (3.15), we introduce a comparison functional. By the definition of $Q_0(u)$ it can be shown [16] that for $u = u^+ + u^- \in E = E^+ \oplus E^-$,

$$J(u) \ge \frac{1}{2}\|u^+\|_E^2 - \frac{1}{2}\|u^-\|_E^2 - \frac{a_0}{p}\|u^+\|_p^p - \frac{a_0}{p}\|u^-\|_p^p - a_1, \tag{4.1}$$

where $a_0 > 0$, $a_1 > 0$ are constants independent of u. For $u \in E^+$, set

$$K(u) = \frac{1}{2}\|u^+\|_E^2 - \frac{a_0}{p}\|u^+\|_p^p \in C^2(E^+, \mathbb{R}).$$

Then we can easily see the following.

Lemma 4.1. *(i) $J(u) \geq K(u) - a_1$ for all $u \in E^+$. (ii) $K(u)$ satisfies the (P.S.) on E^+.*

For $m > k$, $k, m \in \mathbb{N}$, set

$$A_k^m = \{\sigma \in C(S^{m-k}, E_m^+) : \sigma(-x) = -\sigma(x) \text{ for all } x \in S^{m-k}\}$$

and

$$\beta_k^m = \sup_{\sigma \in A_k^m} \min_{x \in S^{m-k}} K(\sigma(x)).$$

We can show that there exists a subsequence $\{m_j\}$ such that for all k,

$$\beta_k = \lim_{j \to \infty} \beta_k^{m_j} \in \mathbb{N}.$$

exists. We list the following important properties of β_k:

Proposition 4.2. *(i) β_k's are critical values of $K \in C^2(E^+, \mathbb{R})$ for each $k \in \mathbb{N}$; (ii) $\beta_k \leq \beta_{k+1}$ for all $k \in \mathbb{N}$; (iii) $\beta_k \to \infty$ as $k \to \infty$.*

The proof of the above proposition is same as in [16] except for some adjustment for n-dimensional consideration. To estimate b_k we establish the following relation between b_k and β_k using topological linking lemmas.

Proposition 4.3. *For all $k \in \mathbb{N}$,*

$$b_k \geq \beta_k - a_1, \tag{4.2}$$

where a_1 is the number in Lemma 4.1.

For a proof of the above proposition, see [16].

5. Estimate of β_k using Morse Index

In this section some index properties of β_k are discussed. The lower bound for the index of K'' obtained here and the upper bound estimate in the next section give the growth estimate (3.15) that we are looking for.

Definition. For $u \in E^+$, we define an index of $K''(u)$ by

$$\begin{aligned} \operatorname{index} K''(u) &= \text{the number of nonpositive eigenvalues of } K''(u) \\ &= \max\{\dim S; S \leq E^+ \text{ such that } \langle K''(u)h, h\rangle \leq 0 \text{ for all } h \in S\}. \end{aligned}$$

Here $A \leq B$ in the bracket means A is a subspace of B.

Proposition 5.1. *Suppose $\beta_k < \beta_{k+1}$. Then there exists $u_k \in E^+$ such that*

$$K(u_k) \leq \beta_k, \quad K'(u_k) = 0, \quad \operatorname{index} K''(u_k) \geq k.$$

The proof of the above proposition can be done using a theorem from Morse theory; see [16].

6. Proof of the existence of the solutions

By Propositions 3.4 and 3.6, we know that (3.15), the growth estimate on b_k's ensures the existence of an unbounded sequence of critical values. In view of (4.3), however, we now need the growth estimate on β_k's. First note by Proposition 5.1 that there exists $\{u_{k_j}\}$ such that

$$\beta_{k_j} \geq K(u_{k_j}) = \frac{1}{2}\|u_{k_j}\|_E^2 - \frac{a_0}{p}\|u_{k_j}\|_p^p = (\frac{1}{2} - \frac{1}{p})a_0\|u_j\|_p^p. \tag{6.1}$$

Thus, by Proposition 5.1 again, we need to get an upper bound of index $K''(u_{k_j})$ in terms of $\|u_{k_j}\|_p^p$ in proving (3.15). For $u,h,w \in E^+$, $k''(u)$ is given by

$$\langle K''(u)w,h\rangle = \langle w,h\rangle - (p-1)a_0(|u|^{p-2}h,h).$$

Thus by the definition of index,

$$\operatorname{index} K''(u) = \max\{\dim S : S \le E^+, (p-1)a_0(|u|^{p-2}h,h) \ge \|h\|_E^2, h \in S\}.$$

Define an operator $D : L^2 \to E^+$ such that for $v(x,t) = \sum v_{l,j,m}\phi_{l,m}e^{ijt}$,

$$(Dv)(x,t) = \sum_m \sum_{\lambda(l,j)>0} |\lambda(l,j)|^{-1/2} v_{l,j,m}\phi_{l,m}e^{ijt}.$$

Remark 6.1. D is an isometry from

$$L_+^2 = \overline{\operatorname{span}}_{L^2}\{\phi_{l,m}e^{ijt}; \lambda(l,j) > 0\}$$

to E^+ and $D = 0$ on $\overline{\operatorname{span}}_{L^2}\{\phi_{l,m}e^{ijt}; \lambda(l,j) \le 0\}$.

Remark 6.2. *Setting $h = Dv$ in the above expression of index, we get*

$$\begin{aligned} \operatorname{index} K''(u) &= \max\{\dim S : S \le L^2 \ \textit{s.t.}\ (p-1)a_0(|u|^{p-2}Dv, Dv) \ge \|v\|_2^2, v \in S\} \\ &= \#\{\mu_j : \mu_j \ge 1,\ \textit{eigenvalues of } D^*((p-1)a_0|u|^{p-2})D\}. \end{aligned}$$

Proposition 6.3. *There exist $C > 0$ such that for $u \in E^+$,*

$$\operatorname{index} K''(u_j) \le C\|u\|_s^r,$$

where $r = \frac{2(p-2)nq}{n+1-(n-1)q}$ and $s = \frac{(p-2)q}{q-1}$.

Proof. We try to find a big enough l such that

$$(p-1)a_0(|u|^{p-2}Dv, Dv) \le \|v\|_2^2, \text{on } E^+\backslash E_{l-1}^+,$$

which implies index $K''(u) \le l$. First we have the following estimate on $E^+\backslash E_{l-1}^+$

$$\begin{aligned} \int_\Omega |Dv|^2|u|^{p-2} &\le C\Big(\int \Omega|Dv|^{2q}\Big)^{1/q}\Big(\int \Omega|u|^{(p-2)\frac{q}{q-1}}\Big)^{\frac{q-1}{q}}, \\ &= C\|Dv\|_{2q}^2\|u\|_{(p-2)\frac{q}{q-1}}^{p-1}, \\ &\le C\|Dv\|_2^{2s}\|Dv\|_{\bar q}^{2(1-s)}\|u\|_{(p-2)\frac{q}{q-1}}^{p-2}, \\ &\le C\frac{1}{\lambda_l^s}\|v\|_2^{2s}\|v\|_2^{2(1-s)}\|u\|_{(p-2)\frac{q}{q-1}}^{p-2}, \\ &= C\frac{1}{\lambda_l^s}\|v\|_2^2\|u\|_{(p-2)\frac{q}{q-1}}^{p-2}, \end{aligned}$$

where $\bar q = \frac{2n+2}{n-1}$, $\frac{1}{2q} = \frac{s}{2} + \frac{1-s}{\bar q}$ and to get the second last inequality, we used the facts $\|Dv\|_E^2 \le |\lambda_l|^{-1}\|v\|_{L^2}^2$ on $E^+\backslash E_{l-1}^+$ and $\|Dv\|_E^2 = \|v\|_{L^2}^2$ and the compact embedding theorem 2.1. Thus to have $\int |Dv|^2|u|^{p-2} \le \|v\|_2^2$, we need

$$\|u\|_{(p-2)\frac{q}{q-1}}^{p-2} \le |\lambda_l|^s \sim C|l|^{s/n}, \quad s = (n+1-(n-1)q)/2q,\ \textit{i.e.},$$

$$\alpha := C\|u\|_{(p-2)\frac{q}{q-1}}^{(p-2)n\frac{2q}{(n+1)-(n-1)q}} \sim l.$$

Let $l = [\alpha + 1]$. Then

$$\int |Dv|^2|u|^{p-1} \le \|v\|_{L^2}^2 \text{ for all } v \in E^+\backslash E_{l-1}^+.$$

and therefore

$$\operatorname{index} K''(u) \le l = [\alpha + 1] \le C\alpha = C\|u\|_{(p-2)\frac{q}{q-1}}^{(p-2)\frac{2nq}{n+1-(n-1)q}}.$$

□

We now prove (3.15); i.e., $b_{k_j} > C\, k_j^{\frac{p}{p-1-\varepsilon}}$. From Proposition 6.3 and Proposition 5.1 we have

$$j \le \operatorname{index} K''(u_{k_j}) \le C\|u_{k_j}\|_{(p-2)\frac{q}{q-1}}^{(p-2)\frac{2nq}{(n+1)-(n-1)q}}, \quad 2 < p < \frac{2(n+2)}{n-1}.$$

Note that

$$\|u_{k_j}\|_p^p \ge C\|u_{k_j}\|_{(p-2)\frac{q}{q-1}}^p \quad \text{if } q \ge \frac{p}{2},$$

so that

$$\|u_{k_j}\|_p^p \ge j^{p/((p-2)\frac{2nq}{(n+1)-(n-1)q}} \quad \text{if } q \ge \frac{p}{2}.$$

In order to have (3.15), we need

$$(p-2)\frac{2nq}{(n+1)-(n-1)q} < (p-1).$$

Since $\frac{2nq}{(n+1)-(n-1)q}$ is an increasing function of q, choose $q = \frac{p}{2}$. Then we finally obtain

$$2 < p < \frac{7n+1+\sqrt{25n^2-2n+9}}{2(3n-1)},$$

for which (3.15) is satisfied.

We remark that this upper bound of p may not be optimal and we are still trying to improve it.

Now there exists a sequence $u_k \subset E$ of critical points of $I(u)$ such that as $k \to \infty$

$$I(u_k) = \frac{1}{2}\|u_k^+\|_E^2 - \frac{1}{2}\|u_k^-\|_E^2 - \frac{1}{p}\|u_k + v(u_k)\|_p^p - (f, u_k + v(u_k)) \to \infty.$$

Since $I'(u_k) = 0$, we have

$$\langle I'(u_k), u_k\rangle = \|u_k^+\|_E^2 - \|u_k^-\|_E^2 - (|u_k + v(u_k)|^{p-2}(u_k + v(u_k)) + f, u_k + v(u_k)) = 0.$$

Above two equations combined gives

$$(\frac{1}{2} - \frac{1}{p})\|u_k + v(u_k)\|_p^p + \frac{1}{2}(f, u_k + v(u_k)) \to \infty \quad \text{as } n \to \infty. \tag{6.2}$$

By direct calculation we can easily see that the $\{u_k + v(u_k)\}$ are critical points of $F(u)$, so it follows from (6.2) that

$$\|u_k + v(u_k)\|_p \to \infty \quad \text{as } k \to \infty.$$

This ensures the existence of a unbounded sequence of critical points for $F(u)$, which is a unbounded sequence of the weak solutions of the nonlinear wave equation (1.1) on S^n.

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