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EXISTENCE OF THREE SOLUTIONS FOR A HIGHER-ORDER BOUNDARY-VALUE PROBLEM

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Abstract. We consider a higher-order multi-point boundary-value problem with a nonlinear boundary condition. Sufficient conditions are obtained for the existence of three solutions. In our problem, the differential equation has dependence on all lower order derivatives of the unknown function and the boundary condition covers many multi-point boundary conditions studied earlier by other authors. Our results extend some recent work in the literature.

1. Introduction

In this paper, we are concerned with the existence of solutions of the nth order boundary value problem (BVP) consisting of the equation

$$u^{(n)} + f\big(t, u, u', \dots, u^{(n-1)}\big) = 0, \quad t \in (0,1), \tag{1.1}$$

and the general multi-point boundary conditions (BC)

$$\begin{gathered} u^{(i)}(0) = g_i\big(u^{(i)}(t_1), \dots, u^{(i)}(t_m)\big), \quad i = 0, \dots, n-2, \\ u^{(n-2)}(1) = g_{n-1}\big(u^{(n-2)}(t_1), \dots, u^{(n-2)}(t_m)\big), \end{gathered} \tag{1.2}$$

where $n \geq 2$ and $m \geq 1$ are integers, $t_j \in [0,1]$ for $j = 1, \dots, m$ with $0 \leq t_1 < t_2 < \dots < t_m \leq 1$, $f \in C((0,1) \times \mathbb{R}^n)$, and $g_i \in C(\mathbb{R}^m)$ for $i = 0, \dots, n-1$. By a solution of (1.1), (1.2), we mean a function $u \in C^{n-1}[0,1] \cap C^n(0,1)$ such that $u(t)$ satisfies (1.1) on $(0,1)$, and satisfies (1.2).

We observe that (1.2) covers many multi-point BCs studied in the literature. In recent years, the existence of solutions of BVPs with various multi-point linear BCs have been extensively investigated by numerous researchers using a variety of methods and techniques. For a small sample of such work, we refer the reader to [8, 10, 12, 13, 14] for results on second order problems and [2, 4, 6] on higher order ones. BVPs with two-point or multi-point nonlinear BCs have also been studied in the literature, for example, in [1, 3, 5, 7, 11]. In particular, using the lower and upper solution method, the present authors [5] studied (1.1), (1.2) and found several sufficient conditions for the existence of a solution. This paper may be regarded

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as a continuation of our work in [5]. Here, we prove a result on the existence of multiple solutions of (1.1), (1.2).

The approach used in this paper is motivated by Henderson and Thompson [9]. Our result is under the assumption that there exist two strict lower solutions and two strict upper solutions of (1.1), (1.2) satisfying certain relations and that f satisfies a Nagumo growth condition. We use the lower and upper solutions to obtain a modified problem of (1.1), (1.2) and find a priori bounds on solutions of this problem. Then, we employ degree theory to show that (1.1), (1.2) has three distinct solutions. Recently, this method has been developed in [1, 2, 10] to study other types of BVPs. Specifically, Du, Liu, and Lin [2] studied the BVP consisting of (1.1) and the three-point BC

$$u^{(i)}(0) = 0, \quad i = 0, \dots, n-2, \quad u^{(n-2)}(1) = \xi u^{(n-2)}(\eta), \tag{1.3}$$

where $\xi > 0$, $0 < \eta < 1$ with $0 < \xi\eta < 1$, and discussed the existence of three solutions. Our work is an improvement and extension of the result in [2]. In fact, our BC (1.2) is much more general than BC (1.3); even for the special case of BC (1.3), our result is new and better since the restriction $0 < \xi\eta < 1$ is removed; i.e., our result works not only for the nonresonance case covered in [2] but also for the resonance case.

In the next section, we present our main theorem together with an illustrative example. The proof of the main theorem is given a separate section.

2. Main Result

In the sequel, for any $u \in C[0,1]$, we define $\|u\|_\infty = \max_{t\in[0,1]} |u(t)|$. Let

$$\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty, \dots, \|u^{(n-1)}\|_\infty\}$$

and

$$\|u\|_p = \begin{cases} (\int_0^1 |u(t)|^p dt)^{1/p}, & 1 \le p < \infty, \\ \inf\{M : \text{meas}\{t : |u(t)| > M\} = 0\}, & p = \infty, \end{cases}$$

stand for the norms in $C^{n-1}[0,1]$ and $L^p(0,1)$, respectively, where mess$\{\cdot\}$ denotes the Lebesgue measure of a set.

We first define strict lower and upper solutions of (1.1), (1.2) and a Nagumo condition.

Definition 2.1. A function $\alpha \in C^{n-1}[0,1] \cap C^n(0,1)$ is said to be a strict lower solution of (1.1), (1.2) if

$$\alpha^{(n)}(t) + f\big(t, \alpha(t), \alpha'(t), \dots, \alpha^{(n-1)}(t)\big) > 0 \quad \text{on } (0,1), \tag{2.1}$$

and

$$\begin{gathered} \alpha^{(i)}(0) < g_i\big(\alpha^{(i)}(t_1), \dots, \alpha^{(i)}(t_m)\big), \quad i = 0, \dots, n-2, \\ \alpha^{(n-2)}(1) < g_{n-1}\big(\alpha^{(n-2)}(t_1), \dots, \alpha^{(n-2)}(t_m)\big). \end{gathered} \tag{2.2}$$

A function $\beta \in C^{n-1}[0,1] \cap C^n(0,1)$ is said to be a strict upper solution of (1.1), (1.2) if

$$\beta^{(n)}(t) + f\big(t, \beta(t), \beta'(t), \dots, \beta^{(n-1)}(t)\big) < 0 \quad \text{on } (0,1), \tag{2.3}$$

and

$$\begin{gathered} \beta^{(i)}(0) > g_i\big(\beta^{(i)}(t_1), \dots, \beta^{(i)}(t_m)\big), \quad i = 0, \dots, n-2, \\ \beta^{(n-2)}(1) > g_{n-1}\big(\beta^{(n-2)}(t_1), \dots, \beta^{(n-2)}(t_m)\big). \end{gathered} \tag{2.4}$$

Definition 2.2. Let α, $\beta \in C^{n-1}[0,1]$ satisfy

$$\alpha^{(i)}(t) \le \beta^{(i)}(t) \quad \text{for } t \in [0,1] \text{ and } i = 0, \dots, n-2. \tag{2.5}$$

We say that f satisfies a Nagumo condition with respect to α and β if for

$$\xi = \max\big\{\beta^{(n-2)}(1) - \alpha^{(n-2)}(0),\ \beta^{(n-2)}(0) - \alpha^{(n-2)}(1)\big\}, \tag{2.6}$$

there exist a constant $C = C(\alpha, \beta)$ with

$$C > \max\big\{\xi,\ \|\alpha^{(n-1)}\|_\infty,\ \|\beta^{(n-1)}\|_\infty\big\} \tag{2.7}$$

and functions $\phi \in C[0,\infty)$ and $w \in L^p(0,1)$, $1 \le p \le \infty$, such that $\phi > 0$ on $[0,\infty)$,

$$|f(t, x_0, \dots, x_{n-1})| \le w(t)\phi(|x_{n-1}|) \quad \text{on } (0,1) \times \prod_{i=0}^{n-2}[\alpha^{(i)}(t), \beta^{(i)}(t)] \times \mathbb{R}, \tag{2.8}$$

and

$$\int_\xi^C \frac{v^{(p-1)/p}}{\phi(v)} dv > \|w\|_p \eta^{(p-1)/p}, \tag{2.9}$$

where $(p-1)/p \equiv 1$ for $p = \infty$ and

$$\eta = \max_{t\in[0,1]} \beta^{(n-2)}(t) - \min_{t\in[0,1]} \alpha^{(n-2)}(t). \tag{2.10}$$

Remark 2.3. Let α, $\beta \in C^{n-1}[0,1]$ satisfy (2.5). Assume that there exist $w \in L^p(0,1)$, $1 \le p \le \infty$, and $0 \le \sigma \le 1 + (p-1)/p$ such that

$$|f(t, x_0, \dots, x_{n-1})| \le w(t)(1 + |x_{n-1}|^\sigma) \quad \text{on } (0,1) \times \prod_{i=0}^{n-2}[\alpha^{(i)}(t), \beta^{(i)}(t)] \times \mathbb{R}. \tag{2.11}$$

Then f satisfies a Nagumo condition with respect to α and β with $\phi(v) = 1 + v^\sigma$.

Now, we present the main result of this paper.

Theorem 2.4. *Assume that the following conditions hold:*

- (H1) *BVP* (1.1), (1.2) *has two strict lower solutions α_1 and α_2 and two strict upper solutions β_1 and β_2 satisfying*
$$\alpha_1^{(i)}(t) \le \alpha_2^{(i)}(t) \le \beta_2^{(i)}(t),\ \alpha_1^{(i)}(t) \le \beta_1^{(i)}(t) \le \beta_2^{(i)}(t),\ \text{and } \alpha_2^{(i)}(t) \not\le \beta_1^{(i)}(t)$$
for $t \in [0,1]$ and $i = 0, \dots, n-2$;
- (H2) *for $(t, x_0, \dots, x_{n-1}) \in (0,1) \times \prod_{i=0}^{n-3}[\alpha^{(i)}(t), \beta^{(i)}(t)] \times \mathbb{R}^2$, $f(t, x_0, \dots, x_{n-1})$ is nondecreasing in each of the variables $x_0, \dots, x_{n-3}$;*
- (H3) *f satisfies a Nagumo condition with respect to α_1 and β_2 with $C = C(\alpha_1, \beta_2)$ being the constant given in Definition 2.2;*
- (H4) *for $i = 1, \dots, n-1$ and $(y_1, \dots, y_m) \in \mathbb{R}^m$, $g_i(y_1, \dots, y_m)$ is nondecreasing in each of its arguments.*

Then (1.1), (1.2) *has at least three solutions $u_1(t)$, $u_2(t)$, and $u_3(t)$ satisfying*

$$\alpha_j^{(i)}(t) \le u_j^{(i)}(t) \le \beta_j^{(i)}(t) \quad \text{for } t \in [0,1],\ i = 0, \dots, n-2,\ \text{and } j = 1, 2, \tag{2.12}$$

and

$$\alpha_1^{(i)}(t) \le u_3^{(i)}(t) \le \beta_2^{(i)}(t),\ u_3^{(i)}(t) \not\le \beta_1^{(i)}(t),\ \text{and } u_3^{(i)}(t) \not\ge \alpha_2^{(i)}(t) \tag{2.13}$$

for $t \in [0,1]$ and $i = 0, \dots, n-2$.

Remark 2.5. Notice that in (H2) we do not need the monotonicity of f in the last two variables x_{n-2} and x_{n-1}. In particular, for the case when $n = 2$, no monotonicity is required on f.

In the remainder of this section, we provide the following example to illustrate Theorem 2.4. To the best of our knowledge, no existing criteria can be applied to this example.

Example. Consider the BVP consisting of the equation

$$u''' + t^{-1/2}h(u') + (u'')^2 + 1 = 0, \quad t \in (0,1), \tag{2.14}$$

and the BC

$$u(0) = u^{1/3}(1/2) + 1, \quad u'(0) = u'(1) = u^r(1/2), \tag{2.15}$$

where $h \in C(\mathbb{R})$ satisfies

$$\begin{gathered} h(y) \geq 8 \quad \text{for } y \in [-9,-8] \cup [2,3], \\ h(y) \leq -26 \quad \text{for } y \in [-3,-2] \cup [8,9], \end{gathered} \tag{2.16}$$

and $r = a/b \in (\ln 2/\ln 3, \ln 9/\ln 8)$ with a, b odd numbers. Clearly, the function $g(x) = x^r$ is nondecreasing and odd on $\mathbb{R}$. Let

$$\begin{gathered} \alpha_1(t) = -4t^3/3 + 2t^2 - 9t - 2, \\ \alpha_2(t) = -4t^3/3 + 2t^2 + 2t + 1, \\ \beta_1(t) = 4t^3/3 - 2t^2 - 2t + 2, \\ \beta_2(t) = 4t^3/3 - 2t^2 + 9t + 4. \end{gathered} \tag{2.17}$$

We claim that (2.14), (2.15) has at least three solutions $u_1(t)$, $u_2(t)$, and $u_3(t)$ satisfying (2.12) and (2.13) with the above $\alpha_1(t)$, $\alpha_2(t)$, $\beta_1(t)$, and $\beta_2(t)$.

In fact, with $n = 3$, $m = 1$, $t_1 = 1/2$, $f(t, x_0, x_1, x_2) = t^{-1/2}h(x_1) + x_2^2 + 1$, $g_0(x) = x^{1/3} + 1$, and $g_1(x) = g_2(x) = x^r$, we see that BVP (2.14), (2.15) is of the form of (1.1), (1.2). Clearly, (H2) and (H4) hold.

From (2.17), we have that for $t \in [0,1]$

$$\begin{gathered} -9 \leq \alpha_1'(t) = -4t^2 + 4t - 9 \leq -8, \\ 2 \leq \alpha_2'(t) = -4t^2 + 4t + 2 \leq 3, \\ -3 \leq \beta_1'(t) = 4t^2 - 4t - 2 \leq -2, \\ 8 \leq \beta_2'(t) = 4t^2 - 4t + 9 \leq 9, \end{gathered} \tag{2.18}$$

and

$$\begin{gathered} -4 \leq \alpha_1''(t) = -8t + 4 \leq 4, \\ -4 \leq \alpha_2''(t) = -8t + 4 \leq 4, \\ -4 \leq \beta_1''(t) = 8t - 4 \leq 4, \\ -4 \leq \beta_2''(t) = 8t - 4 \leq 4. \end{gathered} \tag{2.19}$$

It follows from (2.18) and (2.19) that

$$\alpha_1^{(i)}(t) \leq \alpha_2^{(i)}(t) \leq \beta_2^{(i)}(t), \; \alpha_1^{(i)}(t) \leq \beta_1^{(i)}(t) \leq \beta_2^{(i)}(t), \text{ and } \alpha_2^{(i)}(t) \not\leq \beta_1^{(i)}(t)$$

for $t \in [0,1]$ and $i = 0, 1$. Moreover, from (2.16)–(2.19), it is easy to verify that $\alpha_1(t)$ and $\alpha_2(t)$ are strict lower solutions of (2.14), (2.15) and $\beta_1(t)$ and $\beta_2(t)$ are strict upper solutions of (2.14), (2.15). Hence, (H1) holds.

In view of (2.18), we see that

$$|f(t,x_0,x_1,x_2)| \le (1+t^{-1/2}\max_{y\in[-9,9]}|h(y)|)(1+x_2^2)$$

on $(0,1)\times[\alpha_1(t),\beta_2(t)]\times[\alpha_1'(t),\beta_2'(t)]$. Then, by Remark 2.3, (H3) holds. The conclusion now follows from Theorem 2.4.

Remark 2.6. One example of a continuous function h satisfying (2.16) is

$$h(y)=\begin{cases}-34y/5-232/5, & y\in(-\infty,0),\\ 136y/5-232/5, & y\in[0,3],\\ -306y/25+1798/25, & y\in(3,\infty).\end{cases}$$

3. Proof of the Main Result

In this section, we give a proof to Theorem 2.4. Assume (H1)–(H4) hold. Let α and β be strict lower and upper solutions of (1.1), (1.2), respectively, satisfying (2.5). Let $C=C(\alpha,\beta)$ be given in Definition 2.2 and f satisfy a Nagumo condition with respect to α and β. For $u\in C^{n-1}[0,1]$, define

$$\tilde u^{[i]}(\alpha,\beta)(t)=\max\big\{\alpha^{(i)}(t),\min\big\{u^{(i)}(t),\ \beta^{(i)}(t)\big\}\big\},\quad i=0,\dots,n-2 \tag{3.1}$$

and

$$\tilde u^{[n-1]}(\alpha,\beta)(t)=\max\big\{-C(\alpha,\beta),\ \min\big\{u^{(n-1)}(t),\ C(\alpha,\beta)\big\}\big\}. \tag{3.2}$$

Then, for $i=0,\dots,n-1$, $\tilde u^{[i]}(\alpha,\beta)(t)$ is continuous on $[0,1]$,

$$\begin{gathered}\tilde\alpha^{[i]}(\alpha,\beta)(t)=\alpha^{(i)}(t),\ \tilde\beta^{[i]}(\alpha,\beta)(t)=\beta^{(i)}(t),\\ \alpha^{(i)}(t)\le\tilde u^{[i]}(\alpha,\beta)(t)\le\beta^{(i)}(t)\end{gathered} \tag{3.3}$$

for $t\in[0,1]$ and $i=0,\dots,n-2$, and

$$-C(\alpha,\beta)\le\tilde u^{[n-1]}(\alpha,\beta)(t)\le C(\alpha,\beta)\quad\text{on }[0,1]. \tag{3.4}$$

Define a functional $F_{\alpha,\beta}:(0,1)\times C^{n-1}[0,1]\to\mathbb{R}$ by

$$\begin{aligned}F_{\alpha,\beta}(t,u(\cdot))=f\Big(t,\tilde u^{[0]}(\alpha,\beta)(t),\tilde u^{[1]}(\alpha,\beta)(t),\dots,\tilde u^{[n-1]}(\alpha,\beta)(t)\Big)\\ +\frac{\tilde u^{[n-2]}(\alpha,\beta)(t)-u^{(n-2)}(t)}{1+(u^{(n-2)}(t))^2}.\end{aligned} \tag{3.5}$$

Then, for $u\in C^{n-1}[0,1]$ and $t\in(0,1)$, $F_{\alpha,\beta}(t,u(\cdot))$ is continuous in u, and from (2.8), (3.3), and (3.4), we see that

$$|F_{\alpha,\beta}(t,u(\cdot))|\le w(t)\max_{y\in[0,C(\alpha,\beta)]}\phi(y)+\|\alpha\|+\|\beta\|+1. \tag{3.6}$$

Consider the BVP consisting of the equation

$$u^{(n)}+F_{\alpha,\beta}(t,u(\cdot))=0,\quad t\in(0,1), \tag{3.7}$$

and the BC

$$\begin{gathered}u^{(i)}(0)=g_i\Big(\tilde u^{[i]}(\alpha,\beta)(t_1),\dots,\tilde u^{[i]}(\alpha,\beta)(t_m)\Big),\quad i=0,\dots,n-2,\\ u^{(n-2)}(1)=g_{n-1}\Big(\tilde u^{[n-2]}(\alpha,\beta)(t_1),\dots,\tilde u^{[n-2]}(\alpha,\beta)(t_m)\Big).\end{gathered} \tag{3.8}$$

It is well known that the Green's function for the BVP

$$-u''(t)=0\quad\text{on }(0,1),\quad u(0)=u(1)=0,$$

is

$$G(t,s)=\begin{cases} t(1-s), & 0\le t\le s\le 1,\\ s(1-t), & 0\le s\le t\le 1.\end{cases}$$

Let $G_1(t,s)=G(t,s)$ and for $j=2,\dots,n-1$, recursively define

$$G_j(t,s)=\int_0^t G_{j-1}(v,s)dv. \tag{3.9}$$

Lemma 3.1 below is taken from [5, Lemma 3.2] and Lemma 3.2 follows from [5, Lemmas 3.4 and 3.5].

Lemma 3.1. *The function $u(t)$ is a solution of (3.7), (3.8) if and only if $u(t)$ is a solution of the integral equation*

$$u(t)=\sum_{i=0}^{n-1} g_i\big(\tilde u^{[i]}(\alpha,\beta)(t_1),\dots,\tilde u^{[i]}(\alpha,\beta)(t_m)\big)p_i(t)+\int_0^1 G_{n-1}(t,s)F_{\alpha,\beta}(s,u(\cdot))ds,$$

where

$$\begin{gathered} p_i(t)=\frac{t^i}{i!},\quad i=0,\dots,n-3,\\ p_{n-2}(t)=\frac{t^{n-2}}{(n-2)!}-\frac{t^{n-1}}{(n-1)!},\\ p_{n-1}(t)=\frac{t^{n-1}}{(n-1)!},\end{gathered}$$

and $G_{n-1}(t,s)$ is given by (3.9) with $j=n-1$.

Lemma 3.2. *If $u(t)$ is a solution of (3.7), (3.8), then $u(t)$ satisfies*

$$\alpha^{(i)}(t)\le u^{(i)}(t)\le \beta^{(i)}(t)\quad \text{for } t\in[0,1] \text{ and } i=0,\dots,n-2, \tag{3.10}$$

and

$$|u^{(n-1)}(t)|\le C(\alpha,\beta)\quad \text{for } t\in[0,1]. \tag{3.11}$$

Consequently, $u(t)$ is a solution of (1.1), (1.2).

Proof of Theorem 2.4. Let F_{α_1,β_2} be defined by (3.5) with (α,β) replaced by (α_1,β_2). Define an operator $T_{\alpha_1,\beta_2}:C^{n-1}[0,1]\to C[0,1]$ by

$$\begin{aligned} T_{\alpha_1,\beta_2}u(t)=&\sum_{i=0}^{n-1} g_i\left(\tilde u^{[i]}(\alpha_1,\beta_2)(t_1),\dots,\tilde u^{[i]}(\alpha_1,\beta_2)(t_m)\right)p_i(t)\\ &+\int_0^1 G_{n-1}(t,s)F_{\alpha_1,\beta_2}(s,u(\cdot))ds.\end{aligned} \tag{3.12}$$

Then, by Lemma 3.1, $u(t)$ is a solution of (3.7), (3.8) with $(\alpha,\beta)=(\alpha_1,\beta_2)$ if and only if u is a fixed point of T_{α_1,β_2}. In the following, we show that T_{α_1,β_2} is compact. Clearly, T_{α_1,β_2} is continuous. Let $S\subseteq C^{n-1}[0,1]$ be a bounded set; we will show that $T_{\alpha_1,\beta_2}(S)$ is relatively compact. For $u\in S$, in view of (3.3) where $(\alpha,\beta)=(\alpha_1,\beta_2)$, there exists $d>0$ such that

$$\left|g_i\left(\tilde u^{[i]}(\alpha_1,\beta_2)(t_1),\dots,\tilde u^{[i]}(\alpha_1,\beta_2)(t_m)\right)\right|\le d\quad \text{for } i=0,\dots,n-1. \tag{3.13}$$

From (3.9), we see that

$$0\le G_j(t,s)\le 1\quad \text{for } (t,s)\in[0,1]\times[0,1] \text{ and } j=1,\dots,n-1. \tag{3.14}$$

For $p_i(t)$ defined in Lemma 3.1, $\|p_i\| \le 1$, $i = 0, \dots, n-1$. From (3.6) with $(\alpha, \beta) = (\alpha_1, \beta_2)$ and (3.12)–(3.14), we obtain

$$\begin{aligned} |(T_{\alpha_1,\beta_2}u)^{(j)}(t)| &\le \sum_{i=0}^{n-1} |g_i\left(\tilde{u}^{[i]}(\alpha_1,\beta_2)(t_1), \dots, \tilde{u}^{[i]}(\alpha_1,\beta_2)(t_m)\right)|\ |p_i^{(j)}(t)| \\ &\quad + \int_0^1 G_{n-1-j}(t,s)|F_{\alpha_1,\beta_2}(s,u(\cdot))|ds \\ &\le nd + \int_0^1 |F_{\alpha_1,\beta_2}(s,u(\cdot))|ds \\ &\le nd + \max_{y\in[0,C(\alpha_1,\beta_2)]} \phi(y) \int_0^1 w(s)ds + \|\alpha\| + \|\beta\| + 1 < \infty \end{aligned} \tag{3.15}$$

for $j = 0, \dots, n-2$, and

$$\begin{aligned} |(T_{\alpha_1,\beta_2}u)^{(n-1)}(t)| &\le \sum_{i=0}^{n-1} |g_i\left(\tilde{u}^{[i]}(\alpha_1,\beta_2)(t_1), \dots, \tilde{u}^{[i]}(\alpha_1,\beta_2)(t_m)\right)|\ |p_i^{(j)}(t)| \\ &\quad + \int_0^t s|F_{\alpha_1,\beta_2}(s,u(\cdot))|ds + \int_t^1 (1-s)|F_{\alpha_1,\beta_2}(s,u(\cdot))|ds \\ &\le nd + \int_0^1 |F_{\alpha_1,\beta_2}(s,u(\cdot))|ds \\ &\le nd + \max_{y\in[0,C(\alpha_1,\beta_2)]} \phi(y) \int_0^1 w(s)ds + \|\alpha\| + \|\beta\| + 1 < \infty. \end{aligned} \tag{3.16}$$

Then, T_{α_1,β_2} is uniformly bounded on S and $(T_{\alpha_1,\beta_2}u)^{(j)}(t)$ is equicontinuous on $[0,1]$ for $j = 0, \dots, n-2$. Moreover, since

$$\begin{aligned} (T_{\alpha_1,\beta_2}u)^{(n-1)}(t) &= -g_{n-2}\left(\tilde{u}^{[n-2]}(\alpha_1,\beta_2)(t_1), \dots, \tilde{u}^{[n-2]}(\alpha_1,\beta_2)(t_m)\right) \\ &\quad + g_{n-1}\left(\tilde{u}^{[n-2]}(\alpha_1,\beta_2)(t_1), \dots, \tilde{u}^{[n-2]}(\alpha_1,\beta_2)(t_m)\right) \\ &\quad + \int_t^1 F_{\alpha_1,\beta_2}(s,u(\cdot))ds - \int_0^1 sF_{\alpha_1,\beta_2}(s,u(\cdot))ds, \end{aligned}$$

the equicontinuity of $(T_{\alpha_1,\beta_2}u)^{(n-1)}(t)$ follows from the absolute continuity of the integrals. Thus, by the Arzelà-Ascola theorem, T_{α_1,β_2} is compact.

Let M be large enough so that

$$M > \max\big\{C(\alpha_1,\beta_2),\ nd + \max_{y\in[0,C(\alpha_1,\beta_2)]} \phi(y) \int_0^1 w(s)ds + \|\alpha\| + \|\beta\| + 1\big\}.$$

Define

$$\Omega = \{u \in C^{n-1}[0,1] : \|u\| < M\}.$$

For any $u \in \overline{\Omega}$, (3.15) and (3.16) still hold. Then

$$\|T_{\alpha_1,\beta_2}u\| \le nd + \max_{y\in[0,C(\alpha_1,\beta_2)]} \phi(y) \int_0^1 w(s)ds + \|\alpha\| + \|\beta\| + 1 < M.$$

Thus,

$$\deg(I - T_{\alpha_1,\beta_2}, \Omega, 0) = 1. \tag{3.17}$$

Let

$$\Omega_{\alpha_2} = \{u \in \Omega : u^{(i)}(t) > \alpha_2^{(i)}(t) \text{ for } t \in [0,1] \text{ and } i = 0, \dots, n-2\}$$

and

$$\Omega_{\beta_1} = \{u \in \Omega : u^{(i)}(t) < \beta_1^{(i)}(t) \text{ for } t \in [0,1] \text{ and } i = 0, \dots, n-2\}.$$

Since $\alpha_2^{(i)}(t) \not\leq \beta_1^{(i)}(t)$, $\alpha_2^{(i)}(t) \geq \alpha_1^{(i)}(t) > -M$, and $\beta_1^{(i)}(t) \leq \beta_2^{(i)}(t) < M$ for $t \in [0,1]$ and $i = 0, \dots, n-2$, it follows that

$$\Omega_{\alpha_2} \neq \emptyset \neq \Omega_{\beta_1}, \quad \overline{\Omega}_{\alpha_2} \cap \overline{\Omega}_{\beta_1} = \emptyset, \quad \Omega \setminus \{\overline{\Omega_{\alpha_2} \cup \Omega_{\beta_1}}\} \neq \emptyset.$$

We claim that

(i) if $u(t)$ is a solution of (3.7), (3.8) with $(\alpha, \beta) = (\alpha_1, \beta_2)$ and satisfies

$$u^{(i)}(t) \geq \alpha_2^{(i)}(t) \quad \text{for } t \in [0,1] \text{ and } i = 0, \dots, n-2, \tag{3.18}$$

then we have the strict inequalities

$$u^{(i)}(t) > \alpha_2^{(i)}(t) \quad \text{for } t \in [0,1] \text{ and } i = 0, \dots, n-2; \tag{3.19}$$

(ii) if $u(t)$ is a solution of BVP (3.7), (3.8) with $(\alpha, \beta) = (\alpha_1, \beta_2)$ and satisfies

$$u^{(i)}(t) \leq \beta_1^{(i)}(t) \quad \text{for } t \in [0,1] \text{ and } i = 0, \dots, n-2,$$

then we have the strict inequalities

$$u^{(i)}(t) < \beta_1^{(i)}(t) \quad \text{for } t \in [0,1] \text{ and } i = 0, \dots, n-2.$$

We first prove (i). By Lemma 3.2, $u(t)$ is a solution of (1.1), (1.2) satisfying (3.10) where $(\alpha, \beta) = (\alpha_1, \beta_2)$. Then, from (1.2), (2.2), (H4), and (3.18), we have

$$\begin{aligned} \alpha_2^{(i)}(0) &< g_i\big(\alpha_2^{(i)}(t_1), \dots, \alpha_2^{(i)}(t_m)\big) \\ &\leq g_i\big(u^{(i)}(t_1), \dots, u^{(i)}(t_m)\big) = u^{(i)}(0), \quad i = 0, \dots, n-2, \end{aligned}$$

and

$$\begin{aligned} \alpha_2^{(n-2)}(1) &< g_{n-1}\big(\alpha_2^{(n-2)}(t_1), \dots, \alpha_2^{(n-2)}(t_m)\big) \\ &\leq g_{n-1}\big(u^{(n-2)}(t_1), \dots, u^{(n-2)}(t_m)\big) = u^{(n-2)}(1); \end{aligned}$$

i.e.,

$$u^{(i)}(0) > \alpha_2^{(i)}(0), \ i = 0, \dots, n-2, \ u^{(n-2)}(1) > \alpha_2^{(n-2)}(1). \tag{3.20}$$

We now show that

$$u^{(n-2)}(t) > \alpha_2^{(n-2)}(t) \quad \text{for } t \in [0,1]. \tag{3.21}$$

If (3.21) does not hold, then, in view of (3.18) and (3.20) with $i = n-2$, there exists $t^* \in (0,1)$ such that $u^{(n-2)}(t) - \alpha_2^{(n-2)}(t)$ has the minimum value 0 at t^*. Thus, $u^{(n-2)}(t^*) = \alpha_2^{(n-2)}(t^*)$, $u^{(n-1)}(t^*) = \alpha_2^{(n-1)}(t^*)$, and $u^{(n)}(t^*) \geq \alpha_2^{(n)}(t^*)$. On the other hand, from (1.1), (2.1), (H2), and (3.18), we obtain that

$$\begin{aligned} u^{(n)}(t^*) &= -f\big(t^*, u(t^*), u'(t^*), \dots, u^{(n-1)}(t^*)\big) \\ &\leq -f\big(t^*, \alpha_2(t^*), \alpha_2'(t^*), \dots, \alpha_2^{(n-1)}(t^*)\big) \\ &< \alpha_2^{(n)}(t), \end{aligned}$$

which is a contradiction. Thus, (3.21) holds. Integrating (3.21) and using (3.20), we see that $u(t)$ satisfies (3.19). The proof for (ii) is similar and hence is omitted.

Now, by the claim (see (3.18)–(3.19)), BVP (3.7), (3.8) has no solution on $\partial\Omega_{\alpha_2}\cup\partial\Omega_{\beta_1}$. Hence,

$$\begin{aligned}\deg(I-T_{\alpha_1,\beta_2},\Omega,0) &= \deg(I-T_{\alpha_1,\beta_2},\Omega\setminus\{\overline{\Omega_{\alpha_2}\cup\Omega_{\beta_1}}\},0)\\ &\quad+\deg(I-T_{\alpha_1,\beta_2},\Omega_{\alpha_2},0)+\deg(I-T_{\alpha_1,\beta_2},\Omega_{\beta_1},0).\end{aligned}\tag{3.22}$$

Next, we show that

$$\deg(I-T_{\alpha_1,\beta_2},\Omega_{\alpha_2},0)=\deg(I-T_{\alpha_1,\beta_2},\Omega_{\beta_1},0)=1.\tag{3.23}$$

Let F_{α_2,β_2} be defined by (3.5) with (α,β) replaced by (α_2,β_2). Define an operator $T_{\alpha_2,\beta_2}: C^{n-1}[0,1]\to C[0,1]$ by

$$\begin{aligned}T_{\alpha_2,\beta_2}u(t) &= \sum_{i=0}^{n-1} g_i\left(\tilde{u}^{[i]}(\alpha_2,\beta_2)(t_1),\dots,\tilde{u}^{[i]}(\alpha_2,\beta_2)(t_m)\right)p_i(t)\\ &\quad+\int_0^1 G_{n-1}(t,s)F_{\alpha_2,\beta_2}(s,u(\cdot))ds.\end{aligned}$$

Then, by Lemma 3.1, $u(t)$ is a solution of (3.7), (3.8) with $(\alpha,\beta)=(\alpha_2,\beta_2)$ if and only if u is a fixed point of T_{α_2,β_2}. It can also be shown that T_{α_2,β_2} is compact. Arguing as before, it follows that $u(t)$ is a solution of (3.7), (3.8) with $(\alpha,\beta)=(\alpha_2,\beta_2)$ only if $u\in\Omega_{\alpha_2}$. Then,

$$\deg(I-T_{\alpha_2,\beta_2},\Omega\setminus\overline{\Omega}_{\alpha_2},0)=0.$$

Moreover, as in (3.15) and (3.16), it is easy to see that $T_{\alpha_2,\beta_2}(\overline{\Omega})\subseteq\Omega$, which in turn implies that

$$\deg(I-T_{\alpha_2,\beta_2},\Omega,0)=1.$$

Then,

$$\begin{aligned}\deg(I-T_{\alpha_1,\beta_2},\Omega_{\alpha_2},0) &= \deg(I-T_{\alpha_2,\beta_2},\Omega_{\alpha_2},0)\\ &= \deg(I-T_{\alpha_2,\beta_2},\Omega\setminus\overline{\Omega}_{\alpha_2},0)+\deg(I-T_{\alpha_2,\beta_2},\Omega_{\alpha_2},0)\\ &= \deg(I-T_{\alpha_2,\beta_2},\Omega,0)=1.\end{aligned}$$

Similarly, we can show that

$$\deg(I-T_{\alpha_1,\beta_2},\Omega_{\beta_1},0)=1.$$

Thus, (3.23) holds. From (3.17), (3.22), and (3.23), we reach the conclusion that

$$\deg(I-T_{\alpha_1,\beta_2},\Omega\setminus\{\overline{\Omega_{\alpha_2}\cup\Omega_{\beta_1}}\},0)=-1.\tag{3.24}$$

From (3.23), (3.24), and Lemma 3.2, it follows that (1.1), (1.2) has three solutions in Ω_{α_2}, Ω_{β_1}, and $\Omega\setminus\{\overline{\Omega_{\alpha_2}\cup\Omega_{\beta_1}}\}$, respectively, satisfying (2.12) and (2.13). This completes the proof of the theorem. $\square$

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